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Integral Inequalities in Higher Dimensional Spaces

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Abstract. Integral inequalities play an important role in many different areas including differential equations, integral equations, variational calculus, etc. In this work, we present some new higher dimensional integral inequalities involving monotonic or convex functions in higher dimensional spaces. These are then applied to solve directly some Calculus of Variations problems for optimal solutions, effectively.

Keywords: *Integral inequalities, calculus of variations*

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1. Introduction and basic integral inequalities

In this paper, we discuss some integral inequalities of functions in higher dimensional spaces and some integral inequalities of vector-valued functions. By extending Cheung's idea in [1] to higher dimensional spaces, we first derive some basic integral inequalities involving functions of several variables with certain kinds of monotonicity, and then we extend these results to inequalities involving convex functions. Finally, we give some applications to the calculus of variations.

In this section, we derive some integral inequalities in higher dimensional spaces involving functions on a bounded rectangle. Generally speaking, such integral inequalities require some kind of monotonicity of the functions involved; but as seen in Lemma 1 and Theorem 2 below, this is not strictly necessary in

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certain situations. Some results in this section will be generalized to more general settings in Section 3.

Let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{R}^n$. In the following, let $D = \prod_{i=1}^n [a_i, b_i]$, $M = \prod_{i=1}^n (b_i - a_i)$, $\mathbf{x} = (x_1, \cdots, x_n) \in D$, $d\mathbf{x} = dx_1 dx_2 \cdots dx_n$ and $\mathbf{h}_j = (h_1^j, \cdots, h_n^j) \in D$.

Lemma 1. *Let $f_j, g_j : D \to \mathbb{R}$, $j = 1, \cdots, m$, be continuous functions. Let $\mathbf{f} = (f_1, \cdots, f_m)$ and $\mathbf{g} = (g_1, \cdots, g_m)$. Suppose that for each j, $1 \le j \le m$, $\mathbf{h}_j \in D$ is a point such that*

$$f_j(\mathbf{h}_j) = \frac{1}{M} \int_D f_j(\mathbf{x})\, d\mathbf{x}.$$

Let

$$\begin{aligned}
\bar{\mathbf{f}}(\mathbf{x}) &= \big(\bar{f}_1(\mathbf{x}), \cdots, \bar{f}_m(\mathbf{x})\big) = \big(f_1(\mathbf{x}) - f_1(\mathbf{h}_1), \cdots, f_m(\mathbf{x}) - f_m(\mathbf{h}_m)\big) \\
\bar{\mathbf{g}}(\mathbf{x}) &= \big(\bar{g}_1(\mathbf{x}), \cdots, \bar{g}_m(\mathbf{x})\big) = \big(g_1(\mathbf{x}) - g_1(\mathbf{h}_1), \cdots, g_m(\mathbf{x}) - g_m(\mathbf{h}_m)\big).
\end{aligned}$$

Then the following inequaliets hold:

(i) *If $\langle \bar{\mathbf{f}}, \bar{\mathbf{g}} \rangle \le 0$, then*

$$\Big\langle \int_D \mathbf{f}(\mathbf{x})\, d\mathbf{x}, \int_D \mathbf{g}(\mathbf{x})\, d\mathbf{x} \Big\rangle \ge M \int_D \big\langle \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \big\rangle\, d\mathbf{x}. \tag{1}$$

(ii) *If $\langle \bar{\mathbf{f}}, \bar{\mathbf{g}} \rangle \ge 0$, then*

$$\Big\langle \int_D \mathbf{f}(\mathbf{x})\, d\mathbf{x}, \int_D \mathbf{g}(\mathbf{x})\, d\mathbf{x} \Big\rangle \le M \int_D \big\langle \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \big\rangle\, d\mathbf{x}. \tag{2}$$

Furthermore, in both cases the equality holds if and only if $\langle \bar{\mathbf{f}}, \bar{\mathbf{g}} \rangle \equiv 0$.

