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Global Asymptotic Stability of High-Order Delay Hopfield Neural Networks with Time-Varying Coefficients

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Abstract. In this paper, the problem of global asymptotic stability of the high-order delay neural networks with time-varying coefficients is investigated. Sufficient conditions are obtained for the existence and global asymptotic stability of the equilibrium of such neural networks by using Brouwer's fixed point theorem and Liapunov method.

Keywords: *High-order delay, Hopfield neural networks, global asymptotic stability, Liapunov functional, time-varying coefficients*

MSC 2000: 34K20,34K25.

1. Introduction

Hopfield neural networks have been intensively discussed in recent years, see the references and therein. When the networks are applied to reality, the hardware makes signals delay which are transmitted between neurons. From this point, the investigation of delay Hopfield network is very meaningful [1, 3 – 4, 9 – 10]. High-order delay networks are prior to lower ones such as better approach ability, faster convergence, more storage capability and stronger fault tolerance, etc. So, more and more attentions have been paid to the study of high-order neural networks [2, 7, 10].

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Xu and Liao [10] consider a second order delay Hopfield neural networks as follows:

$$\begin{aligned} C_i\frac{du_i(t)}{dt} &= -\frac{u_i(t)}{R_i} + I_i(t) + \sum_{j=1}^{n} W_{ij}g_j(u_j(t-\tau)) \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} W_{ijk}g_j(u_j(t-\tau))g_k(u_k(t-\tau)), \ \ i=1,\dots,n, \end{aligned} \tag{1}$$

with initial conditions

$$u_i(s) = \varphi_i(s), \qquad s\in[-\tau,0], \ i=1,\dots,n, \tag{2}$$

where $C_i > 0$, $R_i > 0$ and I_i represent the input capacitance, input resistance of the ith neural cell and external input signals to the network, respectively. W_{ij}, W_{ijk} stimulate the first and second order connections between the set of the n neurons (or synaptic strengths), respectively. Transmission delay τ is a positive constant; $\varphi_i \in C([-\tau,0],R)$, $i=1,2,\dots,n$. By using the Liapunov second method, Xu and Liao [10] present some sufficient conditions of the global asymptotic stability (GAS) for the above systems.

Moreover, along with the changing of the time, environment and the aging of the network and so on, the input capacitance, input resistance, interconnections and external input may change, too. Under this circumstance, we consider the second order delay neural networks with time-varying coefficients

$$\begin{aligned} C_i(t)\frac{du_i(t)}{dt} &= -\frac{u_i(t)}{R_i(t)} + I_i(t) + \sum_{j=1}^{n} W_{ij}(t)g_j(u_j(t-\tau)) \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} W_{ijk}(t)g_j(u_j(t-\tau))g_k(u_k(t-\tau)), \ \ i=1,\dots,n \end{aligned} \tag{3}$$

with initial conditions

$$u_i(s) = \varphi_i(s), \qquad s\in[-\tau,0], \ i=1,\dots,n, \tag{4}$$

where $C_i(t) > 0$, $R_i(t) > 0$, $I_i(t)$, $W_{ij}(t)$, and $W_{ijk}(t)$ represent the same meanings at time t, respectively. The functions g_i are bounded and satisfy a Lipschitz condition, that is, there exist positive real numbers L_i such that $|g_i(x)-g_i(y)| \leqslant L_i|x-y|$, and $\varphi_i \in C([-\tau,0],R)$, $i=1,\dots,n$.

In this paper, some sufficient conditions are given for the GAS of system (3)-(4) by constructing another Liapunov functional together with a Razumikhin-type theorem which generalizes some results in [2, 9 – 10].

