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Regularity of Flows of a Non-Newtonian Fluid subject to Dirichlet Boundary Conditions

P. Kaplický

Abstract. We study a planar flow of a generalized Newtonian fluid under the Dirichlet boundary condition. The fluid is characterized by a nonlinear dependence of the stress tensor on the symmetric part of the velocity gradient. We prove that the unique weak solution of this problem has a Hölder continuous gradient provided the growth of the stress tensor is of order $p-1$ for a certain $p \in \langle 2, 4)$. The result is global in time and in space.

Keywords: *Generalized Newtonian fluid, initial boundary value problem, regularity, $C^{1,\alpha}$ solutions*

MSC 2000: 35B65, 35K50

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $I := (0,T)$ for some $T > 0$, $Q := I \times \Omega$. We investigate the existence of a regular solution $u : Q \to \mathbb{R}^2$, $\pi : Q \to \mathbb{R}$ of the following two dimensional initial value problem:

$$\begin{gathered} \partial_t u + u_i \frac{\partial u}{\partial x_i} - \operatorname{div}(\mathcal{T}(Du)) + \nabla \pi = f, \quad \operatorname{div} u = 0 \text{ in } Q, \\ \int_\Omega \pi(t) = 0 \text{ for a.e. } t \in I, \quad u = u_0 \text{ in } \{0\} \times \Omega \end{gathered} \tag{1}$$

under the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } I \times \partial\Omega. \tag{2}$$

P. Kaplický: Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic; kaplicky@karlin.mff.cuni.cz
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Here, and also in the whole article, we use the standard summation convention, i.e., $u_i \partial u/\partial x_i = \sum_{i=1}^{2} u_i \partial u/\partial x_i$. The symmetric gradient of a function u is denoted Du. Clearly, $Du \in \mathbb{S}$, the set of symmetric 2×2 matrices. We assume that $\mathcal{T}_{ij}(D) = \partial_{ij} F(|D|^2) := \partial F(|D|^2)/\partial D_{ij}$ for all $D \in \mathbb{S}$ and a given potential $F : \langle 0, +\infty) \to \langle 0, +\infty)$. We assume that F is a C^2 function such that $F(0) = 0$ and there exist $p \geq 1$ and $C_2 \geq C_1 > 0$ such that for all $D, E \in \mathbb{S}$

$$C_1(1 + |D|^2)^{\frac{p-2}{2}} |E|^2 \leq \partial_{ij} \mathcal{T}_{kl}(D) E_{ij} E_{kl} \leq C_2(1 + |D|^2)^{\frac{p-2}{2}} |E|^2. \qquad (3)$$

If we interpret u as the velocity field, π as the pressure and f as a net applied force, then the first equation in (1) expresses the balance of linear momentum for an incompressible fluid. Incompressibility of the fluid is captured by the second equation in (1). Specific material properties of the fluid are described by the stress tensor $\mathcal{T}$. We have, mainly, in mind the fluids with shear-dependent viscosities with $\mathcal{T}(D) = 2\mu(|D|^2)D$ for all $D \in \mathbb{S}$ and a given generalized viscosity $\mu : \langle 0, +\infty) \to \langle 0, +\infty)$. Note that then $F(|D|^2) := \int_0^{|D|^2} \mu(s)\mathrm{d}s$ is the potential to $\mathcal{T}$. Typical example of $\mathcal{T}$ satisfying (3) is $\mathcal{T}(D) = (1+|D|^2)^{(p-2)/2} D$.

The existence and uniqueness of a global solution to the problem (1), (2) with (3) were proved in [15] provided $p = 2$.

To our knowledge, the first successful attempt to obtain global in time $C^{1,\alpha}$ regularity of the solution of (1), (2) is due to Seregin [20]. He proved the regularity of the solution in the interior of Ω assuming the boundedness of the third derivatives of the potential F. His method is based on the fact that in this case the third derivatives of u are in $L^2_{\mathrm{loc}}(Q)$. Under the same condition on F it is proved in [16] that the gradient of u is even Lipschitz continuous.

A different approach was used in [19] to show that every weak solution $u : Q \to \mathbb{R}^2$ of the problem

$$\partial_t u - \operatorname{div}(a(\nabla u)) = 0 \quad \text{in } Q \qquad (4)$$

has a locally Hölder continuous gradient, provided $a : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ satisfies similar conditions as $\mathcal{T}$ in (3) with $p = 2$. They first show regularity of $\partial_t u$ and then at every time level they use the stationary L^q theory to (4) with $\partial_t u$ on the right hand side. Their technique was later used in [4] to obtain the regularity result also for a having the growth exponent $p \in (1, 2)$.