Proof. If $\langle \bar{\mathbf{f}}, \bar{\mathbf{g}} \rangle \le 0$, then we have

$$\begin{aligned}
\Big\langle \int_D \mathbf{f}(\mathbf{x})\, d\mathbf{x}, \int_D \mathbf{g}(\mathbf{x})\, d\mathbf{x} \Big\rangle &= \sum_{j=1}^m \Big[\int_D f_j(\mathbf{x})\ d\mathbf{x} \Big] \Big[\int_D g_j(\mathbf{x})\, d\mathbf{x} \Big] \\
&= M \sum_{j=1}^m \int_D f_j(\mathbf{h}_j) g_j(\mathbf{x})\, d\mathbf{x} \\
&= M \sum_{j=1}^m \int_D f_j(\mathbf{x}) g_j(\mathbf{x})\, d\mathbf{x} - M \sum_{j=1}^m \int_D \bar{f}_j(\mathbf{x}) \bar{g}_j(\mathbf{x})\, d\mathbf{x} \\
&= M \int_D \big\langle \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \big\rangle\, d\mathbf{x} - M \int_D \big\langle \bar{\mathbf{f}}(\mathbf{x}), \bar{\mathbf{g}}(\mathbf{x}) \big\rangle\, d\mathbf{x} \\
&\ge M \int_D \big\langle \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \big\rangle\, d\mathbf{x},
\end{aligned}$$

and it is obvious that the equality holds if and only if $\langle \bar{\mathbf{f}}, \bar{\mathbf{g}} \rangle \equiv 0$. This shows case (i). The proof of case (ii) is analogous. ∎

Definition 1. Let $f, g : D \to \mathbb{R}$ be real-valued functions. f is said to be *parallel to* g if $f(\mathbf{x}) = f(\mathbf{y})$ whenever $g(\mathbf{x}) = g(\mathbf{y})$ and $f(\mathbf{x}) < f(\mathbf{y})$ whenever $g(\mathbf{x}) < g(\mathbf{y})$. f is said to be *anti-parallel to* g if $f(\mathbf{x}) = f(\mathbf{y})$ whenever $g(\mathbf{x}) = g(\mathbf{y})$ and $f(\mathbf{x}) < f(\mathbf{y})$ whenever $g(\mathbf{x}) > g(\mathbf{y})$.

It is clear that the 'parallelism' is an equivalence relation in the set of all real-valued functions on D.

Theorem 1. *Let $f, g : D \to \mathbb{R}$ be continuous. Suppose that f is anti-parallel to g. Then*

$$\left[\int_D f(\mathbf{x})\, d\mathbf{x}\right] \left[\int_D g(\mathbf{x})\, d\mathbf{x}\right] \geq M \int_D f(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x}, \tag{3}$$

where the equality holds if and only if $f \equiv const$ or $g \equiv const$.

Proof. Let $\mathbf{h} \in D$ be a point such that $f(\mathbf{h}) = \frac{1}{M} \int_D f(\mathbf{x})\, d\mathbf{x}$. Let $\bar{f}(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{h})$, $\bar{g}(\mathbf{x}) = g(\mathbf{x}) - g(\mathbf{h})$. By the parallelism of f and g, $\bar{f} \cdot \bar{g} \leq 0$ on D, and thus from Lemma 1 it follows that

$$\left[\int_D f(\mathbf{x})\, d\mathbf{x}\right] \left[\int_D g(\mathbf{x})\, d\mathbf{x}\right] \geq M \int_D f(\mathbf{x}) g(\mathbf{x})\ d\mathbf{x},$$

where the equality holds if and only if $\bar{f} \cdot \bar{g} \equiv 0$.

It remains to show that $\bar{f} \cdot \bar{g} \equiv 0$ if and only if $f \equiv const$ or $g \equiv const$. It is trivial that if $f \equiv const$ or $g \equiv const$, then $\bar{f} \cdot \bar{g} \equiv 0$. Conversely, suppose that $\bar{f} \cdot \bar{g} \equiv 0$. Since $f : D \to \mathbb{R}$ is continuous, the same is true for $\bar{f}$. Hence, there are $\mathbf{h}_1, \mathbf{h}_2 \in D$ such that for all $\mathbf{x} \in D$,

$$\bar{f}(\mathbf{h}_1) \leq \bar{f}(\mathbf{x}) \leq \bar{f}(\mathbf{h}_2), \tag{4}$$

and thus

$$\bar{f}(\mathbf{h}_1) \leq \bar{f}(\mathbf{h}) = 0 \leq \bar{f}(\mathbf{h}_2). \tag{5}$$

