Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 24 (2005), No. 2, 327–340

Mackey Topologies on Vector-Valued Function Spaces

Marian Nowak

Abstract. Let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ), and let $(X, \|\cdot\|_X)$ be a real Banach space. Let $E(X)$ be a subspace of the space $L^0(X)$ of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. Let $E(X)^\sim_n$ stand for the order continuous dual of $E(X)$. We examine the Mackey topology $\tau(E(X), E(X)^\sim_n)$ in case when it is locally solid. It is shown that $\tau(E(X), E(X)^\sim_n)$ is the finest Hausdorff locally convex-solid topology on $E(X)$ with the Lebesgue property. We obtain that the space $(E(X), \tau(E(X), E(X)^\sim_n))$ is complete and sequentially barreled whenever E is perfect. As an application, we obtain the Hahn-Vitali-Saks type theorem for sequences in $E(X)^\sim_n$. In particular, we consider the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)^\sim_n)$ on Orlicz-Bochner spaces $L^\Phi(X)$. We show that the space $(L^\Phi(X), \tau(L^\Phi(X), L^\Phi(X)^\sim_n))$ is complete iff L^Φ is perfect. Moreover, it is shown that the Mackey topology $\tau(L^\infty(X), L^\infty(X)^\sim_n)$ is a mixed topology.

Keywords: *Vector-valued function spaces, Orlicz-Bochner spaces, locally solid topologies, Lebesgue topologies, Mackey topologies, mixed topologies, sequential barreledness*

MSC 2000: 46E40, 46E30, 46A20, 46A70

1. Introduction and preliminaries

Given a topological vector space (L, ξ) by $(L, \xi)^*$ we will denote its topological dual. We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L, K\rangle$. In the theory of topological function spaces the Mackey topology $\tau(E, E^\sim_n)$ on a function space E is of importance (see [8, 7, 14]). It is well known that $\tau(E, E^\sim_n)$ is the finest Hausdorff locally convex-solid topology on E with the Lebesgue property.

Marian Nowak: Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, ul. Szafrana 4A, 65–516 Zielona Góra, Poland;
M.Nowak@wmie.uz.zgora.pl

ISSN 0232-2064 / $ 2.50

In this paper we consider the Mackey topology $\tau(E(X), E(X)_n^\sim)$ on a vector-valued function space $E(X)$ whenever E is an ideal of L^0 (over a σ-finite measure space), X is a Banach space and $E(X)_n^\sim$ stand for the order continuous dual of $E(X)$. In Section 2 we examine some properties of solid sets in the order continuous dual $E(X)_n^\sim$ of $E(X)$. We examine the properties of $\tau(E(X), E(X)_n^\sim)$ in case it is locally solid. In Section 3 we show that $\tau(E(X), E(X)_n^\sim)$ is the finest Hausdorff locally convex-solid topology on $E(X)$ with the Lebesgue property (see Theorem 3.2). We obtain that the space $(E(X), \tau(E(X), E(X)_n^\sim))$ is complete and sequentially barreled whenever E is perfect (see Theorem 3.3 and Theorem 3.5). As an application, we obtain that $E(X)_n^\sim$ is $\sigma(E(X)_n^\sim, E(X))$-sequentially complete (see Theorem 3.6). In Section 4 we consider the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)_n^\sim)$ on Orlicz-Bochner spaces $L^\Phi(X)$ (Φ is not necessarily convex). It is shown that the space $(L^\Phi(X), \tau(L^\Phi(X), L^\Phi(X)_n^\sim))$ is complete if and only if L^Φ is perfect (see Theorem 4.4). In particular, we obtain that $\tau(L^\infty(X), L^\infty(X)_n^\sim)$ is a mixed topology (see Theorem 4.5).

First we establish terminology concerning function spaces (see [2, 10, 27]). Let (Ω, Σ, μ) be a complete σ-finite measure space. Let L^0 denote the space of μ-equivalence classes of all Σ-measurable real valued functions defined and finite a.e. on Ω. For a subset M of L^0 by $\operatorname{supp} M$ we denote the support of M, i.e., the smallest set in Σ containing (a.e.) the supports of all $u \in M$ (see [10, Chapter 1.6]). Let χ_A stand for the characteristic function of a set A, and let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all natural and real numbers.

Let E be an ideal of L^0 with $\operatorname{supp} E = \Omega$, and let E' stand for the Köthe dual of E, i.e., $E' = \{v \in L^0 : \int_\Omega |u(\omega)v(\omega)| \, d\mu < \infty \text{ for all } u \in E\}$. Throughout the paper we assume that $\operatorname{supp} E' = \Omega$. Let $E^\sim$, $E_n^\sim$ and $E_s^\sim$ stand for the order dual, the order continuous dual and the singular dual of E, respectively. Then $E_n^\sim$ separates points of E and it can be identified with E' through the mapping: $E' \ni v \mapsto \varphi_v \in E_n^\sim$, where $\varphi_v(u) = \int_\Omega u(\omega)v(\omega) \, d\mu$ for all $u \in E$. E is said to be *perfect* whenever the natural embedding from E into $(E_n^\sim)_n^\sim$ is onto, i.e., $E'' = E$.

Now we collect notation along with some basic facts concerning vector-valued function spaces $E(X)$ and locally solid topologies on $E(X)$ as set out in [3 – 5], [9] and [19 – 21].

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let S_X and B_X denote the unit sphere and the unit ball in X. Let X^* stand for the Banach dual of X. By L^0 we will denote the set of μ-equivalence classes of strongly Σ-measurable functions $f : \Omega \to X$. For $f \in L^0(X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{ f \in L^0(X) : \ \widetilde{f} \in E \}.$$

A subset H of $E(X)$ is said to be *solid* whenever $\widetilde{f_1} \le \widetilde{f_2}$ and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $E(X)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets. A linear topology on $E(X)$ that is at the same time locally solid and locally convex will be called a *locally convex-solid topology* on $E(X)$. A pseudonorm ϱ on $E(X)$ is called *solid* if $\varrho(f_1) \le \varrho(f_2)$ whenever $f_1, f_2 \in E(X)$ and $\widetilde{f_1} \le \widetilde{f_2}$. It is known that a linear topology τ on $E(X)$ is locally solid (resp. locally convex-solid) if and only if it is generated by some family of solid pseudonorms (resp. solid seminorms) defined on $E(X)$ (see [9, Theorems 2.2 and 2.4]).

