

# Cauchy and Nonlocal Multi-Point Problems for Distributed Order Pseudo-Differential Equations, Part One

Sabir Umarov and Rudolf Gorenflo

**Abstract.** We treat the question of existence, uniqueness and construction of a solution to the Cauchy and multi-point problems for a general linear evolution equation with (in general) temporal fractional derivatives with distributed orders. Such equations have met great interest in recent years among researchers in viscoelasticity and in anomalous diffusion processes, and there are numerical analysts who consider them as a challenge. So, we find it desirable to put their theory on a strong and general mathematical basis. After an outline of relevant function spaces and the duality structure generated by them we treat, by Laplace-Fourier techniques, first the Cauchy problem, then the general multi-point problem (where the values of linear combinations of the unknown solution at different instants of time are prescribed). We condense our results in theorems on strong and on weak solutions.

**Keywords:** *Distributed order fractional differential equation, pseudo-differential operator, Cauchy problem, multi-point value problem, Caputo fractional derivative*

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## 1. Introduction

This paper is devoted to distributed order fractional differential equations. In recent years such equations have met interest of several researchers (see, for example [2], [5]–[8], [22], [30] and references therein) who have treated relevant physical and numerical problems, mainly of evolutionary character. Our main purpose here is the mathematical treatment of such equations. We will study the solvability and uniqueness issues of initial and some boundary value problems

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Sabir Umarov: Computer Technologies Department, National University of Uzbekistan, Tashkent, Uzbekistan; [sabir0725@yahoo.com](mailto:sabir0725@yahoo.com)

Rudolf Gorenflo: Fachbereich Mathematik und Informatik, Freie Universität Berlin, Berlin, Germany; [gorenflo@mi.fu-berlin.de](mailto:gorenflo@mi.fu-berlin.de)

for them in different function spaces. The general formulation of the problems can be settled in the following framework: introduction of the distributed time-fractional differential operator equation and study of the Cauchy and some other boundary value problems for it.

Our distributed order time fractional differential-operator equation has the general form

$$\int_0^m A(r)D_*^r u(t)dr = Bu(t), \quad t > 0, \tag{1}$$

where  $A(r)$  (for a fixed  $r \in [0, m]$ ) and  $B$  are linear closed operators defined in a certain locally convex topological vector space  $X$ , the function  $u(t)$  is unknown and belongs to the space  $C^{(m)}(0, T; X)$  with some  $T > 0$  and  $D_*^r$  is the operator of fractional differentiation of order  $r$  in the Caputo sense (see, for example, [4], [12], [27])

$$D_*^r f(t) = \frac{1}{\Gamma(m-r)} \int_0^t (t-\tau)^{m-r-1} f^{(m)}(\tau) d\tau, \quad t > 0, \quad m-1 < r < m \tag{2}$$

$$D_*^m f(t) = f^{(m)}(t)$$

for  $m \in \mathbf{N}$ . By  $\mathbf{N}$  we mean, as usual, the set of positive integers. An essential distinctive feature of this model is that integration in (1) is performed over the variable  $r$ , the order of differentiation. Such models arise in a natural way in the kinetic theory (see [7] and [30]) when the exact scaling is lacking or in the theory of elasticity [22] for description of rheological properties of composite materials (see also the examples below). We will not present here results for such general case but will restrict ourselves to considering some particular cases of  $A(r)$ ,  $B$  and  $X$ , leaving the general case for a separate paper.

In this paper we will consider the following distributed order time-fractional differential equation with spatial pseudo-differential operators

$$\int_0^m A(r; D)D_*^r u(t, x)dr = B(D)u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \tag{3}$$

with the Cauchy conditions

$$\frac{\partial^k u(0, x)}{\partial t^k} = \varphi_k(x), \quad x \in \mathbb{R}^n, \quad k = 0, \dots, m-1, \tag{4}$$

where the  $\varphi_k$ ,  $k = 0, \dots, m-1$ , are given functions in certain spaces described later,  $D = (D_1, \dots, D_n)$ ,  $D_j = -i\frac{\partial}{\partial x_j}$ ,  $j = 1, \dots, n$ ,  $A(r; D)$  (for every fixed value of the parameter  $r \in [0, m]$ ) and  $B(D)$  are pseudo-differential operators with the symbols  $A(r; \xi)$  and  $B(\xi)$ , respectively, which are continuous functions of  $\xi$  defined in an open domain  $G \subset \mathbb{R}^n$ . The pseudo-differential operators used in this paper are defined in Section 2 by the formula (9) below. Some

spatial pseudo-differential operators frequently used to describe anomalous diffusion processes with infinite second moment of the spatially zero-centered fundamental solution also have continuous symbols. So,  $A(r; D)$  and  $B(D)$  can be fractional order spatial operators as well.

For our further considerations we need some properties of pseudo-differential operators with symbols singular in dual variables. These operators were studied by Dubinskij [10], [11] in the case of analytic symbols defined in a domain  $G$  and used by several authors including [32], [33].

In constructions we use in the present paper, in order to include operators with non-analytic symbols, we choose a direct approach for definition of such operators, which is distinct from the above mentioned and does not require glueing of local elements of pseudo-differential operators used in case of analytic symbols. This approach (see [16], [34], [35]) allows us to consider non-analytic symbols which may have non-integrable or other type singularities on the boundary of  $G$  or in its exterior.

We also outline briefly some results for solution of the more general multi-point value problem for equation (3) with the conditions

$$\sum_{j=0}^{m-1} \Gamma_{kj}(D) \frac{\partial^k u(t_{kj}, x)}{\partial t^k} = \varphi_k(x), \quad k = 0, \dots, m - 1, \quad x \in \mathbb{R}^n, \quad (5)$$

where the  $\Gamma_{kj}(D)$ ,  $k, j = 0, \dots, m - 1$ , are pseudo-differential operators whose symbols  $\Gamma_{kj}(\xi)$ ,  $k, j = 0, \dots, m - 1$ , are continuous functions in  $G$ ,  $t_{kj} \in [0, T]$ ,  $0 < T < \infty$ , and the  $\varphi_k$ ,  $k = 0, \dots, m - 1$ , are given functions. We note that the Cauchy problem (3), (4) is a particular case of the multi-point value problem (3), (5).