Only recently this method was modified in [13] and applied to (1) with periodic boundary conditions. It is shown there that if (3) holds with $p \in (4/3, 2\rangle$, $u_0 = 0$ and f is smooth enough, then there exists a solution u of (1) with the periodic boundary conditions that has Hölder continuous gradient, and this solution is unique within the weak solutions satisfying the energy inequality. The lower bound for p is due to the fact that the differences between (1) and (4) do not allow to show the regularity of $\partial_t u$ first, but the regularity of $\partial_t u$ and ∇u

have to be inferred at once. The results from [13] were extended to electrorheological fluids and nonzero initial value in [2].

In this article we transfer the method from [13] to problem (1) with homogeneous Dirichlet boundary conditions (2). Our main result is the following theorem.

Theorem 1.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with* $C^{2+\mu}$ *boundary,* $\mu \in (0,1)$*. Let* (3) *hold for some* $p \in \langle 2,4)$ *and*

$$f \in L^\infty(I, L^2(\Omega)), \quad f(0) \in L^2(\Omega), \quad \partial_t f \in L^2(I, (W^{1,2}_{0,\mathrm{div}}(\Omega))^*) \tag{5}$$

$$u_0 \in W^{r,2}(\Omega) \cap W^{1,2}_{0,\mathrm{div}}(\Omega), \quad r > 2 \ (r = 2 \text{ if } p = 2). \tag{6}$$

Then the unique weak solution u, π *of* (1), (2) *with* (3) *satisfies*

$$\begin{aligned} u, \nabla u, \pi \in L^\infty(I, W^{1,s}(\Omega)) \quad &\text{for all } s \in (1,2) \ (s = 2 \text{ if } p = 2), \\ \partial_t u \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}_{0,\mathrm{div}}(\Omega)). & \end{aligned} \tag{7}$$

Moreover, if there is a $\tilde{q} > 2$ *such that*

$$f \in L^\infty(I, L^{\tilde{q}}(\Omega)), \quad \partial_t f \in L^{\tilde{q}}(I, W^{-1,\tilde{q}}(\Omega)), \tag{8}$$

then there exist $q > 2$ *and* $\alpha > 0$ *such that for all* $\epsilon \in (0,T)$, $s \in \big(0, 1/2\big)$

$$\begin{aligned} \nabla u, \pi \in L^\infty((\epsilon,T), W^{1,q}(\Omega)), \qquad & \nabla u, \pi \in C^{0,\alpha}(\langle \epsilon, T\rangle \times \overline{\Omega}) \\ \partial_t u \in L^q((\epsilon,T), W^{1,q}(\Omega)), \qquad & \pi \in W^{s,q}((\epsilon,T), L^q(\Omega)). \end{aligned} \tag{9}$$

To prove this theorem we construct the solution u as a limit of smooth solutions u^A to approximating problems. The decisive step is to show a priori estimates for u^A in $L^\infty(I, W^{2,q}(\Omega))$ for certain $q > 2$ independent of A. They rely on careful stationary and evolutionary estimates for systems of the Stokes type with bounded measurable coefficients. Evolutionary estimates are used to get a bound of the norm of $\partial_t u^A$ in $L^\infty(I, L^q(\Omega))$ for certain $q > 2$. For the nonlinear system (1) this bound depends on the L^∞ norm of ∇u^A. Then we use the stationary estimates to the approximating equation with the term $\partial_t u^A$ on the right hand side in order to estimate ∇u^A in $L^\infty(I, W^{1,q}(\Omega))$ for certain $q > 2$ (cf. [10, 11, 12]).

In spite of the fact that we closely follow [13] there are some important differences caused by the presence of the boundary. The first difference is hidden in the L^q theory for the evolutionary Stokes system. In the case of the periodic boundary conditions the regularity of the pressure can be obtained first and the L^q theory is then reduced to the L^q theory for the heat equation. This is not possible in the case of the Dirichlet boundary conditions and more complicated methods must be used (cf. [5, 14]). The second difference comes from the fact that we do not know how to get the regularity statement in (7) up to the boundary $\partial\Omega$ if $p < 2$. It is possible to get similar interior estimates, but then we would also need the estimates of $\partial_t \pi$ and $\partial_t^2 u$ in the proof of (9) and we do not know how to solve this problem (see [9] for details). This is why we restrict ourselves to the case $p \geq 2$ in this paper. It should be mentioned that while in the case of the periodic boundary conditions it is easy to get (7) if $p > 2$, this step is nontrivial in the case of the Dirichlet boundary conditions, giving also the upper bound for p.