Recall that a locally solid topology τ on $E(X)$ is said to be a *Lebesgue topology* whenever for a net (f_α) in $E(X)$, $\widetilde{f_\alpha} \xrightarrow{(0)} 0$ in E implies $f_\alpha \xrightarrow{\tau} 0$ (see [9, Definition 2.2]).

In the case when E is provided with a locally solid topology (resp. locally convex-solid topology) ξ one can topologize $E(X)$ as follows. Let $\{p_t : t \in T\}$ be a family of Riesz pseudonorms (resp. Riesz seminorms) on E that generates ξ. By putting

$$\overline{p}_t(f) := p_t(\tilde{f}\,) \quad \text{for } f \in E(X) \quad (t \in T\,)$$

we obtain a family $\{\,\overline{p}_t : t \in T\,\}$ of solid pseudonorms (resp. solid seminorms) on $E(X)$ that defines a locally solid (resp. locally convex-solid) topology $\bar{\xi}$ on $E(X)$ (called the *topology associated* with ξ).

Now we recall "vector valued analogues" of $E^\sim$, $E_n^\sim$ and $E_s^\sim$ as set out in [5, 20].

For a linear functional F on $E(X)$ let us set

$$|F|(f) = \sup\{\,|F(h)| : h \in E(X),\ \tilde{h} \le \tilde{f}\,\} \quad \text{for all } f \in E(X).$$

Then the set

$$E(X)^\sim = \{\, F \in E(X)^\# : |F|(f) < \infty \quad \text{for all } f \in E(X)\,\}.$$

will be called the *order dual* of $E(X)$ (here $E(X)^\#$ denotes the algebraic dual of $E(X)$) (see [5, §§3, 18]).

It is well known that the Mackey topology $\tau(E, E^\sim)$ is locally solid (see [1]). Moreover, one can show that the Mackey topology $\tau(E(X), E(X)^\sim)$ is locally solid and $\tau(E(X), E(X)^\sim) = \overline{\tau(E, E^\sim)}$ (see [21, Theorem 3.3]).

Making use of the concept of $|F|$ we can define in a natural way a positive linear functional φ_F on E. Let $F \in E(X)^\sim$ and $x_0 \in S_X$ be fixed. For $u \in E^+$ let us set

$$\varphi_F(u) := |F|(u \otimes x_0) = \sup\{|F(h)| : h \in E(X),\ \tilde{h} \le u\,\},$$

where $(u \otimes x_0)(\omega) := u(\omega)x_0$ for $\omega \in \Omega$. Clearly $|F|(f) = \varphi_F(\tilde{f})$ for all $f \in E(X)$. Then $\varphi_F : E^+ \to \mathbb{R}^+$ is an additive mapping and φ_F has a unique positive extension to a linear mapping from E to $\mathbb{R}$ (denoted by φ_F again) and given by

$$\varphi_F(u) := \varphi_F(u^+) - \varphi_F(u^-) \quad \text{for all } u \in E$$

(see [5, Lemma 7], [2, Lemma 3.1]).

Now we are ready to consider the concept of solidness in $E(X)^\sim$. For $F_1, F_2 \in E(X)^\sim$ we will write $|F_1| \le |F_2|$ whenever $|F_1|(f) \le |F_2|(f)$ for all $f \in E(X)$. A subset A of $E(X)^\sim$ is said to be *solid* whenever $|F_1| \le |F_2|$ with $F_1 \in E(X)^\sim$ and $F_2 \in A$ imply $F_1 \in A$. A linear subspace I of $E(X)^\sim$ will be called an *ideal* of $E(X)^\sim$ whenever I is solid. It is known that $(E(X),\tau)^*$ is an ideal of $E(X)^\sim$ whenever τ is a locally solid topology on $E(X)$ (see [19, Theorem 3.2]).

Every subset A of $E(X)^\sim$ is contained in the smallest (with respect to inclusion) solid set called the *solid hull* of A and denoted by $S(A)$. One can note that $S(A) = \{F \in E(X)^\sim : \ |F| \le |G| \text{ for some } G \in A\}$.

Recall that a functional $F \in E(X)^\sim$ is said to be *order continuous* whenever for a net (f_α) in $E(X)$, $\tilde{f}_\alpha \xrightarrow{(0)} 0$ in E implies $F(f_\alpha) \to 0$. The set $E(X)^\sim_n$ consisting of all order continuous linear functionals on $E(X)$ is called the *order continuous dual* of $E(X)$. $E(X)^\sim_n$ is an ideal of $E(X)^\sim$ (see [19]).

A functional $F \in E(X)^\sim$ is said to be *singular* if there is an ideal B of E with supp $B = \Omega$ and such that $F(f) = 0$ for all $f \in E(X)$ with $\tilde{f} \in B$. The set consisting of all singular functionals on $E(X)$ will be denoted by $E(X)^\sim_s$ and called the *singular dual* of $E(X)$ (see [6, 18]). $E(X)^\sim_s$ is an ideal of $E(X)^\sim$ (see [19]).

Let $L^0(X^*, X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g : \Omega \to X^*$. One can define the so called *abstract norm* $\vartheta : L^0(X^*, X) \to L^0$ by $\vartheta(g) = \sup\{|g_x| : \ x \in B_X\}$, where $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$ and $x \in X$. One can show that $\vartheta(\lambda g) = |\lambda|\vartheta(g)$ and $\vartheta(g_1 + g_2) \le \vartheta(g_1)+\vartheta(g_2)$ for $g, g_1, g_2 \in L^0(X^*, X)$ and $\lambda \in \mathbb{R}$. Then for $f \in L^0(X)$ and $g \in L^0(X^*, X)$ the function $\langle f, g\rangle : \Omega \to \mathbb{R}$ defined by $\langle f, g\rangle(\omega) := \langle f(\omega), g(\omega)\rangle$ is measurable, and $|\langle f, g\rangle| \le \tilde{f}\,\vartheta(g)$. Moreover, $\vartheta(g) = \tilde{g}$ for $g \in L^0(X^*)$.