If the equalities of (5) hold, then $f \equiv$ constant. Otherwise, by the condition $\int_D \bar{f}(\mathbf{x}) d\mathbf{x} = 0$, we have $\bar{f}(\mathbf{h}_1) < 0 < \bar{f}(\mathbf{h}_2)$. From the assumption $\bar{f} \cdot \bar{g} \equiv 0$, it follows that $\bar{g}(\mathbf{h}_1) = \bar{g}(\mathbf{h}_2) = 0$. By (4) and the parallelism of f and g, $\bar{g}(\mathbf{h}_1) \geq \bar{g}(\mathbf{x}) \geq \bar{g}(\mathbf{h}_2)$ for all $\mathbf{x} \in D$. However, then it forces $\bar{g} \equiv 0$ on D, which completes the proof. ∎

The condition of parallelism of functions is not necessary in the following result.

Theorem 2. *Let $f : D \to \mathbb{R}$ be continuous and positive. Then*

$$\left[\int_D f(\mathbf{x})^m\, d\mathbf{x}\right] \left[\int_D \frac{1}{f(\mathbf{x})}\, d\mathbf{x}\right]^m \geq M^{m+1}$$

for all $m \in \mathbb{N}$, where the equality holds if and only if $f \equiv const$.

Proof. Let $p = f^m$, and $q = \frac{1}{f}$. Let $\mathbf{h} \in D$ be a point such that $p(\mathbf{h}) = \frac{1}{M}\int_D p(\mathbf{x})\,d\mathbf{x}$. Let $\bar{p}(\mathbf{x}) = p(\mathbf{x}) - p(\mathbf{h})$ and $\bar{q}(\mathbf{x}) = q(\mathbf{x}) - q(\mathbf{h})$. It is elementary to check that $\bar{p} \cdot \bar{q} \le 0$ on D. By Lemma 1, we have

$$\left[\int_D p(\mathbf{x})\,d\mathbf{x}\right]\left[\int_D q(\mathbf{x})\,d\mathbf{x}\right] \ge M\int_D p(\mathbf{x})q(\mathbf{x})\,d\mathbf{x},$$

where the equality holds if and only if $\bar{p} \cdot \bar{q} \equiv 0$. Hence,

$$\left[\int_D f(\mathbf{x})^m\,d\mathbf{x}\right]\left[\int_D \frac{1}{f(\mathbf{x})}\,d\mathbf{x}\right] \ge M\int_D f(\mathbf{x})^{m-1}\,d\mathbf{x},$$

where the equality holds if and only if $f \equiv const$. As this is true for all $m \in \mathbb{N}$, we have by induction

$$\left[\int_D f(\mathbf{x})^m\,d\mathbf{x}\right]\left[\int_D \frac{1}{f(\mathbf{x})}\,d\mathbf{x}\right]^m \ge M^{m+1},$$

where the equality holds if and only if $f \equiv const$. ∎

Theorem 3. *Let $f, g : D \to \mathbb{R}$ be continuous. Suppose that f is parallel to g. Then*

$$\left[\int_D f(\mathbf{x})\,d\mathbf{x}\right]\left[\int_D g(\mathbf{x})\,d\mathbf{x}\right] \le M\int_D f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}, \tag{6}$$

where the equality holds if and only if $f \equiv const$ or $g \equiv const$.

Proof. The theorem follows immediately by applying Theorem 1 to the functions f and $-g$. ∎

Corollary 1. *Let $f_j : D \to \mathbb{R}$ be nonnegative and continuous for all $j = 1, \cdots, m$. Suppose that f_j, $j = 1, \cdots, m$, are pairwisely parallel. Then*

$$\prod_{j=1}^m \int_D f_j(\mathbf{x})\,d\mathbf{x} \le M^{m-1}\int_D \prod_{j=1}^m f_j(\mathbf{x})\,d\mathbf{x}. \tag{7}$$

Furthermore, if none of the f_j's is the zero function, then the equality holds if and only if at most one of the f_j's is nonconstant.