The equation (3) is a generalization of fractional/non-fractional differential equations and important from the viewpoint of applications. To justify this we consider some frequently used particular cases, putting definitions formally:

**1.** Let  $A(r; D) = \delta(r - \beta)I$  with the Dirac function (distribution)  $\delta$  and the identity operator  $I$ . In this case we get the fractional differential equation

$$D_*^\beta u(t, x) = B(D)u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \quad \beta > 0.$$

**1.1.** *Time-fractional equation.*

Let  $B(D) = \Delta$  be the Laplace operator with the symbol  $B(\xi) = -|\xi|^2$ . Then we have the time-fractional differential equation

$$D_*^\beta u(t, x) = \Delta u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \quad \beta > 0. \quad (6)$$

The Cauchy problem for this equation represents a fractional model of diffusion (sub-diffusion) in the case  $0 < \beta < 1$ , which has an important role in transport

theory and random walk [23], [25], [26]. It represents processes intermediate between diffusion and wave propagation in the case  $1 < \beta < 2$  [31]. The solution has the relaxation property in the case  $0 < \beta < 1$  and the oscillation-relaxation property when  $1 < \beta < 2$  [16].

**1.2. Space-fractional differential equation.**

Let  $A(r; D) = \delta(r - 1)I$  and  $B(D) = D_0^\alpha$ , the latter being the operator with the symbol  $-|\xi|^\alpha, \xi \in \mathbb{R}^n$ , which is a continuous function. Then we come to the equation

$$\frac{\partial u(t, x)}{\partial t} = D_0^\alpha u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \quad \alpha > 0, \quad (7)$$

which describes in the case  $0 < \alpha < 2$  an anomalous (to use physical language) diffusion process with infinite second moment of its spatially zero-centered fundamental solution. We note that  $D_0^\alpha$  is connected with the inversion of the fractional Riesz potential (see, for example, [29]) and can be written in the form

$$D_0^\alpha f(x) = \frac{-1}{c_n(\alpha)} \int_{\mathbb{R}^n} \frac{(\Delta_h^2 f)(x)}{|h|^{n+\alpha}} dh.$$

Here  $\Delta_h^2$  is the second order finite difference with the spatial step  $h \in \mathbb{R}^n$  and  $c_n(\alpha)$  is a constant (for the values of this constant see [29]). In the one-dimensional case and under the condition  $0 < \alpha \leq 2$  this equation describes symmetric Lévy-Feller diffusion processes, which are Markovian [13]. Symmetric and non-symmetric random walk models approximating Lévy-Feller diffusion processes were presented in detail by Gorenflo and Mainardi [13], [14], [15]. In the multidimensional case it is studied by Umarov and Gorenflo [36]. For more general operators on the right hand side of (7), including operators with variable orders, the corresponding Feller semigroups are constructed in [18]–[21]. These results allow to judge on existence of a solution (in some sense) to the corresponding Cauchy problem for such equations.

**1.3. Time- and space-fractional equation.**

Let  $A(r; D) = \delta(r - \beta)I, \beta \in (0, 1)$  and  $B(D) = D_0^\alpha$ . Then we get the time-space fractional differential equation

$$D_*^\beta u(t, x) = D_0^\alpha u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \quad \alpha, \beta > 0,$$

which describes anomalous diffusion processes, in particular such ones with non-Markovian character (see [17], [24], [26]).

**2.** The next example relates to sub-diffusion with retardation, studied in [7]. Let  $A(r; D) = b_1 \delta(r - \beta_1)I + b_2 \delta(r - \beta_2)I$ , with  $0 < \beta_1 < \beta_2 \leq 1, b_1 > 0, b_2 > 0, b_1 + b_2 = 1$  and  $B(D) = k \frac{\partial^2}{\partial x^2}$  ( $n = 1$ ),  $k = \text{const} > 0$ . Then we get the equation

$$b_1 D_*^{\beta_1} u(t, x) + b_2 D_*^{\beta_2} u(t, x) = k \frac{\partial^2}{\partial x^2} u(t, x),$$

which describes a subdiffusion process with retardation. In [7] the Cauchy problem for the equation

$$\int_0^1 \tau^{\beta-1} w(\beta) \frac{\partial^\beta p}{\partial t^\beta} d\beta = k \frac{\partial^2 p}{\partial x^2}$$

referred to as the "normal form" of the distributed order fractional diffusion is also studied. Note that this equation corresponds to the case  $A(r, \xi) = \tau^{r-1} w(r)$  with  $\tau > 0$ ,  $w(r) > 0$  and  $B(D) = k \frac{\partial^2}{\partial x^2}$ .

**3.** The authors of [22] have discussed equations of the form

$$\int_0^2 k(q) D^q y(t) dq + F(y) = f(t),$$

which describe properties of composite materials. Note that  $k(q)$ ,  $F(y)$  and  $f(t)$  are given functions connected with different characteristics of viscoelastic and viscoinertial materials with rheological properties.