This article is divided into four sections. In Section 2 several systems of the Stokes type with bounded coefficients are studied. The proof of Theorem 1.1 is delivered in Section 3 provided $p = 2$ and the last section is devoted to the case $p > 2$.

Now we add some remarks on weak solutions of (1), (2) with (3). Let $f \in L^{p'}(I, (W^{1,p}(\Omega))^*)$, $u_0 \in L^2(\Omega)$. We say that $u : Q \to \mathbb{R}^2$ is *a weak solution* of (1), (2) with (3) if $u \in L^\infty(I, L^2(\Omega)) \cap L^p(I, W^{1,p}_{0,\mathrm{div}}(\Omega))$, $\partial_t u \in L^{p'}(I, (W^{1,p}_{0,\mathrm{div}}(\Omega))^*)$, it satisfies $u(0) = u_0$ in $L^2(\Omega)$ and it holds for all $\varphi \in \mathcal{D}((-\infty, T), \mathcal{N}(\Omega))$ that

$$\int_I \langle \partial_t u, \varphi \rangle + \int_I \int_\Omega \mathcal{T}(Du) D\varphi - (u \otimes u) D\varphi = \int_I \langle f, \varphi \rangle. \tag{10}$$

It is well known that then $u \in C(\overline{I}, L^2(\Omega))$ and the initial value problem is well posed. Note, that in the case $p \geq 2$ we can take even $\varphi \in L^p(I, W^{1,p}_{0,\mathrm{div}}(\Omega))$ as a test function in (10). Due to this fact the uniqueness of the weak solution is obtained via standard arguments for such p.

Moreover, if we know $\partial_t u \in L^{p'}(I, (W^{1,p}_0(\Omega))^*)$ then the pressure π can be reconstructed at almost every time level $t > 0$ (cf. [22, IV,1.4]), it is uniquely determined by the zero mean value condition $\int_\Omega \pi(t) = 0$ for a.e. $t \in I$ and it satisfies $\pi \in L^{p'}(I, L^{p'}(\Omega))$.

Now, we bring together notation used in this introduction and in the whole article. The word domain indicates an open and connected set Ω. We use the standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{n,p}(\Omega)$, $1 \leq p \leq +\infty$, $n \in \mathbb{N}$ equipped with the norms $\|\ldots\|_{p,\Omega}$, $\|\ldots\|_{n,p,\Omega}$, respectively. Similarly, $W^{\alpha,p}(\Omega)$ for $\alpha \in (0,1)$ denote Sobolev-Slobodeckii spaces. Next, $L^p_{0,\mathrm{div}}(\Omega)$, resp. $W^{n,p}_{0,\mathrm{div}}(\Omega)$, stand for the closures of the space $\mathcal{N}(\Omega) = \{\varphi \in \mathcal{D}(\Omega) : \operatorname{div} \varphi = 0\}$ in the norm of the space $L^p(\Omega)$, resp. $W^{n,p}(\Omega)$, and $C^{0,\alpha}(\overline{\Omega})$ represents the spaces of Hölder continuous functions.

The dual space to a Banach space X will be denoted by X^*. The value of $a(b) \in \mathbb{R}$, where $a \in X^*$ and $b \in X$ is $\langle a, b \rangle_{X^*,X}$. If there is no confusion likely, we write only $\langle a, b \rangle$. We also abbreviate $(W^{1,p'}_0(\Omega))^* = W^{-1,p}(\Omega)$. For the definition of the Bochner spaces $L^p(I, X)$ and $W^{\alpha,p}(I, X)$, $\alpha \in (0,1)$, $p \in (1, +\infty)$, and the corresponding norms see [21].

The constant $K > 0$ may vary from line to line, but it is always independent of all solutions and approximation parameters. Let us recall that in the whole paper the summation convention is used, the symmetric gradient of a function $u : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted Du.

2. Results for generalized Stokes system

The L^q theory for linear parabolic and elliptic systems of the Stokes type plays a significant role in the proof of Theorem 1.1. For systems with the periodic boundary conditions and the zero initial condition it was proved in [13, Section 2]. These results were later extended also for the nonzero initial condition in [2, Section 4.2]. The aim of this section is to provide a detailed summary of the L^q theory for the evolutionary problems with the Dirichlet boundary condition by the method developed in [13, Section 2].