Proof. Inequality (7) clearly holds by induction. Next, assume that none of the f_j's is the zero function. It is clear that if at most one of the f_j's is nonconstant, the equality holds. Conversely, suppose that, without loss of generality, f_1 and f_2 are nonconstant, then from Theorem 3, it follows that

$$\left[\int_D f_1(\mathbf{x})\,d\mathbf{x}\right]\left[\int_D f_2(\mathbf{x})\,d\mathbf{x}\right] < M\int_D f_1(\mathbf{x})f_2(\mathbf{x})\,d\mathbf{x}.$$

Since all of f_j's are nonnegative, $\prod_{j=3}^{m} \int_D f_j(\mathbf{x})\, d\mathbf{x} > 0$. By the parallelism of the f_j's, we have

$$\begin{aligned}\prod_{j=1}^{m} \int_D f_j(\mathbf{x})\, d\mathbf{x} &< M\left[\int_D f_1(\mathbf{x}) f_2(\mathbf{x})\, d\mathbf{x}\right]\left[\prod_{j=3}^{m}\int_D f_j(\mathbf{x})\, d\mathbf{x}\right] \\ &\le M^{m-2}\left[\int_D f_1(\mathbf{x}) f_2(\mathbf{x})\, d\mathbf{x}\right]\left[\int_D \prod_{j=3}^{m} f_j(\mathbf{x})\, d\mathbf{x}\right] \\ &\le M^{m-1}\int_D \prod_{j=1}^{m} f_j(\mathbf{x})\, d\mathbf{x},\end{aligned}$$

which implies that the equality does not hold. This completes the proof. ∎

Corollary 2. *Let $f_i : D \to \mathbb{R}$, $i = 1, \cdots, k$, be nonnegative; and $g_j : D \to \mathbb{R}$, $j = 1, \cdots, l$, be nonpositive. Suppose that $f_1, \cdots, f_k, -g_1, \cdots, -g_l$ are pairwisely parallel. Then the following inequalities hold:*

(i) *If l is odd, then*

$$\left[\prod_{i=1}^{k}\int_D f_i(\mathbf{x})\, d\mathbf{x}\right]\left[\prod_{j=1}^{l}\int_D g_j(\mathbf{x})\, d\mathbf{x}\right] \ge M^{k+l-1}\int_D \left[\prod_{i=1}^{k} f_i(\mathbf{x})\right]\left[\prod_{j=1}^{l} g_j(\mathbf{x})\right] d\mathbf{x}.$$

(ii) *If l is even, then*

$$\left[\prod_{i=1}^{k}\int_D f_i(\mathbf{x})\, d\mathbf{x}\right]\left[\prod_{j=1}^{l}\int_D g_j(\mathbf{x})\, d\mathbf{x}\right] \le M^{k+l-1}\int_D \left[\prod_{i=1}^{k} f_i(\mathbf{x})\right]\left[\prod_{j=1}^{l} g_j(\mathbf{x})\right] d\mathbf{x}.$$

Proof. It follows immediately from Corollary 1 when applied to the functions $f_1, \cdots, f_k, -g_1, \cdots, -g_l$. ∎

2. Generalized integral inequalities in two dimensions

In this section, we show some useful integral inequalities involving mean values of convex functions. These can be applied to derive further interesting integral inequalities and some improvements of certain results obtained in Section 1. For the sake of simplicity, we only present these results in a 2-dimensional setting, but the analogue in higher dimensional situations should be transparent.

Theorem 4. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous, $U \subset \mathbb{R}$ be an open interval containing the image of f, and let $F : U \to \mathbb{R}$ be convex (resp., concave). Then*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(f(x,y))\, dydx \\ &\qquad \ge F\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\, dydx\right]\end{aligned} \tag{8}$$

(resp., the reverse inequality). Furthermore, if F is strictly convex (resp., strictly concave), the equality holds if and only if $f \equiv const$.