Let

$$E'(X^*, X) = \{\, g \in L^0(X^*, X) : \ \vartheta(g) \in E' \,\}.$$

Due to A. V. Bukhvalov (see [4, Theorem 4.1]) $E(X)^\sim_n$ can be identified with $E'(X^*, X)$ through the mapping $E'(X^*, X) \ni g \mapsto F_g \in E(X)^\sim_n$, where

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \, \mathrm{d}\mu \quad \text{for all } f \in E(X) \tag{1.1}$$

and moreover,

$$|F_g|(f) = \int_\Omega \tilde{f}(\omega)\vartheta(g)(\omega) \, \mathrm{d}\mu \quad \text{for all } f \in E(X).$$

It is known (see [19, Corollary 2.5]) that for $g_1, g_2 \in E'(X^*, X)$

$$|F_{g_1}| \le |F_{g_2}| \quad \text{if and only if} \quad \vartheta(g_1) \le \vartheta(g_2). \tag{1.2}$$

Due to A. V. Bukhvalov and G.Y. Lozanowskii (see [5, §3, Theorem 2]) the following Yosida-Hewitt type decomposition holds

$$E(X)^\sim = E(X)_n^\sim \oplus E(X)_s^\sim \tag{1.3}$$

and moreover, if $F = F_g + F_s$, where $g \in E'(X^*, X)$ and $F_s \in E(X)_s^\sim$, then $\varphi_F = \varphi_{F_g} + \varphi_{F_s}$, where $\varphi_{F_g}(u) = \int_\Omega u(\omega)\vartheta(g)(\omega)\,\mathrm{d}\mu$ for $u \in E$ and $\varphi_{F_s} \in E_s^\sim$.

Proposition 1.1. *Let E be an ideal of L^0. Then the following statements are equivalent:*

(i) $E(X)^\sim = E(X)_n^\sim$

(ii) $E(X)_s^\sim = \{0\}$

(iii) $E_s^\sim = \{0\}$

(iv) $E^\sim = E_n^\sim$.

Proof. (i) $\iff$ (ii): It follows from (1.3).

(iii) $\iff$(iv): This is obvious, because $E^\sim = E_n^\sim \oplus E_s^\sim$.

(ii) $\Longrightarrow$ (iii): Assume that $E(X)_s^\sim = \{0\}$ and let $\varphi \in E_s^\sim$. Then there is an ideal B of E with $\operatorname{supp} B = \Omega$ and such that $\varphi(u) = 0$ for all $u \in B$. Let $x_0 \in S_X$ and let $x_0^* \in S_{X^*}$ be such that $x_0^*(x_0) = 1$. Define a linear functional F_φ on $E(X)$ by setting $F_\varphi(f) = \varphi(x_0^* \circ f)$ for $f \in E(X)$. To show that $F_\varphi \in E(X)^\sim$, let $u \in E^+$. Then for $f \in E(X)$ with $\tilde{f} \le u$ we have $|x_0^* \circ f| \le \tilde{f}$, so

$$\begin{aligned}\sup\{|F_\varphi(f)| : f \in E(X),\ \tilde{f} \le u\} &= \sup\{|\varphi(x_0^* \circ f)| : f \in E(X),\ \tilde{f} \le u\}\\ &\le \sup\{|\varphi(w)| : w \in E,\ |w| \le u\} < \infty.\end{aligned}$$

It is seen that $F_\varphi(f) = 0$ for $f \in E(X)$ with $\tilde{f} \in B$, because $x_0^* \circ f \in B$. Hence $F_\varphi \in E(X)_s^\sim = \{0\}$, so $F_\varphi = 0$. Then for $u \in E$, we get $\varphi(u) = \varphi(x_0^*(u \otimes x_0)) = F_\varphi(u \otimes x_0) = 0$. Hence $\varphi = 0$, as desired.

(iii) $\Longrightarrow$ (ii): Assume that $E_s^\sim = \{0\}$ and let $F \in E(X)_s^\sim$. Then $\varphi_F \in E_s^\sim = \{0\}$ (see 1.3), so $F = 0$. ∎

2. Solid sets in the order continuous dual

In this section we shall show that the convex hull ($\operatorname{conv} A$) of a solid subset A of $E(X)_n^\sim$ is also solid in $E(X)_n^\sim$. For this purpose we will need the following two lemmas.

Lemma 2.1. *Let $g \in L^0(X^*, X)$ and $g_i \in L^0(X^*, X)$ for $n = 1, 2, \ldots, n$, and assume that $\vartheta(g) \le \vartheta(\sum_{i=1}^n g_i)$. Then there exist $g_i' \in L^0(X^*, X)$ for $i = 1, 2, \ldots, n$ such that $g = \sum_{i=1}^n g_i'$ and $\vartheta(g_i') \le \vartheta(g_i)$ for $i = 1, 2, \ldots, n$.*

Proof. By using induction it is enough to establish this result for $n = 2$. For $i = 1, 2$ let us put

$$u_i(\omega) = \begin{cases} \dfrac{\vartheta(g_i)(\omega)}{\vartheta(g_1)(\omega) + \vartheta(g_2)(\omega)} & \text{if} \quad \vartheta(g_1)(\omega) + \vartheta(g_2)(\omega) > 0, \\ 0 & \text{if} \quad \vartheta(g_1)(\omega) + \vartheta(g_2)(\omega) = 0. \end{cases}$$

It is seen that u_i are μ-measurable, and let $g_i' = u_i g$ for $i = 1, 2$. Then $g_1' + g_2' = u_1 g + u_2 g = g$ and since $\vartheta(g_1 + g_2) \le \vartheta(g_1) + \vartheta(g_2)$ for $i = 1, 2$ we have

$$\begin{aligned} \vartheta(g_i') &= \sup\{\, |(u_i g)_x| : \; x \in B_X \,\} \\ &= \sup\{\, u_i |g_x| : \; x \in B_X \,\} \\ &\le u_i \sup\{\, |g_x| : \; x \in B_X \,\} = u_i\, \vartheta(g) \\ &\le u_i\, \vartheta(g_1 + g_2) \\ &\le u_i\, (\vartheta(g_1) + \vartheta(g_2)) = \vartheta(g_i). \end{aligned}$$

Thus the proof is complete. ∎

Lemma 2.2. *Let $F \in E(X)_n^\sim$ and $F_i \in E(X)_n^\sim$ for $i = 1, 2, \dots, n$, and assume that $|F| \le |\sum_{i=1}^n F_i|$. Then there exist $F_i' \in E(X)_n^\sim$ for $i = 1, 2, \dots, n$ such that $F = \sum_{i=1}^n F_i'$ and $|F_i'| \le |F_i|$ for $i = 1, 2, \dots, n$.*