**4.** Let  $A(r; D) = \sum_{k=1}^m \delta(r - \beta_k) A_k(D)$  with  $k - 1 < \beta_k \leq k$ ,  $A_k(D)$ ,  $k = 0, \dots, m - 1$ , be pseudo-differential operators with symbols  $A_k(\xi)$  continuous in  $G$ . In this case we obtain the equation

$$D_*^{\beta_m} u(t, x) + \sum_{k=1}^{m-1} A_k(D) D_*^{\beta_k} u(t, x) + A_0(D) u(t, x) = 0, \quad t \in (0, T), \quad x \in \mathbb{R}^n. \quad (8)$$

In the case  $\beta_k \in \mathbb{N}$ ,  $k = 1, \dots, m$ , the Cauchy problem for this equation with analytic symbols or with symbols having singularities was studied, for example, by Dubinskij [10], Umarov [33] and Tran Duc Van [32]. Antipko and Borok [1], Borok [3], Ptashnik [28], and Umarov [34], [35] (see also references therein) considered multi-point value problems with integer  $\alpha_k$ . For fractional  $\alpha_k$  the Cauchy and multi-point value problems are studied in [16].

Our paper is organized as follows. In Section 2 we will give briefly the description of the space  $\Psi_{G,p}(\mathbb{R}^n)$  introduced in [34], define pseudo-differential operators with singular symbols and recall some of their properties referring the readers for details to [34], [16]. In Section 3 we will construct a general representation formula for the solution of the Cauchy problem for distributed order differential equations. In Section 4 we will study in detail properties of solution operators obtained in Section 3. Using these properties in Section 5 we will prove solvability theorems for the Cauchy problem (3), (4) in the space  $\Psi_{G,p}(\mathbb{R}^n)$  and in its dual space  $(\Psi_{G,p}(\mathbb{R}^n))^* = \Psi'_{-G,q}(\mathbb{R}^n)$ ,  $q = p/(p - 1)$ . Finally in Section 6 we will consider the general multi-point value problem (3), (5) and outline the related results.

## 2. Function spaces and pseudo-differential operators

In this section we introduce some function spaces and briefly recall their essential properties to be used later for formulation of solvability theorems. For details see [34] and [16].

For a given open domain  $G \subset \mathbb{R}^n$  let  $f$  be a function in  $L_p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , whose Fourier transform

$$\hat{f}(\xi) = Ff(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\xi} dx$$

has a compact support in  $G$ . The set of all such functions endowed with the convergence defined below is denoted by  $\Psi_{G,p}(\mathbb{R}^n)$ : A sequence of functions  $f_m \in \Psi_{G,p}(\mathbb{R}^n)$  is said to converge to an element  $f_0 \in \Psi_{G,p}(\mathbb{R}^n)$  iff:

- (a) there exists a compact set  $K \subset G$  such that  $\text{supp } \hat{f}_m \subset K$  for all  $m \in \mathbb{N}$ ;
- (b)  $\|f_m - f_0\|_p = \left(\int_{\mathbb{R}^n} |f_m - f_0|^p dx\right)^{\frac{1}{p}} \rightarrow 0$  for  $m \rightarrow \infty$ .

In the case  $p = 2$  we write simply  $\Psi_G(\mathbb{R}^n)$  instead of  $\Psi_{G,2}(\mathbb{R}^n)$ . According to the Paley-Wiener-Schwartz theorem, the elements of  $\Psi_{G,p}(\mathbb{R}^n)$  are entire functions of exponential type which, restricted to  $\mathbb{R}^n$ , are in the space  $L_p(\mathbb{R}^n)$ .

Let  $q = p/(p - 1)$ . Denote by  $\Psi'_{-G,q}(\mathbb{R}^n)$  the space of all linear bounded functionals defined on the space  $\Psi_{G,p}(\mathbb{R}^n)$  endowed with the weak topology. Namely, we say that a sequence of functionals  $g_N \in \Psi'_{-G,q}(\mathbb{R}^n)$  converges to an element  $g_0 \in \Psi'_{-G,q}(\mathbb{R}^n)$  in the weak sense if for all  $f \in \Psi_{G,p}(\mathbb{R}^n)$  the sequence of numbers  $\langle g_N, f \rangle$  converges to  $\langle g_0, f \rangle$  as  $N \rightarrow \infty$ . By  $\langle g, f \rangle$  we mean the value of the action of  $g \in \Psi'_{-G,q}(\mathbb{R}^n)$  on an element  $f \in \Psi_{G,p}(\mathbb{R}^n)$ .

Let  $A(\xi)$  be defined and continuous in a domain  $G \subset \mathbb{R}^n$ . This function may have singularities of arbitrary type outside of  $G$  or on its boundary. Define a pseudo-differential operator  $A(D)$  with a symbol  $A(\xi)$  on a function  $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$  by the formula

$$A(D)\varphi(x) = \frac{1}{(2\pi)^n} \int_G A(\xi)F\varphi(\xi)e^{ix\xi} d\xi \tag{9}$$

where  $F\varphi$  is the Fourier transform of  $\varphi$ .

We define the operator  $A(-D)$  acting in the space  $\Psi'_{-G,q}(\mathbb{R}^n)$  by the formula

$$\langle A(-D)f, \varphi \rangle = \langle f, A(D)\varphi \rangle, \quad f \in \Psi'_{-G,q}(\mathbb{R}^n), \quad \varphi \in \Psi_{G,p}(\mathbb{R}^n). \tag{10}$$

We also recall the following statement (see Lemma 3 in [16] and Lemma 3 in [34]), which is important for proof of solvability theorems for the Cauchy and multipoint problems.

**Theorem 2.1.** *An arbitrary pseudo-differential operator  $A(D)$  ( $A(-D)$ ) whose symbol  $A(\xi)$  ( $A(-\xi)$ ) is continuous in a domain  $G \subset \mathbb{R}^n$  ( $-G \subset \mathbb{R}^n$ ) acts from the space  $\Psi_{G,p}(\mathbb{R}^n)$  ( $\Psi'_{-G,q}(\mathbb{R}^n)$ ) into itself. Moreover, if  $A(\xi)g(\xi)$  is a multiplier in  $L_p$  for any  $g \in C_0^\infty(G)$ , then this operator acts continuously.*

### 3. Representation of the solution to the Cauchy problem for distributed order pseudo-differential equations

In this section we will construct the representation formula for a solution of the Cauchy problem for the distributed order pseudo-differential equation

$$\int_0^m A(r; D) D_*^r u(t, x) dr = B(D)u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n \tag{11}$$

$$\frac{\partial^k u(0, x)}{\partial t^k} = \varphi_k(x), \quad x \in \mathbb{R}^n, \quad k = 0, \dots, m - 1. \tag{12}$$

First we find the formal representation of a solution and then, relying on this representation, study its properties.