Proof. Let $(h,k) \in [a,b] \times [c,d]$ be a point such that

$$f(h,k) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Since F is convex [2], there exists $m \in \mathbb{R}$ such that

$$F(f(h,k)+t) \geq F(f(h,k)) + mt \qquad \text{for all} \quad t \in \mathbb{R}. \tag{9}$$

Therefore

$$\begin{aligned}
\int_a^b \int_c^d F(f(x,y))\,dy\,dx &= \int_a^b \int_c^d F(f(h,k) + (f(x,y) - f(h,k)))\,dy\,dx \\
&\geq \int_a^b \int_c^d [F(f(h,k)) + m(f(x,y) - f(h,k))]\,dy\,dx \\
&= (b-a)(d-c)F(f(h,k)),
\end{aligned}$$

and so inequality (8) follows.

It is obvious that the equality holds if $f \equiv const$. Conversely, if F is strictly convex, strict inequality in (9) holds for all $t \neq 0$, that is,

$$F(f(h,k)+t) > F(f(h,k)) + mt \qquad \text{for all} \quad t \neq 0.$$

If $f \neq$ constant, then there exists $(x,y) \in [a,b] \times [c,d]$ such that $f(x,y) - f(h,k) \neq 0$; and by continuity, there is an open sub-rectangle of $[a,b] \times [c,d]$ on which $f(x,y) - f(h,k) \neq 0$. Hence,

$$\begin{aligned}
\int_a^b \int_c^d F(f(x,y))\,dy\,dx &= \int_a^b \int_c^d F(f(h,k) + (f(x,y) - f(h,k)))\,dy\,dx \\
&> \int_a^b \int_c^d [F(f(h,k)) + m(f(x,y) - f(h,k))]\,dy\,dx \\
&= (b-a)(d-c)F(f(h,k)),
\end{aligned}$$

and so the assertion for $f \equiv const$ follows. The case of concavity is analogously shown. ∎

Corollary 3. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous, $U \subset \mathbb{R}$ be an open interval containing the image of f, and let $F : U \to \mathbb{R}$ be C^2 with $F'' \geq 0$ (resp., $F'' \leq 0$). Then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(f(x,y))\,dy\,dx \\
&\qquad \geq F\left[\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx\right]
\end{aligned} \tag{10}$$

(resp., the reverse inequality). Furthermore, if $F'' = 0$ only at isolated points, the equality holds if and only if $f \equiv const$.

Proof. A function F with $F'' \geq 0$ is convex; and a function F with $F'' \geq 0$ and $F'' = 0$ only at isolated points is strictly convex [2]. Hence, this statement follows immediately from Theorem 4. ∎

Corollary 4. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous, $U \subset \mathbb{R}$ be an open interval containing the image of f, and let $\varphi, \psi : U \to \mathbb{R}$ be C^1 functions such that*

(i) *$\varphi\,'$ doesn't change sign and may vanish only at isolated points,*

(ii) *$\psi\,' \geq 0$ and may vanish only at isolated points, and*

(iii) *$\psi \circ \varphi^{-1}$ is convex (resp., concave) on $\varphi(U)$.*

Then

$$\begin{aligned}\psi^{-1}&\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi\circ f(x,y)\,dy\,dx\right]\\ &\qquad\geq \varphi^{-1}\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi\circ f(x,y)\,dy\,dx\right]\end{aligned} \tag{11}$$

(resp., the reverse inequality). Furthermore, if $\psi \circ \varphi^{-1}$ is strictly convex (resp., strictly concave), the equality holds if and only if $f \equiv const$.

Proof. By (i) and (ii), both φ^{-1} and ψ^{-1} exist. Replacing F and f by $\psi \circ \varphi^{-1}$ and $\varphi \circ f$, respectively, from Theorem 4 it follows that

$$\begin{aligned}\frac{1}{(b-a)(d-c)}&\int_a^b\int_c^d (\psi\circ\varphi^{-1})(\varphi\circ f)(x,y)\,dy\,dx\\ &\geq (\psi\circ\varphi^{-1})\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi\circ f(x,y)\,dy\,dx\right].\end{aligned}$$

Since ψ is strictly increasing, this gives inequality (11). Finally, when the convexity of $\psi \circ \varphi^{-1}$ is strict, by Theorem 4 the equality holds if and only if $\varphi \circ f \equiv const$, or equivalently, $f \equiv const$. The case of concavity is analogously proven. ∎