Proof. In view of (1.1) there exist $g \in E'(X^*, X)$ and $g_i \in E'(X^*, X)$ for $i = 1, 2, \dots, n$ such that $F = F_g$ and $F_i = F_{g_i}$ for $i = 1, 2, \dots, n$. Then $|F_g| \le |\sum_{i=1}^n F_{g_i}| = |F_{\sum_{i=1}^n g_i}|$, so $\vartheta(g) \le \vartheta(\sum_{i=1}^n g_i)$ by (1.2). Then in view of Lemma 2.1 there exist $g_i' \in L^0(X^*, X)$ for $i = 1, 2, \dots, n$ such that $g = \sum_{i=1}^n g_i'$ and $\vartheta(g_i') \le \vartheta(g_i)$. Then $g_i' \in E'(X^*, X)$ for $i = 1, 2, \dots, n$ and let $F_i' = F_{g_i'}$ for $i = 1, 2, \dots, n$. Then $F = F_g = F_{\sum_{i=1}^n g_i'} = \sum_{i=1}^n F_{g_i'} = \sum_{i=1}^n F_i'$ and $|F_i'| = |F_{g_i'}| \le |F_{g_i}| = |F_i|$ for $i = 1, 2, \dots, n$. ∎

Now we are ready to state our desired result.

Proposition 2.3. *Let A be a solid subset of $E(X)_n^\sim$. Then $\operatorname{conv} A$ is also a solid set in $E(X)_n^\sim$.*

Proof. Assume that $|F_0| \le |F|$ where $F_0 \in E(X)_n^\sim$ and $F \in \operatorname{conv} A$. Then there exist $F_i \in A$ and $\alpha_i \ge 0$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $F = \sum_{i=1}^n \alpha_i F_i$. Hence by Lemma 2.2 there exist $F_i' \in E(X)_n^\sim$ for $i = 1, 2, \dots, n$ such that $|F_i'| \le |\alpha_i F_i| = \alpha_i |F_i|$ for $i = 1, 2, \dots, n$ and $F_0 = \sum_{i=1}^n F_i'$. Putting $G_i = \alpha_i^{-1} F_i'$ we get $|G_i| \le |F_i|$ for $i = 1, 2, \dots, n$, so $G_i \in A$ for $i = 1, 2, \dots, n$. Hence $F_0 = \sum_{i=1}^n \alpha_i G_i \in \operatorname{conv} A$, and this means that $\operatorname{conv} A$ is solid in $E(X)_n^\sim$. ∎

3. Mackey topologies on vector-valued functions spaces

One can observe that $(E(X),\tau)^* \subset E(X)_n^\sim$ whenever τ is a Lebesgue topology on $E(X)$. Moreover, it is known that a locally convex-solid topology τ on $E(X)$ has the Lebesgue property whenever $(E(X),\tau)^* \subset E(X)_n^\sim$ (see [20, Theorem 2.4]). In [20, Theorem 3.4] it is shown that if an ideal E is perfect and a Banach space X is reflexive, then the Mackey topology $\tau(E(X),E(X)_n^\sim)$ is locally solid and it is the finest Hausdorff locally convex-solid topology on $E(X)$ with the Lebesgue property.

In this section we extend this result to the setting whenever the Mackey topology $\tau(E(X),E(X)_n^\sim)$ is locally solid. This property is characterized by the following result:

Theorem 3.1. *Let E be an ideal of L^0, and let X be a Banach space. Then the following statements are equivalent:*

(i) $\tau(E(X),E(X)_n^\sim)$ *is locally solid.*

(ii) *Every absolutely convex $\sigma(E(X)_n^\sim,E(X))$-compact subset of $E(X)_n^\sim$ is contained in a solid absolutely convex $\sigma(E(X)_n^\sim,E(X))$-compact subset of $E(X)_n^\sim$.*

Proof. It is enough to repeat the reasoning of the proof of [14, Lemma 2.1] and use the fact that the polar sets of subsets of $E(X)$ and $E(X)_n^\sim$ with respect to the dual pair $\langle E(X),E(X)_n^\sim\rangle$ are solid (see [19, Theorem 3.3]). ∎

Remark. In Section 4 we note that for $X = l^1$ the Mackey topology $\tau(L^\infty(X),L^\infty(X)_n^\sim)$ is not locally solid.

Now we are in position to prove our main result.

Theorem 3.2. *Let E be an ideal of L^0 and X be a Banach space. Assume that the Mackey topology $\tau(E(X),E(X)_n^\sim)$ is locally solid. Then $\tau(E(X),E(X)_n^\sim)$ is the finest locally convex-solid on $E(X)$ with the Lebesgue property and*

$$\tau(E(X),E(X)_n^\sim) = \overline{\tau(E,E_n^\sim)}.$$

Proof. We shall show that

$$\tau(E(X),E(X)_n^\sim) = \overline{\tau(E,E_n^\sim)}.$$

Indeed, assume that $\tau(E,E_n^\sim)$ is generated by a family $\{p_t:\ t\in T\}$ of Riesz seminorms on E. In view of [9, Theorem 5.7] $\overline{\tau(E,E_n^\sim)}$ is the finest locally convex Hausdorff Lebesgue topology on $E(X)$. It follows that $\tau(E(X),E(X)_n^\sim) \subset \overline{\tau(E,E_n^\sim)}$.

To prove that $\overline{\tau(E,E_n^\sim)} \subset \tau(E(X),E(X)_n^\sim)$ it is enough to show that $(E(X),\overline{\tau(E,E_n^\sim)})^* = E(X)_n^\sim$. Since $\overline{\tau(E,E_n^\sim)}$ is a Lebesgue topology, it is enough to prove that $E(X)_n^\sim \subset (E(X), \overline{\tau(E, E_n^\sim)}\,)^*$. Indeed, let $F \in E(X)_n^\sim$, i.e., $F(f) = F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu$ for some $g \in E'(X^*, X)$ and all $f \in E(X)$. Since $\varphi_{\vartheta(g)} \in E_n^\sim = (E, \tau(E, E_n^\sim))^*$ there exist $c > 0$ and $t_i \in T$ $(i = 1, 2, \ldots, n)$ such that for $f \in E(X)$

$$|F(f)| \leq \int_\Omega \tilde{f}(\omega)\vartheta(g)(\omega)\, d\mu = \varphi_{\vartheta(g)}(\tilde{f}) \leq c \max_{1\leq i\leq n} p_{t_i}(\tilde{f}) = c \max_{1\leq i \leq n} \overline{p_{t_i}}(f_i).$$

This means that F is $\overline{\tau(E, E_n^\sim)}$-continuous, as desired. ∎

As a consequence of Theorem 3.2 and [20, Theorem 2.6] we get the following result.