Firstly, we give the vector form of system (3)-(4). Let

$$\begin{aligned}
u(t) &= (u_1(t), u_2(t), \ldots, u_n(t)) \\
g(u(t-\tau)) &= (g_1(u_1(t-\tau)), g_2(u_2(t-\tau)), \ldots, g_n(u_n(t-\tau)))^T, \\
D(t) &= \operatorname{diag}\big(C_1(t)R_1(t), C_2(t)R_2(t), \ldots, C_n(t)R_n(t)\big) \\
W(t) &= \left(\frac{W_{ij}(t)}{C_i(t)}\right)_{n\times n} \\
W_i(t) &= \left(\frac{W_{ijk}(t)}{C_i(t)}\right)_{n\times n} \\
\widetilde{W}(t) &= \big([W_1(t)]_s, [W_2(t)]_s, \ldots, [W_n(t)]_s\big)^T \\
G(u(t-\tau)) &= \operatorname{diag}\big(g(u(t-\tau)), g(u(t-\tau)), \ldots, g(u(t-\tau))\big) \\
I(t) &= \left(\frac{I_1(t)}{C_1(t)}, \frac{I_2(t)}{C_2(t)}, \ldots, \frac{I_n(t)}{C_n(t)}\right)^T \\
\varphi(t) &= (\varphi_1(t), \varphi_2(t), \ldots \varphi_n(t))^T,
\end{aligned}$$

where $[W_i(t)]_s = \frac{1}{2}\left(W_i(t) + W_i(t)^T\right)$, $i = 1, \ldots, n$. Then, system (3)-(4) takes the following form:

$$\begin{aligned}
u'(t) = -D^{-1}(t)u(t) + W(t)g(u(t-\tau)) \\
+ G(u(t-\tau))^T \widetilde{W}(t) g(u(t-\tau)) + I(t)
\end{aligned} \tag{5}$$

$$u(s) = \varphi(s), \qquad s \in [-\tau, 0]. \tag{6}$$

Suppose that the following condition holds:

($\mathbf{H_1}$) $D(t), W(t), \widetilde{W}(t), g(t), I(t)$ are all continuous, bounded functions.

2. Preliminaries

For convenience, we give some basic notation and theories which play an important role in the proof of the main results. The norms $|\cdot|$ and $\|\cdot\|_\infty$ are defined by

$$\begin{aligned}
|x| &= \max_{1\leqslant i\leqslant n} \{|\, x_i \,|\}, & x &\in R^n \\
\|x(t)\|_\infty &= \max_{1\leqslant i\leqslant n} \sup_{t\in R} \{|\, x_i(t) \,|\}, & x(t) &= (x_1(t), x_2(t), \ldots, x_n(t))^T \\
\|A(t)\|_\infty &= \max_{1\leqslant i,j\leqslant n} \sup_{t\in R} \{|\, A_{ij}(t) \,|\}, & A(t) &= (A_{ij}(t))_{n\times n}.
\end{aligned}$$

We substitute $\|\cdot\|_\infty$ with $\|\cdot\|$ when there are no confusions.

From the context of functional differential equations (FDEs) [5], we introduce the function space C. Here, $C = C([-\tau, 0], R^n)$ is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence, for example, designating the norm of an element ϕ in C by $\|\phi\| = \sup_{-\tau \leqslant \theta \leqslant 0} |\phi(\theta)|$. For any map $x : R \to R^n$, the notation x_t was defined by $x_t(\theta) = x(t+\theta)$, $-\tau \leqslant \theta \leqslant 0$. Clearly $x_t \in C$.

Consider the retarded functional differential equations (RFDE(f))

$$x'(t) = f(t, x_t), \qquad x(s) = \phi(s),\ s \in [-\tau, 0], \tag{I}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $f : R \times C \to R^n$, $f(t, 0) = 0$, $\phi \in C$. Assume that f satisfies enough conditions to ensure the existence and uniqueness of the solution as well as the continuous dependence on initial values for any solution. Denote by $x(t) = x(\sigma, \phi)(t)$ the solution through (σ, ϕ).

If $V : R \times G \to R (G \subseteq C)$ is continuous on $R \times \overline{G}$, then define

$$V'(t, \phi) = V'_{(I)}(t, \phi) = \limsup_{h \to 0^+} \frac{1}{h} \left[V(t+h, x_{t+h}(t, \phi)) - V(t, \phi) \right],$$

and if $V : R \times R^n \to R$ is a continuous function, then define

$$V'(t, \phi(0)) = V'_{(I)}(t, \phi(0)) = \limsup_{h \to 0^+} \frac{1}{h} [V(t+h, x(t, \phi)(t+h)) - V(t, \phi(0))],$$

where $x(\sigma, \phi)$ is the solution of system (I) through (σ, ϕ).