Corollary 5. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous, $U \subset \mathbb{R}$ be an open interval containing the image of f, and let $\varphi, \psi : U \to \mathbb{R}$ be C^2 functions such that*

(i) *$\varphi\,'$ doesn't change sign and may vanish only at isolated points,*

(ii) *$\psi\,' \geq 0$ and may vanish only at isolated points, and*

(iii) *$\psi \circ \varphi^{-1} \in C^2$, $(\psi \circ \varphi^{-1})'' \geq 0$ (resp., ≤ 0) on $\varphi(U)$.*

Then

$$\psi^{-1}\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi\circ f(x,y)\,dy\,dx\right] \\ \geq \varphi^{-1}\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \varphi\circ f(x,y)\,dy\,dx\right] \tag{12}$$

(resp., the reverse inequality). Furthermore, if $(\psi\circ\varphi^{-1})''=0$ *only at isolated points, the equality holds if and only if* $f\equiv const.$

Proof. The assertion follows immediately from Corollary 4. ∎

Corollary 6. *Suppose that* $f:[a,b]\times[c,d]\to\mathbb{R}$ *is continuous and positive. Then the following inequalities hold:*

(i) *If* $\alpha<0$ *or* $\alpha>1$*, then*

$$[(b-a)(d-c)]^{\alpha-1}\int_a^b\int_c^d f(x,y)^\alpha\,dy\,dx \geq \left[\int_a^b\int_c^d f(x,y)\,dy\,dx\right]^\alpha.$$

(ii) *If* $0<\alpha<1$*, then*

$$[(b-a)(d-c)]^{\alpha-1}\int_a^b\int_c^d f(x,y)^\alpha\,dy\,dx \leq \left[\int_a^b\int_c^d f(x,y)\,dy\,dx\right]^\alpha.$$

Furthermore, in both cases the equality holds if and only if $f\equiv const.$

Proof. Let $F(z)=z^\alpha$, $z>0$. Then $F''(z)=\alpha(\alpha-1)z^{\alpha-2}$, $z>0$. Since $F''(z)>0$ for $\alpha<0$ or $\alpha>1$ and $F''(z)<0$ for $0<\alpha<1$, by Corollary 3 the results follow. ∎

Corollary 7. *Suppose that* $f:[a,b]\times[c,d]\to\mathbb{R}$ *is continuous and positive. Then the following inequalities hold:*

(i) *If* $\alpha>0$ *or* $\alpha<-1$*, then*

$$\left[\int_a^b\int_c^d \frac{1}{f(x,y)}\,dy\,dx\right]^\alpha\left[\int_a^b\int_c^d f(x,y)^\alpha\,dy\,dx\right] \geq [(b-a)(d-c)]^{\alpha+1}.$$

(ii) *If* $-1<\alpha<0$*, then*

$$\left[\int_a^b\int_c^d \frac{1}{f(x,y)}\,dy\,dx\right]^\alpha\left[\int_a^b\int_c^d f(x,y)^\alpha\,dy\,dx\right] \leq [(b-a)(d-c)]^{\alpha+1}.$$

Furthermore, in both cases the equality holds if and only if $f\equiv const.$

Proof. The results follow from Corollary 6 by replacing f by $\frac{1}{f}$ and α by $-\alpha$. ∎

Corollary 8. *If $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, then*

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \exp\left(f(x, y)\right) dy\, dx \geq \exp\left[\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dy\, dx\right],$$

and the equality holds if and only if $f \equiv const.$

Proof. The statement follows from Corollary 3 by taking $F(z) = \exp(z)$. ∎

Corollary 9. *If $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous and positive, then*

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \ln\left(f(x, y)\right) dy\, dx \leq \ln\left[\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dy\, dx\right],$$

and the equality holds if and only if $f \equiv const.$

Proof. Let $F(z) = \ln z$, $z > 0$. Then $F''(z) = -\frac{1}{z^2} < 0$ for all $z > 0$. Hence, the statement follows from Corollary 3. ∎

Remark. It is evident that all results in sections 2 and 3 above are valid in a slightly more general setting, namely, instead of continuous functions on D (resp. $[a, b] \times [c, d]$), it is sufficient to require the functions under consideration to be integrable on a finite measure space (X, μ, Σ) with the property that there exists some $p \in X$ such that $f(p) = \frac{1}{\mu(X)} \int_X f d\mu$, which is easily seen to be satisfied if $f(X)$ is a bounded interval in $\mathbb{R}$. However, in order that this article can be accessible by a broader class of readers including physicists and engineers, we chose the present less general setting instead.