Theorem 3.3. *Let E be a perfect ideal of L^0, and let X be a Banach space. Assume that the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid. Then the space $(E(X), \tau(E(X), E(X)_n^\sim)\,)$ is complete.*

The topological dual of $(E(X), \tau(E(X), E(X)_n^\sim)\,)$ is characterized by the next theorem.

Theorem 3.4. *Let E be an ideal of L^0, and let X be a Banach space. Assume that the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid. Then the following statements are equivalent:*

- **(i)** *F is order continuous, i.e., $F \in E(X)_n^\sim$.*
- **(ii)** *F is sequentially order continuous (i.e., $F(f_n) \to 0$ whenever $\tilde{f}_n \xrightarrow{(0)} 0$ in E for a sequence (f_n) in $E(X)$).*
- **(iii)** *F is $\tau(E(X), E(X)_n^\sim)$-continuous.*
- **(iv)** *F is sequentially $\tau(E(X), E(X)_n^\sim)$-continuous.*

Proof. (i) $\Leftrightarrow$ (ii): This assertion follows from [19, Theorem 2.3].

(i) $\Leftrightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (ii): Assume that F is sequentially $\tau(E(X), E(X)_n^\sim)$-continuous, and let $\tilde{f}_n \xrightarrow{(0)} 0$ in E for a sequence (f_n) in $E(X)$. Then $f_n \to 0$ for $\tau(E(X), E(X)_n^\sim)$ because $\tau(E(X), E(X)_n^\sim)$ is a Lebesgue topology. Hence $F(f_n) \to 0$, as desired. ∎

Recall that a Hausdorff locally convex space (L, ξ) is said to be *sequentially barreled* whenever every $\sigma(L_\xi^*, L)$-convergent to 0 sequence in L_ξ^* is equicontinuous (see [25]).

Theorem 3.5. *Let E be a perfect ideal of L^0, and let X be a Banach space. Assume that the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid. Then the space $(E(X), \tau(E(X), E(X)_n^\sim))$ is sequentially barreled.*

Proof. In view of Theorem 3.4 we have

$$(E(X), \tau(E(X), E(X)_n^\sim))^* = (E(X), \tau(E(X), E(X)_n^\sim))^+ = E(X)_n^\sim$$

(here $(E(X), \tau(E(X), E(X)_n^\sim))^+$ denotes the sequential topological dual of $(E(X), \tau(E(X), E(X)_n^\sim))$. Since the space $(E(X), \tau(E(X), E(X)_n^\sim))$ is complete (see Theorem 3.3), by [25, Proposition 4.3] the space $(E(X), \tau(E(X), E(X)_n^\sim))$ is sequentially barreled. ∎

Note that if $(E, \|\cdot\|_E)$ is a Banach function space with the norm $\|\cdot\|_E$ satisfying the σ-Fatou property (i.e., $0 \leq u_n \uparrow u$ in E implies $\|u_n\|_E \uparrow \|u\|_E$), then the space $(E(X), \tau(E(X), E(X)_n^\sim))$ is barreled if and only if $\|\cdot\|_E$ is order continuous (see [21, Corollary 3.9]).

It is well known that the space $E_n^\sim$ is $\sigma(E_n^\sim, E)$-sequentially complete (see [2, Theorem 20.23], [10, Corollary 10.3.1]). Now, by making use of Theorem 3.5, Theorem 3.4 and [25, Proposition 4.4] we obtain the vector-valued version of this result.

Theorem 3.6. *Let E be a perfect ideal of L^0, and let X be a Banach space. Assume that the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid. Then the space $E(X)_n^\sim$ is $\sigma(E(X)_n^\sim, E(X))$-sequentially complete.*

As an application of Theorem 3.6 and (1.1) we get immediately the Hahn-Vitali-Saks type theorem for sequences in $E(X)_n^\sim$:

Corollary 3.7. *Let E be a perfect ideal of L^0, and let X be a Banach space. Assume that the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid. Let (g_n) be a sequence in $E'(X^*, X)$ such that for each $f \in E(X)$, $\lim_n \int_\Omega \langle f(\omega), g_n(\omega)\rangle \,d\mu$ exists. Then there is a $g \in E'(X^*, X)$ such that*

$$\lim_n \int_\Omega \langle f(\omega), g_n(\omega)\rangle \,d\mu = \int_\Omega \langle f(\omega), g(\omega)\rangle \,d\mu \quad \textit{for every } f \in E(X).$$

4. Mackey topologies on Orlicz-Bochner spaces

In this section we examine the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)_n^\sim)$ on Orlicz-Bochner spaces $L^\Phi(X)$ whenever Φ is an Orlicz function (not necessarily convex) and X is a general Banach space. Throughout this section we will assume that the measure space (Ω, Σ, μ) is atomless.

First we establish notation and basis results concerning Orlicz spaces (see [13, 24] for more details). By an *Orlicz function* we mean here a map $\Phi : [0, \infty) \to [0, \infty)$ that is non-decreasing left continuous, continuous at 0, vanishing only at 0 and $\lim_{t\to\infty} \inf(\Phi(t)/t) > 0$. Let Φ^* stand for the convex Orlicz function complementary to Φ in the sense of Young. Then the function

$\overline{\Phi}(t) = (\Phi^*)^*(t)$ for $t \geq 0$ is called a *convex minorant* of Φ, because it is the largest convex Orlicz function that is smaller than Φ on $[0, \infty)$.

