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Fractal Polynomial Interpolation

M. A. Navascués

Abstract. A general procedure to define non-smooth versions of classical approximants by means of fractal interpolation functions is proposed. A complete and explicit description in the frequency domain of the functions constructed is obtained through their exact Fourier transforms. In particular, the generalization of the polynomial interpolation is developed. The Lagrange basis of the space of polynomials of degree lower or equal than N is generalized to a basis of fractal polynomials. As a consequence of the process, the density of the polynomial fractal interpolation functions with non-null scale vector in the space of continuous functions in a compact interval is deduced. Furthermore, a method for the interpolation of real data is proposed, by the construction of a fractal function coming from any classical approximant. The convergence of the process when the partition is refined is proved, supposing the convergence of the smooth interpolant.

Keywords: *Fractal interpolation functions, iterated function systems, polynomial interpolation*

MSC 2000: 28A80, 65D05

1. Introduction

Up to now, the interpolation and approximation of functions have been performed by means of smooth functions, sometimes indefinitely differentiable. However, the signals coming from the real world do not share the nice aspect of those. In general, the natural phenomena recorded in a time series suggest original functions with abrupt changes, whose derivatives possess sharp steps or even do not exist at all. The fractal interpolation functions represent an important advance because the interpolants considered are not necessarily differentiable and, in some cases, they are not at any point ([5]). In words of M. Barnsley [1]: "... (they) appear ideally suited for the approximation of naturally occurring functions which display some kind of geometrical self-similarity under magnification".

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This paper proposes the construction of fractal interpolants which are perturbations of the classical polynomials. The purpose is to define non-smooth fractal versions of conventional approximants. For that fractal functions, a complete description in the frequency domain is obtained by means of their exact Fourier transforms. This fact is particularly important as the functions of that kind are defined implicitly in the time domain by a functional equation.

The generalization of the polynomial interpolation is also broached here, defining bases of "fractal polynomials". As a consequence of the process, the density of polynomial fractal interpolation functions in the space of continuous functions in a compact interval is deduced. Later on, a method for the interpolation of real data is proposed, by means of the construction of a fractal interpolation function coming from a classical approximant. The convergence of the process when the partition is refined is proved, assuming the convergence of the smooth interpolant.

2. A fractal interpolation operator

2.1. α-Fractal functions. Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N]$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R : \; n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n$, $n \in \{1, 2, ..., N\}$, be contractive homeomorphisms such that

$$L_n(t_0) = t_{n-1}\,, \;\; L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \le l \; |c_1 - c_2| \quad \forall \; c_1, c_2 \in I \tag{2}$$

for some $0 \le l < 1$. Let $-1 < \alpha_n < 1$; $n = 1, 2, ..., N$, $F = I \times [c, d]$ for some $-\infty < c < d < +\infty$ and N continuous mappings $F_n : F \to R$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}\,, \quad F_n(t_N, x_N) = x_n, \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \le |\alpha_n| \, |x - y|, \quad t \in I, \; x, y \in R \tag{4}$$

Now define functions $w_n(t, x) = (L_n(t), F_n(t, x)), n = 1, 2, ..., N$.

Theorem 1 (Barnsley [1]). *The iterated function system (IFS)* [8] $\{F, w_n : n = 1, 2, ..., N\}$ *defined above admits a unique attractor G. G is the graph of a continuous function $f : I \to R$ which obeys $f(t_n) = x_n$ for $n = 0, 1, 2, ..., N$.*

The previous function is called a *fractal interpolation function* (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$.

Let $\mathcal{G}$ be the set of continuous functions $f : [t_0, t_N] \to [c, d]$ such that $f(t_0) = x_0$; $f(t_N) = x_N$. $\mathcal{G}$ is a complete metric space respect to the uniform norm. Define a mapping $T : \mathcal{G} \to \mathcal{G}$ by

$$(Tf)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \quad \forall \; t \in [t_{n-1}, t_n], \; n = 1, 2, ..., N.$$

T is a contraction mapping on the metric space $(\mathcal{G}, \|\cdot\|_\infty)$:

$$\|Tf - Tg\|_\infty \leq |\alpha|_\infty \|f - g\|_\infty, \tag{5}$$

where $|\alpha|_\infty = \max\{|\alpha_n|;\ n = 1, 2, ..., N\}$. Since $|\alpha|_\infty < 1$, T possesses a unique fixed point on $\mathcal{G}$, that is to say, there is $f \in \mathcal{G}$ such that $(Tf)(t) = f(t)$ for all $t \in [t_0, t_N]$. This function is the FIF corresponding to w_n and it is the unique $f \in \mathcal{G}$ satisfying the functional equation [1]:

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)), \quad n = 1, 2, ..., N,\ t \in I_n = [t_{n-1}, t_n]. \tag{6}$$

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t, x) = \alpha_n x + q_n(t), \end{cases} \tag{7}$$

where α_n is called a *vertical scaling factor* of the transformation w_n and $\alpha = (\alpha_1, \ldots, \alpha_N)$ is the scale vector of the IFS. If $q_n(t)$ is a polynomial, the FIF is termed polynomial.

M. Barnsley proposes, in the reference [1], the generalization of a continuous function h by means of a fractal interpolation defined by the IFS (7) with $q_n(t) = h \circ L_n(t) - \alpha_n b(t)$ where b is continuous and such that $b(t_0) = x_0$, $b(t_N) = x_N$. At first we consider here the case $b = h \circ c$, where c is continuous, increasing and $c(t_0) = t_0$, $c(t_N) = t_N$. For instance, the family $c(t) = (e^{\lambda t} - 1)/(e^{\lambda} - 1)$, $\lambda > 0$, can be considered in the interval $[0, 1]$.