3. Applications to the Calculus of Variations

The results in Section 2 can be applied to solving certain Calculus of Variations problems directly for optimal solutions. For the sake of simplicity, we only work on some less intricate cases. However, as the method of treatment is rather algorithmic, it is easily seen that the same techniques can be applied to more complicated situations. The upshot of the treatment is that we can obtain the optimal solution directly without having to go through the classical steps of deriving and solving the Euler-Lagrange equations, which for most of the time is very tedious if not impossible.

Example 1. Let $\varphi : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and positive. Consider the functional

$$I = \int_a^b \int_c^d \varphi(x,y) f_{12}(x,y)^\alpha \, dy \, dx \,, \qquad \alpha \in \mathbb{R},$$

and for all C^2 functions $f : [a,b] \times [c,d] \to \mathbb{R}$ satisfying

$$\begin{aligned} f_{12} &> 0 && \text{on } [a,b]\times[c,d] \\ f(a,y) = f(x,c) &= 0 && \forall x \in [a,b], y \in [c,d] \\ f(b,d) &= M > 0\,, \end{aligned}$$

where, as usual, $f_{12} = \partial^2 f / \partial x \partial y$. Denote by $\Delta = (b-a)(d-c)$.

(i) If $\alpha < 0$ or $\alpha > 1$, then the minimum of I occurs, when

$$f(x,y) = \frac{1}{C} \int_a^x \int_c^y \frac{1}{\varphi(s,t)^{\frac{1}{\alpha}}} \, dt \, ds \,,$$

where $C = \frac{1}{M} \int_a^b \int_c^d \frac{1}{\varphi(s,t)^{\frac{1}{\alpha}}} \, dt \, ds$ and $I_{\min} = \frac{\Delta}{C^\alpha}$.

(ii) If $0 < \alpha < 1$, then the maximum of I occurs when

$$f(x,y) = \frac{1}{C} \int_a^x \int_c^y \frac{1}{\varphi(s,t)^{\frac{1}{\alpha}}} \, dt \, ds \,,$$

where $C = \frac{1}{M} \int_a^b \int_c^d \frac{1}{\varphi(s,t)^{\frac{1}{\alpha}}} \, dt \, ds$ and $I_{\max} = \frac{\Delta}{C^\alpha}$.

Proof. `Case (i):` By Corollary 6,

$$I \geq \frac{1}{\Delta^{\alpha-1}} \left[\int_a^b \int_c^d \varphi(x,y)^{\frac{1}{\alpha}} f_{12}(x,y) \, dy \, dx \right]^\alpha , \tag{13}$$

where the equality holds if and only if $\varphi(x,y)^{\frac{1}{\alpha}} f_{12}(x,y) = \frac{1}{C}$ for some constant $C > 0$. Since

$$f(x,y) = \int_a^x \int_c^y f_{12}(s,t) \, dt \, ds + f(a,y) + f(x,c) - f(a,c),$$

the equality in (13) holds if and only if

$$f(x,y) = \frac{1}{C} \int_a^x \int_c^y \frac{1}{\varphi(s,t)^{\frac{1}{\alpha}}} \, dt \, ds.$$

From the condition $f(b,d) = M$, it follows that $C = \frac{1}{M} \int_a^b \int_c^d \frac{1}{\varphi(s,t)^{\frac{1}{\alpha}}} \, dt \, ds$. Hence, we have

$$I_{\min} = \frac{1}{\Delta^{\alpha-1}} \left[\int_a^b \int_c^d \frac{1}{C} \, dy \, dx \right]^\alpha = \frac{1}{\Delta^{\alpha-1}} \left[\frac{1}{C} \Delta \right]^\alpha = \frac{\Delta}{C^\alpha}.$$