The Orlicz space L^Φ can be equipped with a complete topology τ_Φ of the Riesz F-norm $\|u\|_\Phi := \inf\{\lambda > 0 : \int_\Omega \Phi(|u(\omega)|/\lambda)\,\mathrm{d}\mu \leq \lambda\}$. It is known that $(L^\Phi)' = L^{\Phi^*}$ (see [11]). Clearly L^Φ is perfect if and only if $L^\Phi = L^{\overline{\Phi}}$ (i.e., Φ is equivalent to some convex Orlicz function). It is seen that $(L^{\overline{\Phi}})' = L^{\Phi^*}$ because $\overline{\Phi}^* = \Phi^*$.

The *Orlicz-Bochner space* $L^\Phi(X)$ $(= \{f \in L^0(X) : \tilde{f} \in L^\Phi\})$ can be equipped with the complete topology $\tau_\Phi(X)$ of the solid F-norm $\|f\|_{L^\Phi(X)} := \|\tilde{f}\|_\Phi$ for $f \in L^\Phi(X)$ (i.e., $\tau_\Phi(X) = \overline{\tau_\Phi}$).

For $\varepsilon > 0$ let $V_\Phi(\varepsilon) = \{f \in L^\Phi(X) : \int_\Omega \Phi(\tilde{f}(\omega))\,\mathrm{d}\mu \leq \varepsilon\}$. Then the family of all sets of the form:

$$\bigcup_{n=1}^{\infty} \Big(\sum_{i=1}^{n} V_\Phi(\varepsilon_i) \Big),$$

where (ε_n) is a sequence of positive numbers, forms a local base at 0 (consisting of solid subsets of $L^\Phi(X)$) for a linear topology $\tau_\Phi^\wedge(X)$ on $L^\Phi(X)$, called the *modular topology* (see [9]).

In particular, for $X = \mathbb{R}$ we will write $\tau_\Phi^\wedge$ instead of $\tau_\Phi^\wedge(\mathbb{R})$. The basic properties of the modular topology $\tau_\Phi^\wedge$ are included in the following theorem (see [15, Theorem 1.1], [17, Theorems 2.5 and 3.2], [18, Theorem 2.2]):

Theorem 4.1. *Let Φ be an Orlicz function. Then:*

- **(i)** $\tau_\Phi^\wedge = \tau_\Phi$ *holds if and only if Φ satisfies the Δ_2-condition.*
- **(ii)** $\tau_\Phi^\wedge$ *is the finest Lebesgue topology on L^Φ.*
- **(iii)** *The Mackey topology $\tau(L^\Phi, L^{\Phi^*})$ is the finest of all locally convex topologies on L^Φ that are weaker than $\tau_\Phi^\wedge$. Moreover, $\tau(L^\Phi, L^{\Phi^*}) = \tau_\Phi^\wedge$ whenever Φ is convex.*
- **(iv)** $\tau(L^\Phi, L^{\Phi^*})$ *coincides with the restriction of the Mackey topology $\tau(L^{\overline{\Phi}}, L^{\Phi^*})$ on L^Φ, i.e., $\tau(L^\Phi, L^{\Phi^*}) = \tau(L^{\overline{\Phi}}, L^{\Phi^*})|_{L^\Phi}$.*
- **(v)** *The completion of $(L^\Phi, \tau(L^\Phi, L^{\Phi^*}))$ equals $(L^{\overline{\Phi}}, \tau(L^{\overline{\Phi}}, L^{\Phi^*}))$.*

Now we pass on to Orlicz-Bochner spaces. Then $L^\Phi(X)_n^\sim = \{F_g : g \in L^{\Phi^*}(X^*, X)\}$ and we can write $\tau(L^\Phi(X), L^{\Phi^*}(X^*, X))$ instead of $\tau(L^\Phi(X), L^\Phi(X)_n^\sim)$.

Theorem 4.2. *Let Φ be an Orlicz function and X be a Banach space. Assume that the Mackey topology $\tau(L^\Phi(X), L^{\Phi^*}(X^*, X))$ is locally solid. Then:*

- **(i)** $\tau_\Phi^\wedge(X)$ *is the finest Lebesgue topology on $L^\Phi(X)$.*
- **(ii)** $\tau_\Phi^\wedge(X) = \overline{\tau_\Phi^\wedge}$.
- **(iii)** $(L^\Phi(X), \tau_\Phi^\wedge(X))^* = L^\Phi(X)_n^\sim$.

Proof. (i): The assertion follows from [9, Theorem 6.3].

(ii): Since $\tau_\Phi^\wedge$ is the finest Lebesgue topology on L^Φ (see Theorem 4.1(ii)), by making use of [9, Theorem 5.7] $\overline{\tau_\Phi^\wedge}$ is the finest Lebesgue topology on $L^\Phi(X)$. Hence, in view of (i) $\tau_\Phi^\wedge(X) = \overline{\tau_\Phi^\wedge}$ as desired.

(iii): In view of (i) we have that $(L^\Phi(X), \tau_\Phi^\wedge(X))^* \subset L^\Phi(X)_n^\sim$. On the other hand, by making use of Theorem 3.2 and Theorem 4.1(iii) we get

$$\tau(L^\Phi(X), L^{\Phi^*}(X^*, X)) = \overline{\tau(L^\Phi, L^{\Phi^*})} \subset \overline{\tau_\Phi^\wedge} = \tau_\Phi^\wedge(X).$$

It follows that $L^\Phi(X)_n^\sim \subset (L^\Phi(X), \tau_\Phi^\wedge(X))^*$, and the proof is complete. ∎

Now we are ready to characterize the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)_n^\sim)$.

Theorem 4.3. *Let Φ be an Orlicz function and X be a Banach space. Assume that the Mackey topology $\tau(L^\Phi(X), L^{\Phi^*}(X^*, X))$ is locally solid. Then:*

- **(i)** *$\tau(L^\Phi(X), L^{\Phi^*}(X^*, X))$ is the finest of all locally convex topologies on $L^\Phi(X)$ that are weaker than $\tau_\Phi^\wedge(X)$. In particular, $\tau(L^\Phi(X), L^{\Phi^*}(X^*, X)) = \tau_\Phi^\wedge(X)$ whenever Φ is convex.*
- **(ii)** $\tau(L^\Phi(X), L^{\Phi^*}(X^*, X)) = \tau(L^{\overline{\Phi}}(X), L^{\Phi^*}(X^*, X))|_{L^\Phi(X)} = \tau_{\overline{\Phi}}^\wedge(X)|_{L^\Phi(X)}$.