Although we need the L^q theory only in $\mathbb{R}^2$ we state it in $\mathbb{R}^n$, $n \in \mathbb{N}$, accordingly $\Omega \subset \mathbb{R}^n$, $Q := I \times \Omega \subset \mathbb{R}^{n+1}$ in this section. Let $\mathbb{A} = (A_{ij}^{kl})_{i,j,k,l=1}^n : Q \to \mathbb{R}^{n^4}$, $\gamma_2 \geq \gamma_1 > 0$ such that for all $E \in \mathbb{S}$, $x \in \Omega$, $t \in I$

$$\begin{gathered}\gamma_1 |E|^2 \leq A_{ij}^{kl}(t,x) E_{ij} E_{kl} \leq \gamma_2 |E|^2 \\ A_{ij}^{kl} \in L^\infty(Q), \quad A_{ij}^{kl}(t,x) = A_{kl}^{ij}(t,x).\end{gathered} \tag{11}$$

For a given $g : Q \to \mathbb{R}^n$ and $v_0 : \Omega \to \mathbb{R}^n$ we study a weak solution $v : Q \to \mathbb{R}^n$ of the problem

$$\begin{gathered}\partial_t v - \operatorname{div}(\mathbb{A} Dv) + \nabla \sigma = g, \quad \operatorname{div} v = 0 \quad \text{in } Q \\ v(0) = v_0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } I \times \partial\Omega.\end{gathered} \tag{12}$$

The existence and uniqueness of a weak solution of this problem is well known. It is shown for example in [23, Theorem III.1.1] provided $A_{ij}^{kl} = \delta_{ik}\delta_{jl}$ on Q for all $i, j, k, l = 1, \ldots, n$ (δ_{ik} denotes the Kronecker symbol). We state this in the following lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. If $g = \operatorname{div} G$ for some symmetric matrix $G \in L^2(Q)$ and $v_0 \in L^2_{0,\mathrm{div}}(\Omega)$, then there exists a uniquely defined weak solution $v \in L^2(I, W^{1,2}_{0,\mathrm{div}}(\Omega))$, $\partial_t v \in L^2(I, (W^{1,2}_{0,\mathrm{div}}(\Omega))^*)$ of (12) with $A_{ij}^{kl} = \delta_{ik}\delta_{jl}$ on Q for all $i, j, k, l = 1, \ldots, n$ such that* $\max(\|Dv\|_{2,Q}, \|v\|_{L^\infty(I,L^2(\Omega))}) \leq \|G\|_{2,Q} + \|v_0\|_{2,\Omega}$ *and*

$$\int_0^T \langle \partial_t v, \varphi \rangle + \int_I \int_\Omega D_{ij} v D_{ij} \varphi = -\int_0^T \int_\Omega G_{ij} D_{ij} \varphi \tag{13}$$

holds for all $\varphi \in L^2(I, W^{1,2}_{0,\mathrm{div}}(\Omega))$.

Note, that once the existence of a weak solution v is known, we get the bound in Lemma 2.1 by testing (13) by this solution v.

In the next lemma we collect the facts about the L^q theory for the Stokes problem.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2+\mu}$ boundary, $0 < \mu < 1$. Let $A_{ij}^{kl} \equiv \delta_{ik}\delta_{jl}$ on Q for all i, j, k, l, $2 < r < +\infty$.*

(i) *For all $s \in (0, 1/2)$ there exists a $K > 0$ such that if $g = \operatorname{div} G$ for a symmetric matrix $G \in L^r(Q)$, $v_0 \equiv 0$ on Ω, then the unique weak solution v of (12) satisfies*

$$\|v\|_{L^r(I,W^{1,r}(\Omega))} + \|v\|_{W^{s,r}(I,L^r(\Omega))} \le K \|G\|_{r,Q} .$$

(ii) *If $r < 5/2$ then there is a $K > 0$ such that if $g \equiv 0$ on Q and $v_0 \in W^{(n+1)(r-2)/r,2}(\Omega) \cap L^2_{0,\mathrm{div}}(\Omega)$, then the unique weak solution v of (12) satisfies*

$$\|v\|_{L^r(I,W^{1,r}(\Omega))} + \|v\|_{L^\infty(I,L^r(\Omega))} \le K \|v_0\|_{W^{(n+1)(r-2)/r,2}(\Omega)} .$$

Moreover, $v \in W^{s,r}(I, L^r(\Omega))$ for all $s < 1/2$.