We split the problem (11), (12) into  $m$  Cauchy problems, one for each index  $j \in \{0, 1, \dots, m - 1\}$ , keeping (11) and using the the Cauchy conditions

$$u(0, x) = 0, \dots, \frac{\partial^{j-1} u(0, x)}{\partial t^{j-1}} = 0,$$

$$\frac{\partial^j u(0, x)}{\partial t^j} = \varphi_j(x), \tag{13}$$

$$\frac{\partial^{j+1} u(0, x)}{\partial t^{j+1}} = 0, \dots, \frac{\partial^{m-1} u(0, x)}{\partial t^{m-1}} = 0.$$

If we denote by  $u_j(t, x)$  a solution to (11), (13), then the general solution to (11), (12) due to its linearity, is represented in the form

$$u(t, x) = \sum_{j=0}^{m-1} u_j(t, x).$$

Applying formally the Fourier transform to (11),(13) we get the  $\xi$ -parametrized equation

$$\int_0^m A(r; \xi) D_*^r \hat{u}(t, \xi) dr = B(\xi)\hat{u}(t, \xi) \tag{14}$$

and the corresponding initial conditions

$$\hat{u}(0, \xi) = 0, \dots, \hat{u}^{j-1}(0, \xi) = 0,$$

$$\begin{aligned} \hat{u}^j(0, \xi) &= \hat{\varphi}_j(\xi), \\ \hat{u}^{j+1}(0, \xi) &= 0, \dots, \hat{u}^{m-1}(0, x) = 0. \end{aligned} \tag{15}$$

We rewrite the left hand side of (14) in the form

$$\int_0^m A(r; \xi) D_*^r \hat{u}(t, \xi) dr = \sum_{k=1}^m \int_{k-1}^k A(r, \xi) D_*^r \hat{u}(t, \xi) dr \tag{16}$$

and apply the Laplace transform to both sides. Taking  $k - 1 < r < k$  into account and making use of the well known formula (see [12] and [27])

$$L[D_*^r f(t)](s) = L[f](s) s^r - \sum_{l=0}^{k-1} f^{(l)}(0) s^{r-l-1},$$

where  $L$  stands for the Laplace transform (we will use on equal status the notations  $\tilde{f}(s)$  and  $L[f](s)$  for the Laplace transform of  $f$ ), we have

$$\begin{aligned} \int_{k-1}^k A(r, \xi) L[D_*^r \hat{u}(t, \xi)] dr &= \tilde{u}(s, \xi) \int_{k-1}^k s^r A(r, \xi) dr \\ &\quad - \sum_{l=0}^{k-1} \hat{u}^{(l)}(0, \xi) \frac{1}{s^{l+1}} \int_{k-1}^k s^r A(r, \xi) dr, \end{aligned}$$

where  $\tilde{u}(s, \xi)$  is the Laplace transform of  $\hat{u}(t, \xi)$ . Now summing up by  $k$  from 1 till  $m$  we obtain

$$\tilde{u}(s, \xi) \int_0^m A(r; \xi) s^r dr - \sum_{k=0}^{m-1} \frac{\hat{\varphi}_k(\xi)}{s^{k+1}} \int_k^m A(r; \xi) s^r dr = B(\xi) \tilde{u}(s, \xi). \tag{17}$$

Denote

$$\Phi_0(s, \xi) = 0; \quad \Phi_j(s, \xi) = \int_0^j s^r A(r, \xi) dr, \quad j = 1, \dots, m. \tag{18}$$

Then it follows from (17) and the hypothesis that only the  $j$ -th Cauchy datum is non-zero (see (15)) that

$$\tilde{u}_j(s, \xi) = \frac{\Phi_m(s, \xi) - \Phi_j(s, \xi)}{s^{1+j}[\Phi_m(s, \xi) - B(\xi)]} \hat{\varphi}_j(\xi), \quad j = 0, \dots, m - 1. \tag{19}$$

Let  $G \subseteq R^n$  be an open set and  $\xi \in G$  be fixed. Denote by  $s_0(\xi)$  the greatest positive root of the equation  $\Phi_m(s, \xi) = B(\xi)$ . If  $A(r, \xi)$  preserves its sign for every  $0 < r < m$ , then it follows from the inequality  $s_1^r < s_2^r$  for  $0 < s_1 < s_2$  and  $r > 0$  that  $\Phi_m(s, \xi), s > 0$ , is a monotone function. Hence, the equation



$\Phi_m(s, \xi) = B(\xi)$  can have no more than one positive root for every fixed  $\xi \in G$ . Thus

$$\Psi_j(s, \xi) = \frac{\Phi_m(s, \xi) - \Phi_j(s, \xi)}{s[\Phi_m(s, \xi) - B(\xi)]} \tag{20}$$

is well defined for  $s > s_0(\xi)$  if the equation  $\Phi_m(s, \xi) = B(\xi)$  has a positive root, or for  $s > 0$ , if there is no such a root. It is not difficult to verify that the collection of functions  $\Phi_0(s, \xi), \dots, \Phi_{m-1}(s, \xi)$  is linearly independent. Denote by  $S_j(t, \xi)$  the Laplace preimage of  $\Psi_j(s, \xi)$ , that is

$$S_j(t, \xi) = L^{-1}\left(\frac{\Phi_m(s, \xi) - \Phi_j(s, \xi)}{s[\Phi_m(s, \xi) - B(\xi)]}\right), \quad j = 0, \dots, m - 1. \tag{21}$$