Proposition 1. *Let $h : I = [a, b] \longrightarrow R$ be continuous, $\Delta : a = t_0 < t_1 < ... < t_N = b$, $N > 1$, $\alpha \in R^N$ and such that $|\alpha|_\infty < 1$. The IFS (7), where $a_n = (t_n - t_{n-1})/(t_N - t_0)$, $b_n = (t_N t_{n-1} - t_0 t_n)/(t_N - t_0)$, $q_n(t) = h \circ L_n(t) - \alpha_n h \circ c(t)$ and c an increasing continuous function such that $c(t_0) = t_0$; $c(t_N) = t_N$, defines a FIF $h^\alpha(t)$ such that $h^\alpha(t_n) = h(t_n)$ for all $n = 0, 1, 2, \ldots, N$.*

Proof. In the first place, we check the conditions of the theorem of Barnsley for L_n, F_n. By definition, $L_n(t_0) = t_{n-1}, L_n(t_N) = t_n$ and L_n is a contractive homeomorphism. Let $x_n = h(t_n), n = 0, 1, ...N$:

$$\begin{aligned} F_n(t_0, x_0) &= \alpha_n x_0 + q_n(t_0) \\ &= \alpha_n x_0 + h \circ L_n(t_0) - \alpha_n h \circ c(t_0) \\ &= \alpha_n x_0 + h(t_{n-1}) - \alpha_n x_0 = x_{n-1} \end{aligned}$$

$$\begin{aligned} F_n(t_N, x_N) &= \alpha_n x_N + q_n(t_N) \\ &= \alpha_n x_N + h \circ L_n(t_N) - \alpha_n h \circ c(t_N) \\ &= \alpha_n x_N + h(t_n) - \alpha_n x_N = x_n\,. \end{aligned}$$

F_n is uniformly Lipschitz in the second variable with constant $|\alpha|_\infty < 1$. For $\mathcal{G} = \{g \in \mathcal{C}([a,b]) : g([a,b]) \subset [c,d], g(a) = x_0, g(b) = x_N\}$ define $T_\alpha : \mathcal{G} \longrightarrow \mathcal{G}$ according to

$$T_\alpha f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \qquad (t \in I_n)$$

By Theorem 1, T_α admits a unique fixed point in $\mathcal{G}$, denoted by h^α in the following, continuous on I. The function h^α is defined by the fixed point equation (6), that is $h^\alpha(t) = \alpha_n h^\alpha \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t)$ for all $t \in I_n$. Using the expression of q_n, one has

$$h^\alpha(t) = h(t) + \alpha_n (h^\alpha - h \circ c) \circ L_n^{-1}(t) \qquad (t \in I_n)\,. \tag{8}$$

Now, h^α passes through the points (t_n, x_n) as

$$h^\alpha(t_n) = F_n(L_n^{-1}(t_n), h^\alpha \circ L_n^{-1}(t_n)) = F_n(t_N, h^\alpha(t_N)) = F_n(t_N, x_N) = x_n\,. \quad \blacksquare$$

Definition 1. Let $h \in \mathcal{C}(I)$, Δ, c and α as in Proposition 1. The FIF $h^\alpha_{\Delta c}$ defined in this proposition is termed *α-fractal function* of h with respect to Δ and c (the dependence on Δ and c will be omitted if not necessary). Define the *α-fractal operator* respect to Δ and c by

$$\begin{array}{rccc} \mathcal{F}^\alpha_{\Delta,c} : & \mathcal{C}(I) & \to & \mathcal{C}(I) \\ & h & \hookrightarrow & h^\alpha \end{array}$$

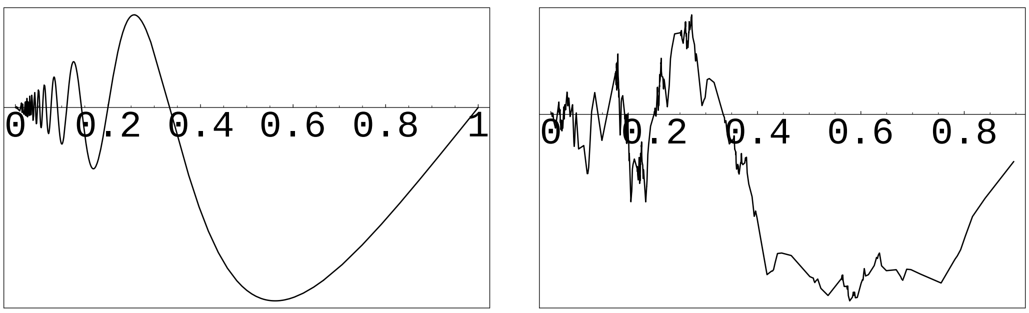

Figure 1: The left figure represents the graph of the function $h(t) = t\,\cos(\frac{\pi}{2t})$ if $t \neq 0$, $h(0) = 0$. The right figure shows the corresponding α-function, with respect to $\Delta : 0 < \frac{1}{8} < \frac{1}{7} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < 1$, c a quadratic in the interval $[0,1]$ and $\alpha_n = 0.2$ for all $n = 1, \ldots, 8$.

Proposition 2. *Let Δ, c and α be defined as in Proposition 1, the operator $\mathcal{F}^\alpha = \mathcal{F}^\alpha_{\Delta c}$ is linear and verifies the following properties:*

2.1. $\mathcal{F}^0 = Id$ *(identity).*

2.2. $1 \in \sigma_p(\mathcal{F}^\alpha)$, *where σ_p is the point spectrum of $\mathcal{F}^\alpha$ for any α fulfiling the hypotheses.*

Proof. The continuous functions h and g give rise to h^α and g^α such that by (8)

$$h^\alpha(t) = h(t) + \alpha_n(h^\alpha - h \circ c) \circ L_n^{-1}(t)$$
$$g^\alpha(t) = g(t) + \alpha_n(g^\alpha - g \circ c) \circ L_n^{-1}(t)$$

for all $t \in I_n$. If the first equation is multiplied by λ and the second one by μ, by the uniqueness of the solution of the fixed point equation, one has $(\lambda h + \mu g)^\alpha = \lambda h^\alpha + \mu g^\alpha$, $\lambda, \mu \in R$. As consequence, $\mathcal{F}^\alpha(f_0) = f_0$, where f_0 is the zero function.

`Case 2.1`: If $\alpha = 0 \in R^N$ is chosen, by the equation (8), $h^\alpha(t) = h(t)$ for all $t \in I$, $\mathcal{F}^0(h) = h$ and $\mathcal{F}^0 = Id$.