Proof. (i): We know that $\tau(L^\Phi(X), L^\Phi(X)_n^\sim) \subset \tau_\Phi^\wedge(X)$ (see the proof of (iii) of Theorem 4.2). Now, let η be a locally convex topology on $L^\Phi(X)$ that is weaker than $\tau_\Phi^\wedge(X)$. Then $(L^\Phi(X), \eta)^* \subset (L^\Phi(X), \tau_\Phi^\wedge(X))^* = L^\Phi(X)_n^\sim$ (see Theorem 4.2 (iii)). Hence $\sigma(L^\Phi(X), (L^\Phi(X), \eta)^*) \subset \sigma(L^\Phi(X), L^\Phi(X)_n^\sim)$ and it follows that $\eta \subset \tau(L^\Phi(X), L^\Phi(X)_n^\sim)$ (see [23, Proposition 3.7.14]).

Moreover, if Φ is convex, then by Theorem 4.1 (iii) we get

$$\tau(L^\Phi(X), L^{\Phi^*}(X^*, X)) = \overline{\tau(L^\Phi, L^{\Phi^*})} = \overline{\tau_\Phi^\wedge} = \tau_\Phi^\wedge(X).$$

(ii): By making use of Theorem 3.2 and Theorem 4.1 (iv) we get:

$$\begin{aligned}\tau(L^\Phi(X), L^{\Phi^*}(X^*, X)) &= \overline{\tau(L^\Phi, L^{\Phi^*})} \\ &= \overline{\tau(L^{\overline{\Phi}}, L^{\Phi^*})|_{L^\Phi}} \\ &= \overline{\tau(L^{\overline{\Phi}}, L^{\Phi^*})}|_{L^\Phi(X)} \\ &= \tau(L^{\overline{\Phi}}(X), L^{\Phi^*}(X^*, X))|_{L^\Phi(X)} \\ &= \tau_{\overline{\Phi}}^\wedge(X)|_{L^\Phi(X)}.\end{aligned}$$ ∎

Theorem 4.4. *Let Φ be an Orlicz function and X be a Banach space. Assume that the Mackey topology $\tau(L^\Phi(X), L^{\Phi^*}(X^*, X))$ is locally solid. Then:*

- **(i)** *The completion of $(L^\Phi(X), \tau(L^\Phi(X), L^{\Phi^*}(X^*, X)))$ equals $(L^{\overline{\Phi}}(X), \tau(L^{\overline{\Phi}}(X), L^{\Phi^*}(X^*, X)))$.*

(ii) *The space* $(L^\Phi(X), \tau(L^\Phi(X), L^{\Phi^*}(X^*, X)))$ *is complete if and only if* L^Φ *is perfect.*

Proof. (i): We know that the space $(L^{\overline{\Phi}}(X), \tau(L^{\overline{\Phi}}(X), L^{\Phi^*}(X^*, X)))$ is complete, because $L^{\overline{\Phi}}$ is perfect (see Theorem 3.3). In view of Theorem 4.3 it is enough to show that $L^\Phi(X)$ is dense in $(L^{\overline{\Phi}}(X), \tau^{\wedge}_{\overline{\Phi}}(X))$. Indeed, let $f \in L^{\overline{\Phi}}(X)$. Then there exists a sequence (Ω_n) in Σ such that $\Omega_n \uparrow \Omega$, $\mu(\Omega_n) < \infty$ and $\chi_{\Omega_n} \in L^\Phi$ for $n \in \mathbb{N}$ (see [27, Theorem 86.2]). For $n \in \mathbb{N}$ let us put

$$f_n(\omega) = \begin{cases} f(\omega) & \text{if } \tilde{f}(\omega) \leq n \text{ and } \omega \in \Omega_n \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f_n \in L^\Phi(X)$ for $n \in \mathbb{N}$ and $\tilde{f}(\omega) \uparrow \tilde{f}(\omega)$ for $\omega \in \Omega$. Moreover, we have

$$\widetilde{f - f_n}(\omega) = \tilde{f}(\omega) - \tilde{f}_n(\omega) = \begin{cases} 0 & \text{if } \tilde{f}(\omega) \leq n \text{ and } \omega \in \Omega_n \\ \tilde{f}(\omega) & \text{elsewhere.} \end{cases}$$

Hence $\widetilde{f - f_n} \downarrow 0$ in E, and since $\tau^{\wedge}_{\overline{\Phi}}(X)$ is a Lebesgue topology on $L^{\overline{\Phi}}(X)$ we get $f_n \to f$ for $\tau^{\wedge}_{\overline{\Phi}}(X)$, as desired.

(ii): Assume that the space $(L^\Phi(X), \tau(L^\Phi(X), L^{\Phi^*}(X^*, X)))$ is complete. Then by (i) $L^\Phi(X) = L^{\overline{\Phi}}(X)$, and this means that L^Φ is perfect. Hence in view of Theorem 3.3 the proof is complete. ∎

Now we consider the Mackey topology $\tau(L^\infty(X), L^\infty(X)^\sim_n)$. The Riesz F-norm

$$\|u\|_0 = \int_\Omega \frac{|u(\omega)|}{1 + |u(\omega)|} w(\omega)\, d\mu \qquad \text{for } u \in L^0,$$

where $w : \Omega \to (0, \infty)$ is a Σ-measurable function with $\int_\Omega w(\omega)\, d\mu = 1$, determines the Lebesgue topology τ_0 on L^0 of the convergence in measure on subsets of finite measure. Recall the mixed topology $\gamma[\tau_\infty, \tau_{0|L^\infty}]$ (briefly γ) is the finest Hausdorff locally convex-solid topology with the Lebesgue property on L^∞, i.e., γ coincides with the Mackey topology $\tau(L^\infty, L^1)$ (see [16]).