Using the theorem on uniqueness of the inverse Laplace transform, we can conclude that the collection  $S_0(t, \xi), \dots, S_{m-1}(t, \xi)$  is linearly independent as well. Thus for  $u_j(t, x)$  we get the representation

$$u_j(t, x) = J^j S_j(t, D)\varphi_j(x),$$

where  $J^j$  is the  $j$ -th power of the common integration operator with lower limit 0,

$$J^j f(t) = \underbrace{\int_0^t \dots \int_0^\tau}_{j\text{-times}} f(\tau_1) d\tau_1 = \int_0^t \frac{(t - \tau)^{j-1}}{\Gamma(j)} f(\tau) d\tau \quad \text{for } j \geq 1, \quad J^0 = I,$$

Here  $I$  denotes the identity operator, and  $S_j(t, D)$  is the pseudo-differential operator with the symbol  $S_j(t, \xi)$ . For the solution of (11), (12) we have the representation

$$u(t, x) = \sum_{j=0}^{m-1} J^j S_j(t, D)\varphi_j(x). \tag{22}$$

**Two notes.** The obtained representation formula is useful both from mathematical and physical point of views.

**1M)** The representation is obtained as the action of the pseudo-differential operators  $J^j S_j(t, \xi), j = 0, \dots, m - 1$ , called *j-th solution operator*, to the given functions. To get solution properties we can use the known properties of these operators. Moreover, all these pseudo-differential operators have a similar structure.

**2Ph)** The formula says that it does not matter how many different fractional orders are taken in the given equation between two consecutive integers. For the initial value problem it is important to have the initial values of the unknown solution just for integer order derivatives. Moreover, from this formula it can be derived that if the maximal order of the derivatives in the equation is

not greater than  $m - 1$ , then the Cauchy problem with  $m$  given data becomes incorrectly posed. Indeed if  $A(r, \xi) = \delta(r - \alpha), \alpha \leq m - 1$ , rewriting (20) for  $j = m - 1$  in the form

$$\Psi_{m-1}(s, \xi) = \frac{\int_{m-1}^m s^r A(r, \xi) dr}{s[\Phi_m(s, \xi) - B(\xi)]},$$

we get  $S_{m-1}(t, \xi) \equiv 0$ .

#### 4. Some auxiliary lemmas

Now we give the definition of solution to the Cauchy problem (11), (12).

**Definition 4.1.** A function  $u(t, x)$  is called a *strong solution* to the Cauchy problem (11), (12) if

1.  $u(t, x) \in C^{(m-1)}(t \geq 0; \Psi_{G,p}(\mathbb{R}^n))$
2.  $u^{(m)}(t, x) \in \Psi_{G,p}(\mathbb{R}^n)$  exists for almost all  $t > 0$  and
3. it satisfies the equation (11) for almost all  $t \in (0, \infty)$  and for all  $x \in \mathbb{R}^n$  including initial conditions (12) pointwise.

**Definition 4.2.** A function  $u(t, x)$  is called a *weak solution* to the Cauchy problem (11), (12) (replacing  $D$  by  $-D$ ) if

1.  $u(t, x) \in C^{(m-1)}(t \geq 0; \Psi'_{-G,q}(\mathbb{R}^n))$
2.  $u^{(m)}(t, x) \in \Psi'_{-G,q}(\mathbb{R}^n)$  for almost all  $t > 0$  and
3. the following equalities hold true for arbitrary  $v \in \Psi_{G,p}(\mathbb{R}^n)$ :

$$\int_0^m \langle D_*^r u(t, x), A(r, D)v(x) \rangle dr = \langle u(t, x), A(D)v(x) \rangle \quad \text{for a.a. } t \in (0, \infty)$$

$$\lim_{t \rightarrow +0} \langle u^{(k)}(t, x), v(x) \rangle = \langle \varphi_k(x), v(x) \rangle, \quad k = 0, \dots, m - 1.$$

To prove solvability theorems we need some auxiliary lemmas, using the abbreviation

$$F_a(k; f) := \int_0^a e^{kt} f(t) dt, \quad k \in \mathbb{R}^1,$$

where  $a$  is a fixed positive real number and  $f$  is a generalized function with  $\text{supp } f \subset [0, a]$ .

**Lemma 4.3.**

1. For a regular distribution  $f(t)$  with  $\text{supp } f \subset [0, d]$ ,

$$|F_a(k; f)| = o(e^{dk}), \quad k \rightarrow \infty.$$

2. For a singular generalized function  $f(t)$  with  $\text{supp } f = \{d\}$ ,

$$|F_a(k; f)| = O(e^{(d+\epsilon)k}), \quad k \rightarrow \infty,$$

where  $\epsilon > 0$  is arbitrarily small.

**Proof.** Let first  $f \in L_\infty(0, a)$ ,  $|f(t)| \leq M < \infty$  and  $\text{supp } f \subset [0, d]$ . Then

$$|F_a(k; f)| = \left| \int_0^d e^{kt} f(t) dt \right| \leq M \int_0^d e^{kt} dt = \frac{M}{k} (e^{kd} - 1).$$

For large  $k$  we have  $|F_a(k; f)| = o(e^{dk})$ . For a regular generalized function  $f$  with  $\text{supp } f \subset [0, d]$  there is a sequence of functions  $f_m \in L_\infty(0, a)$ , all supported in  $[0, d]$  and with common estimation constant  $M$  such that  $f_m \rightarrow f, m \rightarrow \infty$  in the weak sense. For  $f_m$  we have  $k|F_a(k; f_m)| \leq C e^{kd}$  with positive constant  $C$ . Letting  $m \rightarrow \infty$  we obtain the desired result.