`Case 2.2`: If h is a constant function on I, $h(t) = k$ for all $t \in I$, the following equality is verified by (8):

$$h^\alpha(t) = k + \alpha_n h^\alpha \circ L_n^{-1}(t) - \alpha_n k \qquad (t \in I_n)\,.$$

But this equation is fulfiled by $h^\alpha(t) = k$ and, by the uniqueness of the FIF, $\mathcal{F}^\alpha(h) = h$ and the result is deduced. ∎

By the Property 2.1 of Proposition 2, every continuous function can be considered a FIF of Barnsley, with scaling factors equal to zero.

2.2. Error representation of α-fractal functions. Consider the mapping $T : J \times \mathcal{G} \to \mathcal{G}$ according to $(\alpha, f) \to T_\alpha f$ with $J = [0,r] \times [0,r] \times \cdots \times [0,r] \subset R^N$; $0 \le r < 1$, r fixed and $[t_0, t_N] = I$. For $t \in I_n = [t_{n-1}, t_n]$ define

$$T_\alpha f(t) = F_n^{\alpha_n}(L_n^{-1}(t), f \circ L_n^{-1}(t)) = \alpha_n f \circ L_n^{-1}(t) + q_n^{\alpha_n} \circ L_n^{-1}(t)$$

with $q_n^{\alpha_n}(t) = f \circ L_n(t) - \alpha_n f \circ c(t)$; $c(t)$ verifying the conditions described in Proposition 1.

Theorem 2. *The uniform distance between h and $h^\alpha \in \mathcal{C}(I)$ verifies*

$$\|h^\alpha - h\|_\infty \le \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty}\|h\|_\infty\,, \tag{9}$$

where $|\alpha|_\infty = max_{1 \le n \le N}\{|\alpha_n|\}$.

Proof. Let $f \in \mathcal{G}$ be given, then for each value of $t \in I_n$

$$\begin{aligned}
&|T_\alpha f(t) - T_\beta f(t)| \\
&\quad = |\alpha_n f \circ L_n^{-1}(t) + q_n^{\alpha_n} \circ L_n^{-1}(t) - \beta_n f \circ L_n^{-1}(t) - q_n^{\beta_n} \circ L_n^{-1}(t)| \\
&\quad \le |\alpha_n f \circ L_n^{-1}(t) - \beta_n f \circ L_n^{-1}(t)| + |q_n^{\alpha_n} \circ L_n^{-1}(t) - q_n^{\beta_n} \circ L_n^{-1}(t)|\,.
\end{aligned}$$

The first term is bounded by $|\alpha_n - \beta_n|\|f\|_\infty$. The second one is reduced to $|\alpha_n - \beta_n| \cdot |f \circ c \circ L_n^{-1}(t)| \leq |\alpha - \beta|_\infty \|f\|_\infty$, from which it folllows

$$\|T_\alpha f - T_\beta f\|_\infty \leq 2|\alpha - \beta|_\infty \|f\|_\infty \,. \tag{10}$$

On the other hand, h^α is the fixed point of T_α, corresponding to $q_n^{\alpha_n}(t) = h \circ L_n(t) - \alpha_n h \circ c(t)$. Then $\|h^\alpha - h^\beta\|_\infty = \|T_\alpha h^\alpha - T_\alpha h^\beta + T_\alpha h^\beta - T_\beta h^\beta\|_\infty$. Applying the inequalities (5) and (10) we get

$$\|h^\alpha - h^\beta\|_\infty \leq |\alpha|_\infty \|h^\alpha - h^\beta\|_\infty + 2|\alpha - \beta|_\infty \|h^\beta\|_\infty \,,$$

and so

$$\|h^\alpha - h^\beta\|_\infty \leq \frac{2|\alpha - \beta|_\infty}{1 - |\alpha|_\infty} \|h^\beta\|_\infty \,.$$

Setting $\beta = 0 \in R^N$, according to Property 2.1 of Proposition 2, $h^0 = h$ and the result is deduced. ∎

Corollary 1. *$\mathcal{F}^\alpha$ is a linear and continuous operator of $\mathcal{C}(I)$, as by* (9)

$$\|h^\alpha\|_\infty - \|h\|_\infty \leq \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty} \|h\|_\infty \,, \qquad \|\mathcal{F}^\alpha(h)\|_\infty \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty} \|h\|_\infty \,.$$

Furthermore, by Property 2.2

$$1 \leq \|\mathcal{F}^\alpha\| \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty} \,.$$

The set $\{\mathcal{F}^\alpha; |\alpha|_\infty < 1\}$ constitutes a N-parametric family of linear bounded operators of $\mathcal{C}(I)$.

Proposition 3. *Let Δ and c be given as in Proposition* 1, *let $\alpha^m \in R^N$ a sequence of scale vectors such that $|\alpha^m|_\infty < 1$ for all $m \geq 1$ and $\alpha^m \to 0 \in R^N$ as $m \to \infty$. Then, the sequence of operators $\mathcal{F}^{\alpha^m}$ converges in norm towards the identity.*

Proof. The assertion is an immediate consequence of Theorem 2 as

$$\|(\mathcal{F}^{\alpha^m} - Id)h\|_\infty \leq \frac{2|\alpha^m|_\infty}{1 - |\alpha^m|_\infty} \|h\|_\infty \,.$$ ∎

2.3. Construction of non-smooth interpolants. Nowadays, almost all the interpolants being used are smooth. Altough it is obvious that these techniques are very useful for the representation of many phenomena, some essential features of the considered signals are omitted by this kind of procedures. For instance, in the reference [4], a paper of Besicovitch and Ursell proves that if a real function is smooth, the fractal dimension of its graph is one. In this case, this parameter cannot be used as an index of the complexity of the signal. The fractal techniques can represent better the function because they provide numerical characterizations of the geometry of the represented curve, allowing to compare and discriminate experimental processes. Our group has used this procedure in order to detect significant changes in the complexity of electroencephalographic signals during the execution of several cognitive tests and in order to discriminate an attention disorder (see [12]). Other applications can be found in the reference [3]. In this paragraph we approach the problem of constructing non-differentiable approximation functions by means of fractal interpolation.

From here on, we consider the continuity and differentiability of a function in a compact interval globally or laterally depending on the position of the point (inner or extreme) and we write, for instance, $f \in \mathcal{C}^1(I)$ considering only the behaviour of f in the interval.