The constants K are independent of T.

Proof. `Part (i):` Let $P : L^r(\Omega) \to X_r := L^r_{0,\mathrm{div}}(\Omega)$ be the Helmholtz projection. The standard Stokes operator $A_r : \mathcal{D}(A_r) \subset X_r \to X_r$ on $\mathcal{D}(A_r) := W^{2,r}(\Omega) \cap W^{1,r}_{0,\mathrm{div}}(\Omega)$ is defined by $A_r := -P\Delta$. For $\alpha \in (-1, 1)$ we define A^α, $\hat{\mathcal{D}}(A_r^\alpha)$ as in [5]. Let $B := \partial_t : \mathcal{D}(B) \subset L^r(I, X_r) \to L^r(I, X_r)$ with $\mathcal{D}(B) := \{v \in W^{1,r}(I, X_r) : v(0) = 0\}$. In [5, Lemma 4.1] it is proved that if $g \in L^r(I, \hat{\mathcal{D}}(A_r^{-1/2}))$ then there exists a unique weak solution v of (12) satisfying $\|v\|_{\mathcal{D}(B^{1/2})} + \|v\|_{L^r(I,\mathcal{D}(A_r^{1/2}))} \le K\|g\|_{L^r(I,\hat{\mathcal{D}}(A_r^{-1/2}))}$. It holds $\mathcal{D}(A_r^{1/2}) \hookrightarrow W^{1,r}(\Omega)$, see [6, Prop. 1.4], and consequently $(W^{1,r'}(\Omega))^* \hookrightarrow (\mathcal{D}(A_{r'}^{1/2}))^* = \hat{\mathcal{D}}(A_r^{-1/2})$, compare [6, (1.3)]. Defining the operator $g(\varphi) = \int_Q GD\varphi$ for all $\varphi \in L^{r'}(I, W^{1,r'}(\Omega))$, it follows $\|g\|_{L^r(I,\hat{\mathcal{D}}(A_r^{-1/2}))} \le K \|G\|_{r,Q}$. It remains to show that $\mathcal{D}(B^{1/2}) \hookrightarrow W^{s,r}(I, X_r)$ if $s < 1/2$. In fact, B has bounded imaginary powers, see [3], and consequently $\mathcal{D}(B^{1/2}) = (L^r(I, X_r), \mathcal{D}(B))_{[1/2]} \hookrightarrow (L^r(I, X_r), \mathcal{D}(B))_{s,r} \hookrightarrow W^{s,r}(I, X_r)$, compare also [21, proof of Theorem 30]. This finishes the proof of the first part of the lemma. Here we used $(\cdot,\cdot)_{[1/2]}$, $(\cdot,\cdot)_{s,r}$ for the complex and real interpolation methods.

`Part (ii):` First we find in [6, Lemma 1.1] that the operator A_r generates a bounded analytic semigroup on X_r. It follows that if $v_0 \in X_{\theta,r} := (X_r, \mathcal{D}(A_r))_{\theta,r}$ for some $\theta \in (0, 1/2)$, then $v(t) := e^{-A_r t} v_0$ satisfies $\|v(t)\|_{1,r} \le K \min\{(1/t)^{1/2-\theta}, (1/t)^{1/2}\} \|v_0\|_{X_{\theta,r}}$ for $t > 0$. Hence, if $\theta > 1/2 - 1/r$ we have $\|v\|_{L^r(I,W^{1,r}(\Omega))} \le K \|v_0\|_{X_{\theta,r}}$. Moreover, $\|v(t)\|_{X_{\theta,r}} \le K \|v_0\|_{X_{\theta,r}}$ for $t > 0$.