Definition 2.1. A function $a : R^+ \to R^+$ is said to be belong to class K, if a is a continuous nondecreasing function, $a(s) > 0$ on $(0, \infty)$ and $a(0) = 0$.

Theorem 2.2. ([5, Theorem 3.2]) *Suppose $f : R \times G \to R^n$ ($G \subseteq C$ is bounded) and $a,\ b \in K$. If there exists a continuous function $V : R \times C \to R$ such that*

$$a(|\phi(0)|) \leq V(\phi), \qquad V'(t, \phi) \leq -b(|\phi(0)|),$$

then the solution $x = 0$ of (I) *is asymptotically stable. If further $a(s) \to \infty$ as $s \to \infty$, every solution of* (I) *is bounded.*

If $V : R \times R^n \to R$ is a continuous function, then $V'(t, \phi(0))$, the derivative of V along the solution of system (I), is defined to be

$$V'(t, \phi(0)) = V'_{(I)}(t, \phi(0)) = \limsup_{h \to 0^+} \frac{1}{h} [V(t+h, x(t, \phi)(t+h)) - V(t, \phi(0))],$$

where $x(\sigma, \phi)$ is the solution of system (I) through (σ, ϕ).

Theorem 2.3 (Razumikhin-type theorem). ([5, Theorem 4.2]) *Suppose $f : R \times G \to R^n$ ($G \subseteq C$ is bounded) takes $R \times G$ into bounded sets of R^n, and consider system* (I). *Suppose u, v, w belong to K. If there is a continuous nondecreasing function p satisfying $p(s) > s$ for $s > 0$ and a continuous function $V : R \times R^n \to R$ such that*

$$u(|x|) \le V(t,x) \le v(|x|), \qquad t \in R,\ x \in R^n;$$

$$V'(t,\phi(0)) \le -w(|\phi(0)|) \quad \text{if} \quad V(t,\phi(\theta)) \le p\big(V(t,\phi(0))\big)\ \forall\theta \in [-\tau, 0],$$

then the solution $x = 0$ of (I) *is uniformly asymptotically stable. And further, if $u(s) \to \infty$ as $s \to \infty$, the solution $x = 0$ is also a global attractor.*

3. Existence of Equilibrium

Before the main results, we first investigate the existence of the equilibrium for the system (5)-(6) by using the Brouwer's fixed-point theorem. After a coordinate translation, we can see that the GAS property of the equilibrium of (5)-(6) is equivalent that of the zero solution for the corresponding system. And in the next section, we will discuss the GAS of the zero solution for the corresponding system.

Theorem 3.1. *Suppose that* (H_1) *holds, then the system* (5)-(6) *has at least one equilibrium.*

Proof. Consider the operator $F : R^n \to R^n$ according to

$$F(u) = D(t)W(t)g(u) + D(t)G(u)^T\widetilde{W}(t)g(u) + D(t)I(t).$$

It is obvious that F is a continuous operator. Let

$$\Omega = \{u \in R^n \mid \|u - D(t)I(t)\| \leqslant \|D(t)\|\left(\|W(t)\| + L\cdot\|\widetilde{W}(t)\|\right)L\},$$

where $L = \|g(u)\|$, then Ω is a closed bounded subset of R^n. For any $u \in R^n$,

$$\begin{aligned}\|F(u) - D(t)I(t)\| &= \|D(t)W(t)g(u) + D(t)G(u)^T\widetilde{W}(t)g(u)\| \\ &\leqslant \|D(t)\|\cdot\|\left(W(t) + G(u)^T\widetilde{W}(t)\right)\|\cdot\|g(u)\| \\ &\leqslant \|D(t)\|\left(\|W(t)\| + \|g(u)\|\cdot\|\widetilde{W}(t)\|\right)\cdot\|g(u)\| \\ &= \|D(t)\|\left(\|W(t)\| + L\cdot\|\widetilde{W}(t)\|\right)L,\end{aligned}$$

then operator F maps the set Ω into itself. From Brouwer's fixed-point theorem [6], there is at least one fixed-point u^* such that $F(u^*) = u^*$, i.e.,

$$u^* = D(t)W(t)g(u^*) + D(t)G(u^*)^T\widetilde{W}(t)g(u^*) + D(t)I(t).$$

Thus, system (5)-(6) has at least an equilibrium. ∎

Suppose $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ is an equilibrium of the system (3) (or 5). Considering the coordinate translation $x = u - u^* = (x_1, x_2, \ldots, x_n)^T$, we get the following system which is another form of system (3) by using the mean value of Lagrange theorem:

$$\begin{aligned} C_i(t)\frac{dx_i(t)}{dt} &= -\frac{x_i(t)}{R_i(t)} + \sum_{j=1}^{n} W_{ij}(t) f_j(x_j(t-\tau)) \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}(W_{ijk}(t) f_j(x_j(t-\tau)) \cdot f_k(x_k(t-\tau)) \\ &\quad + f_k(x_k(t-\tau)) g_j(u_j^*) + f_j(x_j(t-\tau)) g_k(u_k^*)) \\ &= -\frac{x_i(t)}{R_i(t)} + \sum_{j=1}^{n} W_{ij}(t) f_j(x_j(t-\tau)) \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{n}(W_{ijk}(t) + W_{ikj}(t)) \cdot \xi_k \cdot f_j(x_j(t-\tau)), \end{aligned} \tag{7}$$

where $f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)$, $i = 1, \ldots, n$. If fact, by using the mean value of Lagrange theorem, we have

$$\begin{aligned} & g_j(u_j(t-\tau)) g_k(u_k(t-\tau)) - g_j(u_j^*) g_k(u_k^*) \\ &= g_j(u_j(t-\tau))[g_k(u_k(t-\tau)) + g_k(u_k^*)] - [g_j(u_j(t-\tau)) + g_j(u_j^*)] g_k(u_k^*) \\ &= g_j(u_j(t-\tau)) \left(\frac{g_k^2(u_k(t-\tau)) - g_k^2(u_k^*)}{g_k(u_k(t-\tau)) - g_k(u_k^*)}\right) - \left(\frac{g_j^2(u_j(t-\tau)) - g_j^2(u_j^*)}{g_j(u_j(t-\tau)) - g_j(u_j^*)}\right) g_k(u_k^*) \\ &= 2\xi_k g_j(u_j(t-\tau)) - 2\xi_j g_k(u_k^*) \\ &= 2\xi_k f_j(x_j(t-\tau)) + 2\xi_k g_j(u_i^*) - 2\xi_j g_k(u_k^*), \end{aligned}$$

and similarly by using the mean value of Lagrange theorem, we also have

$$g_j(u_j(t-\tau)) g_k(u_k(t-\tau)) - g_j(u_j^*) g_k(u_k^*) = 2\xi_j f_k(x_k(t-\tau)) + 2\xi_j g_k(u_k^*) - 2\xi_k g_j(u_j^*),$$

and so

$$g_j(u_j(t-\tau)) g_k(u_k(t-\tau)) - g_j(u_j^*) g_k(u_k^*) = \xi_k f_j(x_j(t-\tau)) + \xi_j f_k(x_k(t-\tau)),$$

and ξ_k belongs to the interval of $g_k(u_k(t-\tau))$ and $g_k(u_k^*), k = 1, \ldots, n$. The initial condition is of the type

$$x_i(s) = \phi_i(s), \qquad s \in [-\tau, 0], \tag{8}$$

where $\phi_i(s) = \varphi_i(s) - u^*$, $i = 1, \ldots, n$. Let

$$\begin{aligned} \Xi &= \frac{1}{2}\mathrm{diag}\left((\xi_1, \xi_2, \ldots, \xi_n)^T, (\xi_1, \xi_2, \ldots, \xi_n)^T, \ldots, (\xi_1, \xi_2, \ldots, \xi_n)^T\right) \\ f(x) &= (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T, \end{aligned}$$

then we have the vector form of Hopfield neural networks model (7) along with the initial conditions (8)

$$x'(t) = -D^{-1}(t)x(t) + (W(t) + \Xi^T\widetilde{W}(t))f(x(t-\tau)) \tag{9}$$

$$x(s) = \phi(s), \qquad s \in [-\tau, 0]. \tag{10}$$

Therefore, the GAS of the equilibrium $u = u^*$ for system (3) (or 5) is equivalent to the GAS of the equilibrium $x = 0$ for system (7) (or 9). Next we will present global convergence criteria for system (9)-(10).