Lemma 1. *If $f \in \mathcal{C}^1[0,1]$, $f'_+(0) \neq 0$ and $c(t) = t^\alpha$, where $0 < \alpha < 1$, then $f \circ c$ is not differentiable (at right) at $t = 0$.*

Proof. Let us consider $g = f \circ c$. Applying the mean-value theorem

$$g'_+(0) = \lim_{t \to 0^+} \frac{f(t^\alpha) - f(0)}{t} = \lim_{t \to 0^+} \frac{f'(\xi) t^\alpha}{t},$$

where $\xi \in (0, t^\alpha)$. As f' is continuous and $f'_+(0) \neq 0$, it follows $g'_+(0) = f'_+(0) \lim_{t \to 0^+} t^{\alpha - 1} = \pm\infty$. ∎

The following proposition illustrates the construction of non-smooth interpolants, coming from classical functions (another approach is given in Section 5).

Proposition 4. *Let $h \in \mathcal{C}^1[a,b]$, where $h'_+(a) \neq 0$ or $h'_-(b) \neq 0$, be a smooth function. For any partition of the interval I, there exists a non-differentiable fractal interpolation function h^α arbitrarily close to h.*

Proof. Let Δ be a partition of the interval with $N+1$ points ($N > 1$) and let $\alpha \in (-1,1)^N$ be such that $\alpha_n \neq 0$ for all $n = 1, \cdots, N$.

Let us assume that $h'_+(t_0) \neq 0$. Without loss of generality we can assume that $I = [0,1]$. Let $c(t) = t^\alpha$, where $0 < \alpha < 1$, be the function chosen to construct h^α. By Lemma 1, $h \circ c$ is not differentiable at right at $t = 0$.

Let us assume that h^α is differentiable at every point of the interval I. For $t \in [0, \delta)$ and any $n \in \{1, 2, \cdots, N\}$, $L_n(t) \in I_n$ (see (1), (7)), and we can apply equation (8) for $L_n(t)$ obtaining

$$h \circ c(t) = h^\alpha(t) + \frac{1}{\alpha_n}(h - h^\alpha) \circ L_n(t).$$

As the right hand is differentiable at any point, $h \circ c$ would be differentiable at right at $t = 0$. As a consequence, h^α can not be differentiable on I. The vector α can be chosen small enough (in norm) in order to define h^α so close to h as required (Theorem 2).

If $h'_+(t_0) = 0$ but $h'_-(t_N) \neq 0$, we can choose $c(t) = 1 - (1-t)^\alpha$, where $0 < \alpha < 1$, giving a similar result for the upper extreme. ∎

3. α-Fractal polynomials

Let $\mathcal{P}_m[a,b]$ be the set of polynomials of degree lower or equal than m on $I = [a,b]$ and $\mathcal{P}[a,b] = \bigcup_{m=1}^{\infty} \mathcal{P}_m[a,b]$. The set $\{1, t, t^2, \dots\}$ constitutes a basis of $\mathcal{P}[a,b]$.

Definition 2. Let Δ, c and α be given as in Proposition 1, an *α-fractal polynomial* is an element $p^\alpha(t) \in \mathcal{C}(I)$ such that there is polynomial $p \in \mathcal{P}[a,b]$ with $\mathcal{F}^\alpha(p) = p^\alpha$. If p has degree m, then p^α is an α-fractal polynomial of *degree m*.

Notation: $\mathcal{P}_m^\alpha[a,b] = \mathcal{F}^\alpha(\mathcal{P}_m[a,b])$, $\mathcal{P}^\alpha[a,b] = \mathcal{F}^\alpha(\mathcal{P}[a,b])$.

By the properties described in Proposition 2, $\mathcal{P}_m^\alpha[a,b]$ is linearly spanned by $\{1, t^\alpha, (t^2)^\alpha, \dots, (t^m)^\alpha\}$ and consequently $\dim(\mathcal{P}_m^\alpha[a,b]) < +\infty$. $\mathcal{P}_m^\alpha[a,b]$ is a closed and complete linear subspace of $\mathcal{C}[a,b]$.

In the following, the theorem of uniform approximation of Weierstrass is generalized to α-fractal polynomials.

Theorem 3. *Let $h \in \mathcal{C}[a,b]$ be given. For all $\epsilon > 0$, any partition Δ of the interval I with $N+1$ points ($N > 1$) and any function c verifying the hypotheses of Proposition 1, there exists an α-fractal polynomial p^α with $\alpha \neq 0 \in R^N$ such that*

$$|h(t) - p^\alpha(t)| < \epsilon \qquad (t \in I)\,.$$

Proof. For any $\epsilon > 0$, the quantity $\frac{\epsilon}{2} > 0$ is considered. Applying the theorem of Weierstrass [6], there is a $p \in \mathcal{P}[a,b]$ such that

$$|h(t) - p(t)| < \frac{\epsilon}{2} \qquad (t \in I)\,. \tag{11}$$

For a partition $\Delta : a = t_0 < t_1 < \dots < t_N = b$, we choose $\alpha \in R^N$, $\alpha \neq 0$, such that $|\alpha|_\infty < 1$ and

$$\frac{2|\alpha|_\infty}{1 - |\alpha|_\infty} \|p\|_\infty < \frac{\epsilon}{2} \tag{12}$$

Then, by (9), (11) and (12) it follows

$$|h(t) - p^\alpha(t)| \leq |h(t) - p(t)| + |p(t) - p^\alpha(t)| < \frac{\epsilon}{2} + \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty} \|p\|_\infty < \epsilon\,. \quad ∎$$

If the function c is polynomial, p^α is a polynomial FIF. As a consequence of Theorem 3 one has

Theorem 4. *The set of polynomial FIFs with non-null scale vector is dense in the set of continuous functions* $\mathcal{C}[a,b]$.