Case (ii) is analogous to Case (i). ∎

Example 2. Let $\varphi : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and positive. Among all C^2 functions $f : [a,b] \times [c,d] \to \mathbb{R}$ with

$$\begin{aligned} f_{12} &> 0 && \text{on } [a,b] \times [c,d] \\ f(a,y) = f(x,c) &= 0 && \forall x \in [a,b], y \in [c,d] \\ f(b,d) &= M > 0\,, \end{aligned}$$

the functional

$$I = \int_a^b \int_c^d \varphi(x,y) e^{f_{12}(x,y)}\, dy\, dx$$

attains its minimum when

$$f(x,y) = C(x-a)(y-c) - \int_a^x \int_c^y \ln \varphi(s,t)\, dt\, ds$$

where $C = \frac{1}{\Delta}\left[\int_a^b \int_c^d \ln \varphi(s,t)\ dtds + M\right]$, and $I_{\min} = \Delta \exp(C)$.

Proof. By Corollary 8,

$$I \geq \Delta \exp\left[\frac{1}{\Delta} \int_a^b \int_c^d (\ln \varphi(x,y) + f_{12}(x,y))\, dy\, dx\right], \tag{14}$$

where the equality holds if and only if $\ln \varphi(x,y) + f_{12}(x,y) = C$, that is $f_{12}(x,y) = C - \ln \varphi(x,y)$ for some constant C. Since

$$f(x,y) = \int_a^x \int_c^y f_{12}(s,t)\, dt\, ds + f(a,y) + f(x,c) - f(a,c),$$

the equality in (14) holds if and only if

$$f(x,y) = C(x-a)(y-c) - \int_a^x \int_c^y \ln \varphi(s,t)\, dt\, ds.$$

From the condition $f(b,d) = M$, it follows that $M = C\Delta - \int_a^b \int_c^d \ln \varphi(s,t)\, dt\, ds$, and thus $C = \frac{1}{\Delta}\big[\int_a^b \int_c^d \ln \varphi(s,t)\, dt\, ds + M\big]$. Hence, we have

$$I_{\min} = \Delta \exp(C)\,. \qquad \blacksquare$$

Example 3. Let $\varphi : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and positive. Among all C^2 functions $f : [a,b] \times [c,d] \to \mathbb{R}$ with

$$\begin{aligned} f_{12} &> 0 && \text{on } [a,b] \times [c,d] \\ f(a,y) = f(x,c) &= 0 && \forall x \in [a,b], y \in [c,d] \\ f(b,d) &= M > 0\,, \end{aligned}$$

the functional

$$I = \int_a^b \int_c^d \ln(\varphi(x,y) f_{12}(x,y))\, dy\, dx$$

attains its maximum when

$$f(x,y) = \frac{1}{C} \int_a^x \int_c^y \frac{1}{\varphi(s,t)}\, dt\, ds\,,$$

where $C = \frac{1}{M} \int_a^b \int_c^d \frac{1}{\varphi(s,t)}\, dt\, ds$ and $I_{\max} = -\Delta \ln C$.

Proof. By Corollary 9,

$$I \le \Delta \ln \left[\frac{1}{\Delta} \int_a^b \int_c^d \varphi(x,y) f_{12}(x,y)\, dy\, dx \right], \tag{15}$$

where the equality holds if and only if $\varphi(x,y) f_{12}(x,y) = \frac{1}{C}$ for some constant $C > 0$. Since

$$f(x,y) = \int_a^x \int_c^y f_{12}(s,t)\, dt\, ds + f(a,y) + f(x,c) - f(a,c),$$

the equality in (15) holds if and only if

$$f(x,y) = \frac{1}{C} \int_a^x \int_c^y \frac{1}{\varphi(s,t)}\, dt\, ds.$$

From the condition $f(b,d) = M$, it follows that $C = \frac{1}{M} \int_a^b \int_c^d \frac{1}{\varphi(s,t)}\, dt\, ds$. Hence, we have

$$I_{\max} = \Delta \ln \left[\frac{1}{\Delta} \int_a^b \int_c^d \frac{1}{C}\, dy\, dx \right] = \Delta \ln \left(\frac{1}{C} \right) = -\Delta \ln C. \qquad \blacksquare$$

Remark. These results cannot be obtained by using the classical approach.

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