4. Global asymptotic stability

Suppose that

($\mathbf{H_2}$) f_i are continuous, and for any $x_i \in R \setminus \{0\}$ let $x_i f_i(x_i) > 0$ and

$$\int_0^{x_i} f_i(s)ds \to +\infty \qquad \text{as } |x_i| \to +\infty \quad (i = 1, \ldots, n).$$

Theorem 4.1. *Suppose that* (H$_1$) *and* (H$_2$) *hold, and there exits a positive diagonal matrix P such that*

$$P(W(t) + \Xi^T\widetilde{W}(t)) \cdot [P(W(t) + \Xi^T\widetilde{W}(t))]^T + E$$

is negative definite, then the equilibrium solution $x = 0$ of (9)-(10) *is asymptotically stable and every solution of* (9)-(10) *is bounded.*

Proof. First, it is obvious that the assumption of (H$_1$) and (H$_2$) ensures the solutions' existence of the FDEs (9)-(10). It is a standard theory that solutions of (9) exist on $[0, \delta)$ for some $\delta > 0$. Furthermore, if the solution remains bounded, then $\delta = \infty$.

The Liapunov functional $V : C \to R$ is defined by

$$V(\phi) = 2\sum_{i=1}^{n} p_i \int_0^{\phi_{i(0)}} f_i(\theta)d\theta + \int_{-\tau}^{0} f(\phi(\theta))^T f(\phi(\theta))d\theta,$$

where the constants p_i, $i = 1, \ldots, n$, are the entries of the diagonal matrix P. We can get the derivative of V along with the trajectory of system (9) as following noticing the condition (H_2) and the initial condition (10):

$$\begin{aligned} V'(\phi) &= 2f(\phi(0))^T P\phi'(0) + f(\phi(0))^T f(\phi(0)) - f(\phi(-\tau))^T f(\phi(-\tau)) \\ &= -2f(\phi(0))^T PD(t)^{-1}\phi(0) + 2f(\phi(0))^T P(W(t) \\ &\quad + \Xi^T \widetilde{W}(t)) f(\phi(-\tau)) + f(\phi(0))^T f(\phi(0)) - f(\phi(-\tau))^T f(\phi(-\tau))\,. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} V'(\phi) \leqslant\ & 2f(\phi(0))^T P(W(t) + \Xi^T \widetilde{W}(t)) f(\phi(-\tau)) \\ & + f(\phi(0))^T f(\phi(0)) - f(\phi(-\tau))^T f(\phi(-\tau)). \end{aligned}$$

Let

$$\begin{aligned} X &= P(W(t) + \Xi^T \widetilde{W}(t)) \cdot [P(W(t) + \Xi^T \widetilde{W}(t))]^T + E \\ Y &= f(\phi(-\tau)) - [P(W(t) + \Xi^T \widetilde{W}(t))]^T f(\phi(0)), \end{aligned}$$

then we get $V'(\phi) \leqslant f(\phi(0))^T X f(\phi(0)) - Y^T Y$. Since X is negative definite, $V'(\phi) < 0$ for $\phi \neq 0$ and $V'(0) = 0$. Set

$$a_i(x) = \min\left\{ \int_0^x f_i(\theta)d\theta, \int_0^{-x} f_i(\theta)d\theta \right\}, \qquad (i = 1, \ldots, n).$$

It is obviously that $a_i(0) = 0$ and $a_i(x) = a_i(|x|)$. From (H_2), $a_i(x) > 0$ for any $x > 0$, and $a_i(x) \to +\infty$ as $x \to \infty$. Let $p_0 = \min_{1 \leqslant i \leqslant n}\{p_i\}$, $a = \min_{1 \leqslant i \leqslant n}\{a_i\}$, then we have

$$\begin{aligned} V(\phi) &= 2\sum_{i=1}^{n} p_i \int_0^{\phi_i(0)} f_i(\theta)d\theta + \int_{-\tau}^{0} f(\phi(\theta))^T f(\phi(\theta))d\theta \\ &\geqslant 2\sum_{i=1}^{n} p_0 a_i(\phi_i(0)) \\ &\geqslant 2\sum_{i=1}^{n} p_0 a(|\phi_i(0)|) \\ &\geqslant 2p_0 a(|\phi(0)|), \end{aligned}$$

that is, $V(\phi)$ achieves the lower bound by a positive, radially unbounded function of $|\phi(0)|$. Thus, all solutions of the system (9)-(10) are asymptotically stable and bounded. This completes the proof. ∎

When the bounds of D and f_i are strengthened, we can get another result.