4. Lagrange fractal interpolation

Let $\Delta : a = t_0 < t_1 < \ldots < t_N = b$ be given. The basis of Lagrange associated to the mesh is given by

$$\varphi_{i,N}(t) = \frac{(t-t_0)(t-t_1)\ldots(t-t_{i-1})(t-t_{i+1})\ldots(t-t_N)}{(t_i-t_0)(t_i-t_1)\ldots(t_i-t_{i-1})(t_i-t_{i+1})\ldots(t_i-t_N)}\,.$$

The interpolant polynomial of Lagrange with respect to data $\{(t_n, x_n); n = 0, 1, \ldots, N\}$ adopts the expression $p_N(t) = \sum_{i=0}^N x_i \varphi_{i,N}(t)$. We define the α-fractal interpolant of Lagrange as

$$p_N^\alpha(t) = \mathcal{F}^\alpha(p_N) = \sum_{i=0}^N x_i \varphi_{i,N}^\alpha(t)\,,$$

where $\varphi_{i,N}^\alpha$ is the α-fractal polynomial of $\varphi_{i,N}$ with respect to the partition Δ. The function p_N^α passes through the points (t_n, x_n) according to Proposition 1.

If $\mathcal{L}_N$ represents the Lagrange operator, which assigns to a function f its interpolant polynomial with respect to $\{(t_n, f(t_n)\}_{n=0}^N$, then $p_N^\alpha = \mathcal{F}^\alpha \circ \mathcal{L}_N(f)$. The basis polynomials of Lagrange $\varphi_{i,N}$ are orthogonal with respect to the form $\langle f, g\rangle = \sum_{n=0}^N f(t_n)g(t_n)$ This property is inherited by $\varphi_{i,N}^\alpha$ as

$$\left\langle \varphi_{i,N}^\alpha, \varphi_{j,N}^\alpha \right\rangle = \sum_{n=0}^N \varphi_{i,N}(t_n)\varphi_{j,N}(t_n) = \sum_{n=0}^N \delta_i^n \delta_j^n\,,$$

where δ_i^n is the delta of Kronecker. If $p^\alpha \in \mathcal{P}_N^\alpha[a,b]$, by the linearity of the operator $\mathcal{F}^\alpha$, $p^\alpha = \sum_{i=0}^N \lambda_i \varphi_{i,N}^\alpha$. Furthermore, the orthogonality of $\varphi_{i,N}^\alpha$ implies the linear independence, so that $\{\varphi_{i,N}^\alpha\}$ constitutes a basis for the space $\mathcal{P}_N^\alpha[a,b]$ of α-fractal polynomials on the partition Δ. The fact that $\mathcal{P}_N^\alpha[a,b]$ is finite-dimensional allows to obtain for each $h \in \mathcal{C}[a,b]$ a $p^{\alpha*}$ such that $\|h - p^{\alpha*}\|_\infty = \inf\{\|h - p^\alpha\|_\infty; p^\alpha \in \mathcal{P}_N^\alpha[a,b]\}$.

4.1. Bounding the interpolation error. In this section, the error of interpolation is bounded in terms of the derivatives of the original function, the scaling factors of the transformation and the Lebesgue constant of the associated partition.

Theorem 5 (Cauchy remainder for polynomial interpolation [7]). *Let $f \in \mathcal{C}^N[a,b]$ and suppose that $f^{(N+1)}(t)$ exists at each point of (a,b). If $a \leq t_0 < t_1 < ... < t_N \leq b$, then*

$$f(t) - p_N(t) = \frac{(t-t_0)(t-t_1)\dots(t-t_N)}{(N+1)!} f^{(N+1)}(\xi)\,,$$

where $\min(t, t_0, t_1, \dots, t_N) < \xi < \max(t, t_0, t_1, \dots, t_N)$. *The point ξ depends upon $t, t_0, t_1, \dots, t_N$ and f.*

In practice, to estimate this error it is necessary to have an expression for the derivatives of high order and to obtain a bound for them. Another kind of inequalities can be found in the book of Davis [7, Chapters II – IV]. The size of the interpolation error depends on the properties of the function to be approximated as well as the distribution of the nodes.

Definition 3. [15] A scheme of interpolation nodes

$$K = \begin{pmatrix} t_{10} & t_{11} & & & \\ t_{20} & t_{21} & t_{22} & & \\ t_{30} & t_{31} & t_{32} & t_{33} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is fixed a priori (independently of the functions to be approximated) is called a *node matrix.*

Definition 4. [15] The *Lebesgue function* of K of order N is given by

$$\Lambda_N(t;K) = \sum_{i=0}^{N} |\varphi_{i,N}(t)|\,,$$

and $\Lambda_N(K) = \|\Lambda_N(t;K)\|_\infty$ is the *Lebesgue constant* of K of order N.

Theorem 6. *Let $h \in \mathcal{C}^{N+1}[a,b]$ be given, $\|h^{(N+1)}\|_\infty = M_{N+1}$, $w_{N+1}(t) = (t-t_0)(t-t_1)\dots(t-t_N)$; $\alpha \in R^N$ such that $|\alpha|_\infty < 1$ and p_N^α the α-fractal polynomial of interpolation respect to $\Delta : a = t_0 < t_1 < ... < t_N = b$, then*

$$\|h - p_N^\alpha\|_\infty \leq \frac{M_{N+1}}{(N+1)!}\|w_{N+1}\|_\infty + \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|h\|_\infty \Lambda_N(K)\,.$$

Proof. For any $t \in [a,b]$, $|h(t) - p_N^\alpha(t)| \leq |h(t) - p_N(t)| + |p_N(t) - p_N^\alpha(t)|$. Theorem 5 bounds the first term. For the second, according to Theorem 2,

$$|p_N(t) - p_N^\alpha(t)| \leq \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|p_N\|_\infty\,.$$

If $x_n = h(t_n)(n = 0, 1, \dots, N)$, then

$$\|p_N\|_\infty = \max_{t\in I} |\sum_{i=0}^{N} x_i \varphi_{i,N}(t)| \leq \max_{t\in I} \sum_{i=0}^{N} |x_i||\varphi_{i,N}(t)| \leq \|h\|_\infty \Lambda_N(K)\,,$$

and the result is deduced. ∎

4.2. The condition of function values. The condition of function values describes the sensitivity of the values $p_N(t)$ to perturbations of the data. This condition is, unlike the condition of the coefficients, independent of the basis used (Lagrange, Bernstein, Chebyshev).