If  $f$  is a singular generalized function with support  $\text{supp } f = \{\alpha\}, 0 < \alpha < a$ , then  $f$  is a finite linear combination of  $\delta^{(j)}(t - \alpha)$ , where  $\delta$  is the Dirac function. It is readily seen that  $F_a(k; f) = O(e^{(\alpha+\epsilon)k})$  for any  $\epsilon > 0$  as  $k \rightarrow \infty$  in this case. ■

**Lemma 4.4.** Let  $\Phi_j(s, \xi), j = 0, \dots, m$ , be as in (18).

1. If  $A(r, \xi)$  is a regular distribution with respect to  $r$  with  $\text{supp } A(r, \xi) \subset [0, d]$ , then  $\Phi_j(s, \xi) = o(s^\nu), s \rightarrow \infty$ , where  $\nu = \min\{d, j\}$ .
2. If  $A(r, \xi)$  is a singular distribution with respect to  $r$  with  $\text{supp } A(r, \xi) = \{d\}$ , then  $\Phi_j(s, \xi) = O(s^{d+\epsilon}), (\epsilon \text{ is arbitrarily small}) s \rightarrow \infty$ , in the case  $d \leq j$  and  $\Phi_j(s, \xi) = 0$  when  $d > j$ .

**Remark.** Hereafter  $\text{supp } A(r, \xi)$  means the support of  $A(r, \xi)$  with respect to the variable  $r$ .

**Proof.** We have

$$\Phi_j(s, \xi) = \int_0^j s^r A(r, \xi) dr = \int_0^j e^{r \ln s} A(r, \xi) dr =: F_j(\ln s; A(r, \xi)), \quad s > s_0(\xi).$$

Now it is an easy exercise to apply Lemma 4.1 and get the asymptotics in the two cases. ■

**Corollary.** For every  $j = 0, \dots, m - 1$ ,  $\Psi_j(s, \xi) = O(\frac{1}{s}), s \rightarrow \infty$ , where  $\Psi_j(s, \xi)$  is given by (20).

**Lemma 4.5.** For every  $j = 0, \dots, m - 1$ , and  $k = 0, 1, \dots$ ,

$$\Phi_j^{(k)}(s, \xi) = \frac{1}{s^k} \int_0^j r^k s^r A(r, \xi) dr = \frac{1}{s^k} F_j(\ln s; r^k A(r, \xi)), \quad s > s_0(\xi),$$

where  $\xi$  is fixed and derivatives are taken with respect to the variable  $s$ .

**Remark.** Note that  $r$  is a dummy variable in the expressions  $F_j(\ln s; r^k A(r, \xi))$ ,  $k = 0, 1, \dots$

**Corollary.** For every  $j = 0, \dots, m - 1$ ,  $\Psi_j^{(k)}(s, \xi) = O(\frac{1}{s^{k+1}})$ ,  $s \rightarrow \infty$ ,  $k \in \mathbf{N}$ .

**Lemma 4.6.** For  $S_j(t, \xi) = L_{s \rightarrow t}^{-1} \Psi_j(s, \xi)$ ,  $j = 0, \dots, m - 1$ , with any fixed  $\xi \in G$  the following assertions are true:

1.  $S_j(t, \xi) \rightarrow 1$  for  $t \rightarrow +0$ ,  $j = 0, \dots, m - 1$ ;
2.  $\frac{\partial^\ell S_j(t, \xi)}{\partial t^\ell} \rightarrow 0$  for  $t \rightarrow +0$ ,  $j = 0, \dots, m - 1$ ,  $\ell = 1, \dots, m - j - 1$ .

**Proof.** Let  $\xi \in G$  be fixed. It follows from the representation (20) for  $\Psi_j(s, \xi)$  and Lemma 4.2 that  $\Psi_j(s, \xi) = O(\frac{1}{s})$  for large  $s$ , which allows to invert  $\Psi_j(s, \xi)$  (see [37]) with respect to  $s$ . Thus  $S_j(t, \xi)$  exists for all  $j = 0, \dots, m - 1$ . Further we use the well known relation [9], [37]

$$\lim_{s \rightarrow \infty} s \tilde{f}(s) = \lim_{t \rightarrow +0} f(t). \tag{23}$$

Lemma 4.2 implies  $s\Psi_j(s, \xi) \rightarrow 1$  as  $s \rightarrow \infty$ . Hence  $S_j(t, \xi) \rightarrow 1$  when  $t \rightarrow +0$ . We have proved (i). The Laplace transform of  $\frac{\partial S_j(t, \xi)}{\partial t}$  is

$$s\Psi_j(s, \xi) - S_j(+0, \xi) = \frac{\Phi_m(s, \xi) - \Phi_j(s, \xi)}{\Phi_m(s, \xi) - B(\xi)} - 1 = \frac{-\Phi_j(s, \xi) + B(\xi)}{\Phi_m(s, \xi) - B(\xi)}, \quad s > s_0(\xi),$$

which is  $O(\frac{1}{s^{d-j}})$  for large  $s$  with  $d = \max \text{supp } A(r, \xi)$ . Note that  $d > m - 1$ . Otherwise the Cauchy problem is meaningless (see Note 2Ph)). Analogously for the Laplace transform of  $\frac{\partial^\ell S_j(t, \xi)}{\partial t^\ell}$  by induction we get

$$L \left[ \frac{\partial^\ell S_j(t, \xi)}{\partial t^\ell} \right] = s^{\ell-1} (s\Psi_j(s, \xi) - S_j(+0, \xi)) = O \left( \frac{1}{s^{d-j-\ell+1}} \right). \tag{24}$$

Consequently, we have  $sL \left[ \frac{\partial^\ell S_j(t, \xi)}{\partial t^\ell} \right] = O(\frac{1}{s^{d-j-\ell}})$ . Using this and (23) we arrive at (ii). ■

**Lemma 4.7.** For every  $j = 0, \dots, m - 1$  the following assertions hold:

1.  $S_j(t, \xi) \in C^{m-j-1}(t \geq 0; C(G))$ .
2. If the upper bound of  $\text{supp } A(r, \xi) = m$ , then  $S_j^{(m-j)}(t, \xi)$  exists for almost all  $t \in (0, \infty)$ .