Theorem 4.2. *Suppose that* (H$_1$) *and* (H$_2$) *hold, and there exist positive constants* ν_i *and* γ_i *such that, for any* $x_i \neq 0$,

$$D_i^{-1}(t) \geqslant \nu_i, \qquad |f_i(x_i)| \leqslant \gamma_i |x_i|\,,$$

where $D_i^{-1}(t)$ *are the entries of the diagonal matrix* $D^{-1}(t), i = 1, 2, \ldots, n$. *Set* $\beta = \max_{1\leqslant i\leqslant n} \frac{\gamma_i}{\nu_i}$. *If there exists a positive diagonal matrix* $P = \operatorname{diag}(p_1, \ldots, p_n)$ *such that*

$$\beta^2 P(W(t) + \Xi^T \widetilde{W}(t)) \cdot [P(W(t) + \Xi^T \widetilde{W}(t))]^T - 2\beta\nu P\Gamma^{-1} + E$$

is negative definite, where $\nu = \operatorname{diag}(\nu_1, \ldots, \nu_n)$, *and* $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$, *then the equilibrium solution* $x = 0$ *of* (9)-(10) *is asymptotically stable and every solution of* (9)-(10) *is bounded.*

Proof. We establish the Liapunov functional $V : C \to R^n$ defined by

$$V(\phi) = 2\beta \sum_{i=1}^{n} p_i \int_0^{\phi_i(0)} f_i(\theta)d\theta + \beta \int_{-\tau}^{0} f(\phi(\theta))^T f(\phi(\theta))d\theta,$$

then

$$\begin{aligned} V'(\phi) = &-2\beta f(\phi(0))^T P D(t)^{-1}\phi(0) + 2\beta f(\phi(0))^T P(W(t) \\ &+ \Xi^T \widetilde{W}(t)) \cdot f(\phi(-\tau)) + \beta f(\phi(0))^T f(\phi(0)) \\ &- \beta f(\phi(-\tau))^T f(\phi(-\tau)). \end{aligned}$$

Using the assumption and noticing that $p_i > 0$ and $\beta > 0$, we have

$$2\beta p_i f_i(\phi_i(0)) D_i(t)^{-1}\phi_i(0) \geqslant 2\beta p_i \nu_i f_i(\phi_i(0))\phi_i(0) \geqslant 2\beta p_i \frac{\nu_i}{\gamma_i} |f_i(\phi_i(0))|^2$$

for $i = 1, \ldots, n$. Then $2\beta f(\phi(0))^T P D^{-1}(t)\phi(0) \geqslant 2\beta f(\phi(0))^T \nu P \Gamma^{-1} f(\phi(0))$, and furthermore, we have

$$\begin{aligned} V'(\phi) \leqslant &-2\beta f(\phi(0))^T \nu P\Gamma^{-1} f(\phi(0)) + 2\beta f(\phi(0))^T P(W(t) \\ &+ \Xi^T \widetilde{W}(t)) \cdot f(\phi(-\tau)) \\ &+ \beta f(\phi(0))^T f(\phi(0)) - \beta f(\phi(-\tau))^T f(\phi(-\tau)) \\ = &\, f(\phi(0))^T X f(\phi(0)) - Y^T Y, \end{aligned}$$

where

$$\begin{aligned} X &= \beta^2 P(W(t) + \Xi^T \widetilde{W}(t)) \cdot [P(W(t) + \Xi^T \widetilde{W}(t))]^T - 2\beta\nu\Gamma^{-1} + E \\ Y &= f(\phi(-\tau)) - \beta[P(W(t) + \Xi^T \widetilde{W}(t))]^T f(\phi(0)). \end{aligned}$$

Provided that X is negative definite, we are able to complete the proof by using the same arguments presented in Theorem 4.1. ∎

If $W_{ijk}(t) \equiv 0$ $i,j,k = 1,\dots,n$, the following corollaries are obtained directly from Theorem 4.1 and 4.2.