Let $x_0, x_1, ..., x_N$ be unperturbed data and $\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_N$ be perturbed data, $p_N(t) = \sum_{i=0}^N x_i \varphi_{i,N}(t)$ and $\tilde{p}_N(t) = \sum_{i=0}^N \tilde{x}_i \varphi_{i,N}(t)$. The polynomial interpolation operator $\mathcal{L}_N : \mathcal{C}[a,b] \to \mathcal{P}_N[a,b]$ is linear and $\|\mathcal{L}_N(h) - \mathcal{L}_N(\tilde{h})\|_\infty \leq \|\mathcal{L}_N\| \|h - \tilde{h}\|_\infty$. The absolute condition can be characterized by $\|\mathcal{L}_N\|$. The condition number of Lagrange interpolation

$$\|\mathcal{L}_N\| = \sup_{f \neq 0} \frac{\|\mathcal{L}_N f\|_\infty}{\|f\|_\infty}$$

can be estimated using the Lebesgue constant (see, e.g., [15]). Thus $p_N = \sum x_i \varphi_{i,N}$ implies $\|\mathcal{L}_N(h)\|_\infty \leq \|h\|_\infty \Lambda_N(K)$ and $\|\mathcal{L}_N\| \leq \Lambda_N(K)$. Then

$$\|p_N^\alpha - \tilde{p}_N^\alpha\|_\infty = \|\mathcal{F}^\alpha \circ \mathcal{L}_N(h) - \mathcal{F}^\alpha \circ \mathcal{L}_N(\tilde{h})\|_\infty \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty} \Lambda_N(K) \|h - \tilde{h}\|_\infty .$$

4.3. Piecewise polynomial interpolation. An alternative to use a uniformly defined polynomial is the approximation by means of piecewise polynomial functions in $[a,b]$:

$$g(t) = \begin{cases} p_{d_1}^1(t) & t \in [a, t_1) \\ p_{d_2}^2(t) & t \in [t_1, t_2) \\ \vdots & \vdots \\ p_{d_k}^k(t) & t \in [t_{k-1}, b] \end{cases}$$

This is a method of local approximation (the data in an interval have no influence on the others). Choosing a suitable breakdown of the interval, and choosing the degrees $d_1, d_2, \ldots, d_k$ appropriately, the interpolation can be well adapted to local differences in the shape of the function f. In particular, step discontinuities of the derivatives can be simulated.

The following theorem proves that the approximation error of the function and its derivatives can satisfy any precision requirement, assuming that the nodes are suitably chosen and the original function is smooth.

Theorem 7. ([15], [10]) *For a function $f \in \mathcal{C}^{d+1}[a,b]$ with $\|f^{(l)}\|_\infty = M_l$ on $[a,b]$, $l = 1, 2, \ldots, d+1$, and a piecewise polynomial interpolation g with degrees $d_i = d$ and subinterval length h, the following error estimate holds for $j = 0, 1, \ldots, d$:*

$$|g^{(j)}(t) - f^{(j)}(t)| \leq \sum_{l=0}^{j} c_{dlj} h^{d+1-l} M_{d+1+j-l}$$

where the constants c_{dlj} can be consulted in the references [10] *and* [15].

It must be borne in mind that in the case of piecewise polynomial interpolation, the improvement of the approximation precision is not fulfilled by increasing the degree but the number of polynomial pieces. The degree d remains constant, and $h \to 0$ indicates that the number of subintervals, and so the number of pieces, increases.

As a consequence of Theorem 7, the next result for fractal piecewise polynomial interpolation is verified.

Theorem 8. *For a function $f \in \mathcal{C}^{d+1}[a,b]$ with $\|f^{(l)}\|_\infty = M_l$ on $[a,b]$, $l = 1,2,\ldots,d+1$, and a piecewise polynomial interpolation g with degrees $d_i = d$ and subinterval length h, the following error estimate holds:*

$$\|f - g^\alpha\|_\infty \le c_{d00}\, h^{d+1} M_{d+1} + \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|f\|_\infty \Lambda_d(K)\,,$$

where c_{d00} is the constant of Theorem 7.

The proof is similar to that of Theorem 6, setting $j = 0$ in Theorem 7.

5. Interpolation of real data

The description of a procedure to build fractal interpolation functions for a set of real data with equidistant nodes is developed here. This function is constructed as a deformation (perturbation) of a smooth classical interpolant h. h would represent a “long-term trend curve” and h^α would add the irregular oscillations characteristic of the “real-world” signals.

We consider here the general case proposed by M. Barnsley, where the FIF is defined by the IFS (7) with $q_n(t) = h \circ L_n(t) - \alpha_n b(t)$, b continuous and such that $b(t_0) = x_0$ and $b(t_N) = x_N$.

Let $\{(t_n, x_n), n = 0,1,\ldots,N\}$ be a set of data points with equidistant nodes and suppose N to be even. Consider the subset $P = \{(t_{2m}, x_{2m}), m = 0,1,\ldots,N/2\}$, and let h be a classical interpolation function (polynomial, for instance) passing through P. From h, an IFS $w_m = (L_m, F_m)$ is defined by

$$\begin{cases} L_m(t) = a_m t + b_m \\ F_m(t,x) = \alpha_m x + q_m(t) \end{cases}$$

for $m = 1,2,\ldots,N/2$ in such a way that the corresponding FIF passes through the points (t_{2m}, x_{2m}) and $q_m(t) = h \circ L_m(t) - \alpha_m b(t)$. The intermediate data points (odd index) will be used for the definition of α_m in order the FIF to pass through them.

If $I_m = [t_{2(m-1)}, t_{2m}]$ and α_m is the scale factor, we impose the condition of passing through (t_{2m-1}, x_{2m-1}). By (8), for $q_m = h \circ L_m - \alpha_m b$ it holds

$$x_{2m-1} = h^\alpha(t_{2m-1}) = h(t_{2m-1}) + \alpha_m (h^\alpha - b) \circ L_m^{-1}(t_{2m-1})\,.$$

If the nodes are equidistant, then $L_m^{-1}(t_{2m-1}) = t_{N/2}$ and $x_{2m-1} = h(t_{2m-1}) + \alpha_m(h^\alpha - b)(t_{N/2})$. Imposing the condition $h^\alpha(t_{N/2}) = x_{N/2}$ leads to

$$\alpha_m = \frac{x_{2m-1} - h(t_{2m-1})}{x_{N/2} - b(t_{N/2})} . \tag{13}$$

If the interpolant h converges towards the original function, the numerator tends to zero as the partition is refined and $|\alpha_m|_\infty < 1$ can be obtained. In any case $|\alpha_m| = CE(t_{2m-1})$ with $C = |x_{N/2} - b(t_{N/2})|^{-1}$, and $E(t_{2m-1})$ is the absolute value of the interpolation error of h in t_{2m-1}. The function b must be chosen in such a way that the denominator of α_m is large enough in order the scale factor to be lower than 1, and always non-null. If h is piecewise linear and b is a line, the method of Strahle [14] is obtained as a particular case.