**Proof.** In proving the previous lemma we noticed that the  $S_j(t, \xi)$ ,  $j = 0, \dots, m - 1$ , exist. Now we will check their differentiability properties. It is known [38] that if for given  $f(t)$  its Laplace transform  $\tilde{f}(s)$  satisfies  $s\tilde{f}(s) \rightarrow 0$  when  $s \rightarrow \infty$ , then  $f$  is continuous. For fixed  $\xi \in G$  it follows from (24) that

$$sL \left[ \frac{\partial^\ell S_j(t, \xi)}{\partial t^\ell} \right] = O \left( \frac{1}{s^{d-l-j}} \right).$$

Take  $l = m - j - 1$ , which means that  $sL[\frac{\partial^\ell S_j(t,\xi)}{\partial t^\ell}]$  vanishes as  $|s| \rightarrow \infty$ . Hence  $\frac{\partial^\ell S_j(t,\xi)}{\partial t^\ell}$  is continuous. Now assume that  $d = m$ . Then we have  $L[\frac{\partial^{m-j} S_j(t,\xi)}{\partial t^{m-j}}] = O(\frac{1}{s})$ . Thus  $\frac{\partial^{m-j} S_j(t,\xi)}{\partial t^{m-j}}$  exists for almost every  $t \in (0, \infty)$ . ■

**Remark.** If  $m - 1 < d < m$  one can show that  $D_*^{d-j} S_j(t, \xi)$ ,  $j = 0, \dots, m - 1$ , exists and is bounded for almost every  $t$  (compare with Lemma 7 in [16]).

### 5. Solution of the Cauchy problem for distributed order pseudo-differential equations

In Section 3 the formal representations for a solution of the Cauchy problem (11), (12) have been obtained. In this Section we establish that they represent the strong and weak solutions of the Cauchy problem in the spaces  $\Psi_{G,p}(\mathbb{R}^n)$  and  $\Psi'_{-G,q}(\mathbb{R}^n)$ , respectively.

**Theorem 5.1.** *Let  $G$  be a domain of continuity of the symbols  $A(r, \xi)$  ( $r$  fixed) and  $B(\xi)$ . Let  $\varphi_j \in \Psi_{G,p}(\mathbb{R}^n)$ ,  $j = 0, \dots, m - 1$ . Then the Cauchy problem (11), (12) has a unique strong solution. This solution is given by the representation (22).*

**Proof.** Let  $G$  be a domain of continuity of the symbols  $A(r, \xi)$  and  $B(\xi)$  and  $\varphi_j \in \Psi_{G,p}(\mathbb{R}^n)$ ,  $j = 0, \dots, m - 1$ . By construction of the representation (22) each of its terms satisfies the equation (11) and, by virtue of Lemma 4.3, the conditions (12). It follows from Lemma 4.4 that for every  $j = 0, \dots, m - 1$  the inclusion  $J^j B_j(t, \xi) \in C^{m-1}(t \geq 0; C(G))$  holds. Moreover,  $D_*^d J^j B_j(t, \xi)$  is bounded for almost every  $t \in (0, \infty)$ . ■

**Theorem 5.2.** *Let  $\varphi_j \in \Psi'_{-G,q}(\mathbb{R}^n)$ ,  $j = 0, \dots, m - 1$ . Then the Cauchy problem (11), (12) has a unique weak solution given by*

$$u(t, x) = \sum_{j=0}^{m-1} J^j S_j(t, -D)\varphi_j(x), \tag{25}$$

where  $S_j(t, -D)$ ,  $j = 1, \dots, m$ , is the  $j$ -th solution operator with symbol  $B_j(t, \xi)$ .

**Proof.** Let  $\varphi_j \in \Psi'_{-G,q}(\mathbb{R}^n)$ ,  $j = 0, \dots, m - 1$ . It follows from Theorem 2.1 that every term in the right-hand side of (25), namely,  $u_j(t, x) = J^j S_j(t, -D)\varphi_{k-1}(x)$ ,  $j = 0, \dots, m - 1$ , is a functional from the space  $\Psi'_{-G,q}(\mathbb{R}^n)$ . To prove the theorem we have to show that  $u_j(t, x)$ ,  $j = 0, \dots, m - 1$ , satisfies the equation (11) and the initial conditions (12) in the weak sense. Let  $v \in \Psi_{G,p}(\mathbb{R}^n)$  be an

arbitrary function. Then for  $u_j(t, x) = J^j S_j(t, -D)\varphi_j(x)$  we have

$$\begin{aligned} & \int_0^m \langle D_*^r u_j(t, x), A(r, D)v(x) \rangle dr - \langle u_j(t, x), B(D)v(x) \rangle \\ &= \left\langle \int_0^m D_*^r J^j S_j(t, -D)\varphi_j(x), A(r, D)v(x) \right\rangle dr - \langle J^j S_j(t, -D)\varphi_j(x), B(D)v(x) \rangle \\ &= \left\langle \varphi_j(x), \left[ \int_0^m D_*^r A(r, D)J^j B_j(t, -D) dr - B(D)J^j S_j(t, -D) \right] v(x) \right\rangle. \end{aligned}$$

Applying the Fourier transform and remembering the definition of  $S_j(t, D)$  we conclude that the expression in the square brackets in the last relation is zero for almost every  $t \in (0, \infty)$ . Besides, Lemma 4.3 yields

$$\lim_{t \rightarrow +0} \langle u_j^{(k)}(t, x), v(x) \rangle = \langle \delta_{j,k} \varphi_k(x), v(x) \rangle, \quad j, k = 0, \dots, m - 1.$$