Corollary 4.3. *Suppose that* $\widetilde{W}(t) \equiv 0$, (H_1) *and* (H_2) *hold, and there exists a positive diagonal matrix* P *such that* $PW(t)[PW(t)]^T + E$ *is negative definite, then for every delay* $\tau > 0$*, the equilibrium solution* $x = 0$ *of* (9)-(10) *is asymptotically stable and every solution of* (9)-(10)*is bounded.*

Corollary 4.4. *Suppose that* $\widetilde{W}(t) \equiv 0$, (H_1) *and* (H_2) *hold, and there exist positive constants* ν_i *and* γ_i *such that, for any* $x_i \neq 0$,

$$D_i(t) \geqslant \nu_i, \qquad |f_i(x_i)| \leqslant \gamma_i |x_i|,$$

where $D_i(t)^{-1}$ *are the entries of the diagonal matrix* $D(t)^{-1}, i = 1,\dots,n$. *Set* $\beta = \max_{1\leqslant i\leqslant n} \frac{\gamma_i}{\nu_i}$. *If there exists a positive diagonal matrix* $P = \mathrm{diag}\,(p_1,\dots,p_n)$ *such that*

$$\beta^2 PW(t)[P(W(t)]^T + 2\beta\nu P\Gamma^{-1} + E$$

is negative definite, where $\nu = \mathrm{diag}\,(\nu_1,\dots,\nu_n)$*, and* $\Gamma = \; diag(\gamma_1,\dots,\gamma_n)$*, then for every delay* $\tau > 0$*, the equilibrium solution* $x = 0$ *of* (9)-(10) *is asymptotically stable and every solution of* (9)-(10) *is bounded.*

By using the Razumikhin-type theorem, we can get the globally asymptotical stability of system (9)-(10). For this, suppose $d = \min_{1\leq i\leq n} \inf_{t\in R^+} |D_i^{-1}(t)| > 0$, and set $k = n^2|(W(t) + \Xi^T \widetilde{W}(t)|$.

Theorem 4.5. *Suppose the condition* (H_1) *is satisfied and* $|\frac{\partial f}{\partial x}| \leq L$*. If there exist* $q > 1$ *and* $\mu > 0$ *such that* $-d + kLq < -\mu < 0$*, then system* (9)-(10) *is GAS.*

Proof. If $x(t)$ is a solution of system (9)-(10), then we have

$$\begin{aligned} &x'(t) = -D^{-1}(t)x(t) + \big(W(t) + \Xi^T \widetilde{W}(t) f(x(t-\tau)\big) \\ &x(s) = \phi(s), \qquad s \in [-\tau, 0]. \end{aligned}$$

Set $V(t,\phi(0)) = \frac{1}{2}\phi^T(0)\cdot\phi(0)$, then we have

$$\begin{aligned} V'(t,x) &= x^T(t)\cdot x'(t) \\ &= x^T(t)\cdot\big[-D^{-1}(t)x(t) + (W(t) + \Xi^T\widetilde{W}(t))f(x(t-\tau))\big] \\ &\leq -x^T(t)D^{-1}(t)x(t) + x^T(t)\big(W(t) + \Xi^T\widetilde{W}(t)\big)\left|\frac{\partial f}{\partial x}\right| x(t-\tau) \\ &\leq -d|x(t)|^2 + kL|x(t)||x(t-\tau)|. \end{aligned}$$

Set $u(s) = v(s) = \frac{1}{2}s^2$ and $p(s) = q^2 s$, so if $V(t, \phi(\theta)) \le p(V(t, \phi(0)))$ for $\theta \in [-\tau, 0]$, that is, $\phi^T(t-\theta)\phi(t-\theta) \le q^2\phi^T(t)\phi(t)$ for $\theta \in [-\tau, 0]$, we then have

$$\begin{aligned} V'(t, \phi(0)) &\le -d|\phi(0)|^2 + kL|\phi(0)^T||\phi(-\tau)| \\ &\le (-d + kLq)|\phi(0)|^2 \\ &\le -\mu|\phi(0)|^2. \end{aligned}$$

Notice that $x = 0$ is the equilibrium of system (9)-(10), and so $x(t) \to 0$ as $t \to \infty$. While $x = 0$ is a global attractor and so system (9)-(10) is globally asymptotically stable. ∎

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