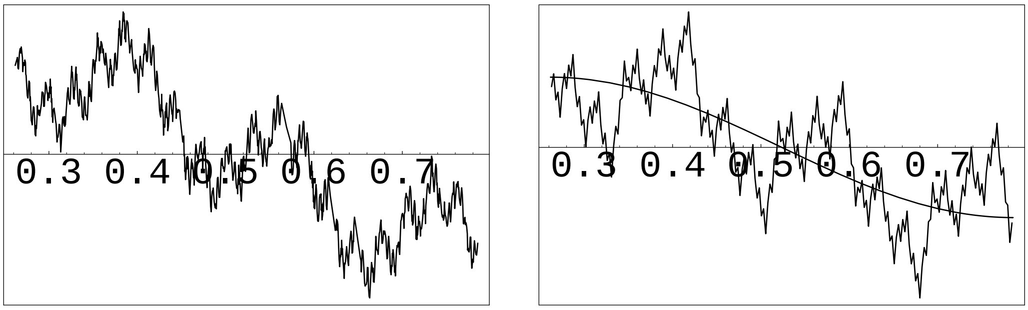

Figure 2: In the left frame, the function $f(t) = \sum_{k=1}^{+\infty} \frac{1}{2^k} \sin(6^k t)$ is represented in the interval $I = [\pi/12, 3\pi/12]$. At right, the polynomial p of Lagrange defined from 7 equidistant nodes (smooth curve) along with the fractal interpolant constructed by the method described in the paragraph 5 (rough line), with $b = p \circ c$ and $c(t) = (e^{2t} - 1)/(e^2 - 1)$.

To construct non-smooth interpolating functions one can proceed in the following way. Let h be a classical (smooth) interpolant of the data. Choose a nowhere differentiable function b (for instance, a Weierstrass's function; see, e.g., [9], [16]) and α_n non-null for all n. For the general case $q_n(t) = h \circ L_n(t) - \alpha_n b(t)$, the fixed point equation (6) defining the fractal interpolation function adopts the expression

$$h^\alpha(t) = h(t) + \alpha_n (h^\alpha - b) \circ L_n^{-1}(t) \qquad (t \in I_n) .$$

As h is smooth, h^α cannot be differentiable at every point because if it were, for any $t \in I$, $L_n(t) \in I_n$ and the former equation can be written as

$$b(t) = h^\alpha(t) + \frac{1}{\alpha_n}(h - h^\alpha) \circ L_n(t) .$$

As a consequence, b would be differentiable at some point.

For non-smooth functions we have the following result.

Proposition 5. *For any partition of the interval, if h is not differentiable in some subinterval I_n^0, there exists h^α arbitrarily close to h which is not differentiable at some point of the interval I.*

Proof. Let h be non-differentiable at some point of I_n^0 and let b be a smooth function (a line for instance). Let $\alpha \in (-1,1)^N$ be any scale vector. The function h^α can not be differentiable in the interval I because if it were, then

$$h(t) = h^\alpha(t) - \alpha_n(h^\alpha - b) \circ L_n^{-1}(t) \qquad (t \in I_n^0)\,,$$

and h would be differentiable in I_n^0. ∎

5.1. Convergence of fractal polynomial interpolants. In some cases, the polynomial interpolation converges towards the function as the number of nodes increases indefinitely. The fact that $\mathcal{P}_N[a,b]$ has a finite dimension allows the existence of minimum distance from any continuous function $f \in \mathcal{C}[a,b]$ to this subspace, $d_N^* = d(f, \mathcal{P}_N[a,b])$. By the theorem of Weierstrass, $d_N^* \to 0$ as $N \to \infty$. In general, one has the next result.

Theorem 9. ([15]) *For a function $f \in \mathcal{C}[a,b]$ and a sequence of polynomial interpolants $\{p_N(f)\}_{N=1}^\infty$ with respect to the node matrix K, the following inequality holds:*

$$\|f - p_N(f)\|_\infty \le d_N^*(1 + \Lambda_N(K)) \qquad (N = 1, 2, \dots)\,.$$

The Lebesgue constant (which depends on the node matrix but not on f) is a measure of the separation of the interpolation error from the minimum error d_N^*. Moreover:

Theorem 10 (Jackson [6]). *For every Lipschitz continuous function f on $[-1,1]$ with a Lispchitz constant L:*

$$d_N^* \le \frac{\pi L}{2N+2}\,.$$

Other convergence results for equidistant and non-equidistant nodes can be consulted in the reference (see [7, Chapter IV]).

Convergence: For fractal polynomial interpolants, we consider $h = p_N$ in order to define the interpolant described at the beginning of this section. If $h = p_N$ converges towards the original signal, then $\alpha \to 0$ in (13) as the partition is refined. As a consequence, if $\mathcal{L}_N(f) \to f$, $\mathcal{F}^\alpha \circ \mathcal{L}_N(f)$ converges to f.