Now we consider the mixed topology $\gamma[\tau_\infty(X), \tau_0(X)|_{L^\infty(X)}]$ (briefly γ_X) on $L^\infty(X)$ (here $\tau_\infty(X)$ stands for the topology of the norm $\|f\|_{L^\infty(X)} := \|\tilde{f}\|_\infty = \operatorname{ess\,sup}_{\omega\in\Omega} \tilde{f}(\omega)$ and $\tau_0(X)$ denotes the topology of the F-norm $\|f\|_{L^0(X)} := \|\tilde{f}\|_0$ on $L^0(X)$). For a sequence (ε_n) of positive numbers and $r > 0$ let

$$W(\varepsilon_n, r) = \bigcup_{n=1}^{\infty} \Big(\sum_{i=1}^{n} (V_X(\varepsilon_i) \cap i B_X(r)) \Big)$$

where $B_X(r) = \{f \in L^\infty(X) : \|f\|_{L^\infty(X)} \leq r\}$ and $V_X(\varepsilon) = \{f \in L^0(X) : \|f\|_{L^0(X)} \leq \varepsilon\}$. Then the family of all such $W(\varepsilon_n, r)$ forms a local base at 0 for γ_X (see [20, 26] for more details). One can show that $\gamma_X = \overline{\gamma}$ (see [20, Theorem 4.2]).

Hence, by Theorem 3.2 we get:

Theorem 4.5. *Assume that the Mackey topology $\tau(L^\infty(X), L^1(X^*, X))$ is locally solid. Then $\tau(L^\infty(X), L^1(X^*, X))$ coincides with the mixed topology γ_X.*

Remark. The Mackey topology $\tau(L^\infty(X), L^1(X^*, X))$ and the mixed topology γ_X on $L^\infty(X)$ are closely related to the theory of operator valued measures $m : \Sigma \to B(X, Y)$, where Y is a Banach space and $B(X, Y)$ stands for the space of all bounded linear operators from X to Y. One can show (see [22]) that if $\tau(L^\infty(X), L^1(X^*, X))$ is locally solid (i.e., $\tau(L^\infty(X), L^1(X^*, X)) = \gamma_X$) then for every Banach space Y an operator valued measure $m : \Sigma \to B(X, Y)$ is countably additive in the uniform operator topology if and only if m is variationally semiregular (see [11] for more details).

On the other hand, I. Dobrakov [6, Example 7] defined a measure $m : 2^N \to B(l^1, c_0)$ which is countably additive in the uniform operator topology but it is not variationally semiregular. It follows that for $X = l^1$ the Mackey topology $\tau(L^\infty(X), L^1(X^*, X))$ is not locally solid.

Acknowledgement. The author wishes to thank the referees for their remarks.

References

[1] Amemiya, I.: *On order topological spaces.* In: Proceedings of the International Symposium on Linear Spaces (Jerusalem 1961). Jerusalem: Jerusalem Academic Press 1961, pp. 14 – 23.

[2] Aliprantis, C. D. and O. Burkinshaw: *Locally Solid Riesz Spaces.* New York: Academic Press 1978.

[3] Aliprantis, C. D. and K. C. Border: *Infinite Dimensional Analysis. A hitchhiker's guide* (2^{nd} ed). Berlin: Springer 1999.

[4] Bukhvalov, A. V.: *On an analytic representation of operators with abstract norm* (in Russian). Izv. Vyss. Vceb. Zaved. 11 (1975), 21 – 32.

[5] Bukhvalov, A. V. and G.Ya. Lozanowskii: *On sets closed in measure in spaces of measurable functions.* Trans. Moscow Math. Soc. 2 (1978), 127 – 148.

[6] Dobrakov, I.: *On integration in Banach spaces*, I. Czechoslovak Math. J. 20 (95) (1970), 511 – 536.

[7] Duhoux, M.: *Mackey topologies on Riesz spaces and weak compactness.* Math. Z. 158 (1978), 199 – 209.

[8] Fremlin, D. H.: *Abstract Köthe spaces*. I. Proc. Camb. Phil. Soc. 63 (1967), 653 – 660.

[9] Feledziak, K. and M. Nowak: *Locally solid topologies on vector-valued function spaces*. Collect. Math. 48 (1997)(4 – 6), 487 – 511.

[10] Kantorovitch, L. V. and A. V. Akilov: *Functional Analysis* (in Russian). Moscow: Nauka 1984 (3^{rd} ed).

[11] Lewis, P. W.: *Some regularity conditions on vector measures with finite semivariation*. Rev. Roumaine Math. Pures Appl. 15 (1970), 375 – 384.

[12] Maligranda, L. and W. Wnuk: *Landau type theorem for Orlicz spaces*. Math. Z. 208 (1991), 57 – 64.

[13] Musielak, J.: *Orlicz Spaces and Modular Spaces*. Lecture Notes in Math. 1034. Berlin: Springer 1983.

[14] Moore, L. C. and J. C. Reber: *Mackey topologies which are locally convex Riesz topologies*. Duke Math. J. 39 (1972), 105 – 119.

[15] Nowak, M.: *On the modular topology on Orlicz spaces*. Bull. Polish Acad. Sci. Math. 36 (1988)(9–10), 553 – 562.

[16] Nowak, M.: *On the finest Lebesgue topology on the space of essentially bounded measurable functions*. Pacific J. Math. 140 (1989)(1), 155 – 161.

[17] Nowak, M.: *Orlicz lattices with modular topology* I. Comment. Math. Univ. Carolinae. 30 (1989)(2), 261 – 270.

[18] Nowak, M.: *On the strongest locally convex Lebesgue topology on Orlicz spaces*. Results Math. 33 (1998), 134 – 138.

[19] Nowak, M.: *Duality theory of vector valued function spaces* I. Comment. Math. = Prace Mat. 37 (1997), 195 – 215.

[20] Nowak, M.: *Lebesgue topologies on vector-valued function spaces*. Math. Japonica 52 (2000)(2), 171 – 182.

[21] Nowak, M.: *Strong topologies on vector-valued function spaces*. Czechosl. Math. J. 50 (125) (2000), 401 – 414.

[22] Nowak, M.: *Operator-valued measures and Mackey topologies* (submitted).

[23] Robertson, A. P. and W. J. Robertson: *Topological Vector Spaces*. London: Cambridge University Press 1973 (2^{nd} ed).

[24] Rao, M. M. and Z. D. Ren: *Theory of Orlicz Spaces*. New York: Marcel Dekker 1991.

[25] Webb, J. H.: *Sequential convergence in locally convex spaces*. Proc. Camb. Phil. Soc. 64 (1968), 341 – 364.

[26] Wiweger, A.: *Linear spaces with mixed topology*. Studia Math. 20 (1961), 47 – 68.

[27] Zaanen, A. C.: *Riesz Spaces* II. Amsterdam: North-Holland Publ. Comp. 1983.

Received 20.04.2004