Hence,  $u(x, t)$  satisfies the equation (11) in the weak sense. ■

### 6. Solution of the multi-point value problem for distributed order differential equations

In this section we deal with the general multi-point value problem (5) for the distributed order pseudo-differential equation (3). As we have seen in Section 2 each of the functions  $S_j(t, \xi), j = 0, \dots, m - 1$ , satisfies the equation

$$\int_0^m A(r; \xi) D_*^r S_j(t, \xi) dr = B(\xi) S_j(t, \xi),$$

and the initial conditions

$$\begin{aligned} S_j(0, \xi) &= 0, \dots, S_j^{j-1}(0, \xi) = 0, \\ S_j^j(0, \xi) &= 1, \\ S_j^{j+1}(0, \xi) &= 0, \dots, S_j^{m-1}(0, x) = 0. \end{aligned}$$

We look for a solution of the multi-point problem in the form

$$U(t, \xi) = \sum_{j=0}^{m-1} c_j S_j(t, \xi), \tag{26}$$

with unknown coefficients  $c_j = c_j(\xi), j = 0, \dots, m - 1$ . Substituting (26) into

$$\sum_{j=0}^{m-1} \Gamma_{kj}(\xi) \frac{\partial^k U(t_{kj}, \xi)}{\partial t^k} = \hat{\varphi}_k(\xi), \quad \xi \in G, \tag{27}$$

which follows from (5) and applying the Fourier transform with respect to  $x$  leads to the system of linear algebraic equations

$$M(\xi)C(\xi) = \hat{\Phi}(\xi). \tag{28}$$

Here  $M(\xi)$  is a square matrix of order  $m$  with the entries

$$m_{kl} = \sum_{j=0}^{m-1} \Gamma_{kj}(\xi)S_l^{(j)}(t_{kj}, \xi), \quad k, l = 0, \dots, m - 1,$$

$C(\xi) = (c_0(\xi), \dots, c_{m-1}(\xi))$  is a vector of unknown coefficients and  $\hat{\Phi}(\xi) = (\hat{\varphi}_0(\xi), \dots, \hat{\varphi}_{m-1}(\xi))$  is the initial data vector. Denote by  $M_0$  the set of all points  $\xi \in G$  such that  $\text{Det}M(\xi) = 0$ . For  $\xi \notin M_0$  the equation (28) has a unique solution

$$C(\xi) = M^{-1}(\xi)\hat{\Phi}(\xi). \tag{29}$$

We note that  $M_0$  is the singular set for the symbols of the solution operators. Inserting the representation (29) of the vector  $C(\xi) = (c_0(\xi), \dots, c_{m-1}(\xi))$  into (26) and applying the inverse Fourier transform we get the solution of the general multi-point value problem (3), (5) as

$$u(t, x) = \sum_{k=0}^{m-1} U_k(t, D)\varphi_k(x), \tag{30}$$

where the  $U_k(t, D)$ ,  $k = 0, \dots, m - 1$ , are solution pseudo-differential operators with the symbols  $U_k(t, \xi) = (M^*)^{-1}(\xi)S(t, \xi)$ , the matrix  $(M^*)^{-1}(\xi)$  is inverse to the Hermitian conjugate of  $M(\xi)$ , and  $S(t, \xi)$  is the transpose of the row vector  $(S_0(t, \xi), \dots, S_{m-1}(t, \xi))$  with the components given by (21).

**Theorem 6.1.** *Let  $\varphi_k \in \Psi_{G',p}(\mathbb{R}^n)$ ,  $k = 0, \dots, m - 1$ ,  $G' = G \setminus M_0$ . Then the multi-point value problem (3), (5) has a unique strong solution in the space  $C^{(m)}((T_1, T_2); \Psi_{G',p}(\mathbb{R}^n))$ . This solution is given by the formula (30).*

The analogous theorem is valid for the dual problem too. With the dual problem we mean the problem obtained by replacing the operators  $A(r, D)$ ,  $B(D)$  and  $\Gamma_{kj}(D)$ ,  $j, k = 0, \dots, m - 1$ , by their dual operators (see formula (10)).

**Theorem 6.2.** *Let  $\varphi_k \in \Psi'_{-G',q}(\mathbb{R}^n)$ ,  $k = 0, \dots, m - 1$ ,  $G' = G \setminus M_0$ . Then the dual multi-point value problem has a unique weak solution in the space  $C^{(m)}((T_1, T_2); \Psi'_{-G',q}(\mathbb{R}^n))$ . This solution is given by the formula*

$$u(t, x) = \sum_{k=0}^{m-1} U_k(t, -D)\varphi_k(x).$$

## 7. Concluding remarks

Complementing previous works of Umarov and of Umarov et al. (see [16], [33] – [36]) we have elaborated how the Cauchy problem for a general linear evolution equation with (fractional) temporal derivatives of distributed orders can be successfully analyzed by the powerful theory of pseudo-differential operators that has been conceived by Yu. A. Dubinskij in 1982. By applying this theory, using Fourier transform in space, Laplace transform in time, we have solved the questions of strong solutions (in classical function spaces) and weak solutions (in corresponding dual spaces), not only for the Cauchy problem, but also for a rather general multi-point problem. Having in mind applications mainly in physics, where in the case of the Cauchy problem initial conditions are given in form of functions and integer order derivatives we have throughout adhered to the Caputo version of fractional derivatives. Our treatment is constructive, so not only answers the questions for existence and uniqueness but in addition renders transparent integral representations for the solution. For mathematical completeness it would be desirable to treat the analogous problems where instead of the fractional derivatives of Caputo type those of Riemann-Liouville type are used. This, of course, can easily be done in case of all initial conditions assumed to be zero (in this case the two types of fractional derivatives coincide) but seems rather tricky otherwise. We leave this problem to another paper or to other authors.

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