5.2. Fourier transform of α-fractal functions. One of the objectives of the interpolation of real functions is the knowledge about the spectral content of the signal to be analyzed. In this paragraph, an explicit formula for the FT of the FIF defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t,x) = \alpha_n x + q_n(t) \end{cases}$$

with $q_n(t) = h \circ L_n(t) - \alpha_n b(t)$, where b is continuous and such that $b(t_0) = x_0$; $b(t_N) = x_N$, is obtained. The FT of h^α defined on I is $\widehat{h^\alpha}(\omega) = \int_I h^\alpha(\sigma) e^{2\pi i \sigma\omega} d\sigma$. By equality (8) we have

$$\begin{aligned}\widehat{h^\alpha}(\omega) &= \sum_{n=1}^{N} \int_{I_n} (h(\sigma) + \alpha_n (h^\alpha - b) \circ L_n^{-1}(\sigma)) e^{2\pi i \sigma\omega}\, d\sigma \\ &= \hat{h}(\omega) + \sum_{n=1}^{N} \alpha_n \int_{I_n} (h^\alpha - b) \circ L_n^{-1}(\sigma) e^{2\pi i \sigma\omega}\, d\sigma\,.\end{aligned}$$

With the change of variable $L_n^{-1}(\sigma) = t$, we get

$$\begin{aligned}\int_{I_n} (h^\alpha - b) \circ L_n^{-1}(\sigma) e^{2\pi i \sigma\omega} d\sigma &= a_n \int_I (h^\alpha - b)(t) e^{2\pi i \omega(a_n t + b_n)} dt \\ &= a_n e^{2\pi i \omega b_n} (\widehat{h^\alpha}(a_n\omega) - \hat{b}(a_n\omega))\end{aligned}$$

and

$$\widehat{h^\alpha}(\omega) = \hat{h}(\omega) + \sum_{n=1}^{N} \alpha_n a_n e^{2\pi i \omega b_n} (\widehat{h^\alpha}(a_n\omega) - \hat{b}(a_n\omega)).$$

In order to simplify the computations, we consider the case of equidistant nodes $a_n = 1/N$, and denoting $s(\omega) = \sum_{n=1}^{N} \alpha_n e^{2\pi i \omega b_n}$ one has

$$\widehat{h^\alpha}(\omega) = \hat{h}(\omega) + \frac{1}{N} s(\omega) \left(\widehat{h^\alpha}\left(\frac{\omega}{N}\right) - \hat{b}\left(\frac{\omega}{N}\right) \right) \tag{14}$$

Applying the equality (14) for $\widehat{h^\alpha}(\frac{\omega}{N})$ and substituting

$$\widehat{h^\alpha}(\omega) = \hat{h}(\omega) + \frac{1}{N} s(\omega) \left(\hat{h}\left(\frac{\omega}{N}\right) - \hat{b}\left(\frac{\omega}{N}\right) \right) + \frac{1}{N^2} s(\omega) s\left(\frac{\omega}{N}\right) \left(\widehat{h^\alpha}\left(\frac{\omega}{N^2}\right) - \hat{b}\left(\frac{\omega}{N^2}\right) \right),$$

the process can be iterated as follows:

$$\begin{aligned}\widehat{h^\alpha}(\omega) = \hat{h}(\omega) &+ \frac{1}{N} s(\omega) \left(\hat{h}\left(\frac{\omega}{N}\right) - \hat{b}\left(\frac{\omega}{N}\right) \right) \\ &+ \frac{1}{N^2} s(\omega) s\left(\frac{\omega}{N}\right) \left(\hat{h}\left(\frac{\omega}{N^2}\right) - \hat{b}\left(\frac{\omega}{N^2}\right) \right) \\ &+ \frac{1}{N^3} s(\omega) s\left(\frac{\omega}{N}\right) s\left(\frac{\omega}{N^2}\right) \left(\widehat{h^\alpha}\left(\frac{\omega}{N^3}\right) - \hat{b}\left(\frac{\omega}{N^3}\right) \right).\end{aligned}$$

Bearing in mind that $|s(\frac{\omega}{N^j})| \le |\alpha|_\infty N$ for all $j = 0, 1, \ldots, p-1$, it is true that

$$0 \le \left| \frac{1}{N^p} s(\omega) s\Big(\frac{\omega}{N}\Big) \ldots s\Big(\frac{\omega}{N^{p-1}}\Big) \right| \le |\alpha|_\infty^p \to 0$$

as $p \to \infty$ and

$$\widehat{h^\alpha}(\omega) = \hat{h}(\omega) + \sum_{p=1}^{+\infty} \frac{1}{N^p} \left(\prod_{j=0}^{p-1} s\Big(\frac{\omega}{N^j}\Big) \right) \left(\hat{h}\Big(\frac{\omega}{N^p}\Big) - \hat{b}\Big(\frac{\omega}{N^p}\Big) \right).$$

This formula provides an explicit and complete description of h^α in the frequency domain. The term of order p in the series is bounded by

$$|\alpha|_\infty^p \left| \hat{h}\Big(\frac{\omega}{N^p}\Big) - \hat{b}\Big(\frac{\omega}{N^p}\Big) \right| \le |\alpha|_\infty^p \|h - b\|_\infty T,$$

where T is the length of the interval. The approximate sum S_p will have an error E_p bounded by

$$|E_p| \le \frac{|\alpha|_\infty^{p+1}}{1 - |\alpha|_\infty} \|h - b\|_\infty T.$$

6. Conclusions

Every continuous function h defined in a real compact interval can be generalized by means of a family of fractal interpolation functions associated to a partition of the interval. Each element h^α of the set constitutes an interpolant of h respect to the mesh given. The uniform distance between h and h^α is bounded by a quantity which depends on the uniform norm of h and α. The distance goes to zero with α.

The operator $\mathcal{F}^\alpha$, which associates to each function h its α-fractal h^α, is linear and continuous in $\mathcal{C}(I)$, and lower and upper bounds of its norm can be defined in terms of α. $\mathcal{F}^\alpha$ converges to the identity when α goes to zero.

The α-fractal polynomials where $\alpha \neq 0$ constitute a family of dense functions in the space of continuous functions $\mathcal{C}(I)$. As a consequence, the density of polynomial fractal interpolation functions with non-null scale vector is proved.

The Lagrange basis of the space of polynomials of degree lower or equal than N can be generalized to a basis of fractal polynomials. In this way, the fractal polynomial interpolation is defined. The condition of the function values is fixed by the product of the norms of Lagrange and fractal operators. Likewise, fractal piecewise polynomial interpolants can be constructed.

For any interpolant of real data, one can fixe a scale vector α so that the corresponding fractal function passes through them. The convergence of this procedure when the step size tends to zero is submitted to the convergence of the chosen interpolant.

The fractal functions h^α constructed from a continuous functions are defined explicitly in the frequency domain by means of their Fourier transforms, in terms of the transform of h.

The preceding facts display the power of the method of fractal interpolation, since any other conventional approximant can be generalized by means of that kind of techniques (see, e.g., [11], [13]). At the same time, the "fractality" adds a more real geometrical shape to the reconstruction and simulation of natural and social phenomena.

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