We conclude the proof by the fact that due to the embedding there is $\theta \in (1/2 - 1/r, 1/(2r))$ such that $W^{(n+1)(r-2)/r,2}(\Omega) \cap X_r \hookrightarrow X_{\theta,r}$. The fact that $v \in W^{s,r}(I, X_r)$ for all $s < 1/2$ follows from [7, Corollary 2.3]. ∎

Defining $S_1 : L^2(Q) \times L^2_{0,\text{div}}(\Omega) \to L^2(Q)$, $S_1(G, v_0) = Dv$ and $S_2 : L^2(Q) \times L^2_{0,\text{div}}(\Omega) \to L^\infty(I, L^2(\Omega))$, $S_2(G, v_0) = v$, where v is the unique weak solution of (12) with $A^{kl}_{ij} \equiv \delta_{ik}\delta_{jl}$ on Q, Lemma 2.1 implies that S_1 and S_2 are continuous linear operators with norms bounded by 1. Restricting S_1 and S_2 on $\mathcal{D}_S := L^r(Q) \times (W^{(n+1)(r-2)/r,2}(\Omega) \cap L^2_{0,\text{div}}(\Omega))$, Lemma 2.2 says that for $r \in (2, 5/2)$ there exists $C_r > 0$ such that S_1 is continuous from $\mathcal{D}_S$ to $L^r(Q)$ and S_2 is continuous from $\mathcal{D}_S$ to $L^\infty(I, L^r(\Omega))$ with the norms bounded by $C_r > 0$ independent of T. Unfortunately, the constant C_r cannot be efficiently computed in the proof of Lemma 2.2. In order to get estimates of this constant we interpolate the estimates in Lemma 2.1 and those in Lemma 2.2. Riesz-Thorin interpolation theorem supplies the statement of the following lemma. Note that in Lemma 2.3 below the upper bound for the constant C_q is shown and this upper bound tends to 1 as q goes to 2. This is extremely important in the proof of Proposition 2.4 below.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2+\mu}$ boundary, $\mu \in (0,1)$, $r \in (2, 5/2)$. Let $A^{kl}_{ij} \equiv \delta_{ij}\delta_{kl}$ on Q, $K_r := C_r^{r/(r-2)}$. Then there exists a $K'_r > 0$ such that for every $q \in (2, r)$, $g = \operatorname{div} G$ with a symmetric matrix $G \in L^q(Q)$, $v_0 \in W^{(n+1)(q-2)/q,2}(\Omega) \cap L^2_{0,\text{div}}(\Omega)$ the unique weak solution v of (12) satisfies*

$$\max \left\{ \|v\|_{L^\infty(I,L^q(\Omega))}, \|Dv\|_{q,Q} \right\} \le K_r^{1-\frac{2}{q}} \left(\|G\|_{q,Q} + K'_r \|v_0\|_{W^{(n+1)(q-2)/q,2}(\Omega)} \right).$$

Finally, Lemma 2.3 allows us to prove the L^q theory also for systems of the Stokes type with bounded measurable coefficients provided $q > 2$ is small enough.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2+\mu}$ boundary, $\mu \in (0,1)$, $r \in (2, 5/2)$. Then there are constants $K > 0$, $L > 0$, which may depend only on r and Ω, such that if $q \in (2, 2 + L\gamma_1/\gamma_2\rangle$ (γ_1, γ_2 occur in (11)), $g \in L^q(I, W^{-1,q}(\Omega))$ and $v_0 \in V_0 := W^{(n+1)(q-2)/q,2}(\Omega) \cup L^2_{0,\text{div}}(\Omega)$, then the unique weak solution v of (12) with (11) satisfies $v \in W^{s,q}(I, L^q(\Omega))$ for all $s < 1/2$ and*

$$\|\nabla v\|_{q,Q} + \gamma_2^{-\frac{1}{q}} \|v\|_{L^\infty(I,L^q(\Omega))} \le \frac{K}{\gamma_1} \Big(\|g\|_{L^q(I,W^{-1,q}(\Omega))} + \gamma_2^{1-\frac{1}{q}} \|v_0\|_{V_0} \Big). \quad (14)$$

Proof. We follow the proof of [13, Proposition 2.1]. It is split into three steps.

`Step 1` *shows that there exists a symmetric tensor $G \in L^q(Q)$ such that for all $\varphi \in \mathcal{D}((-\infty, T), \mathcal{N}(\Omega))$*

$$\int_I \int_\Omega G_{ij} D_{ij}\varphi = \int_I \langle g, \varphi \rangle, \quad \|G\|_{q,Q} \le K \|g\|_{L^q(I,W^{-1,q}(\Omega))}. \quad (15)$$

The main idea is to solve the stationary Stokes problem

$$\int_\Omega D_{ij}w(t)D_{ij}\varphi = \langle g(t),\varphi\rangle \quad \forall\varphi\in\mathcal{N}(\Omega)$$

with the homogeneous Dirichlet boundary conditions for a.e. $t\in I$ and define $G(t)=Dw(t)$. This can be done with the help of the L^q theory for the stationary Stokes system, see for example [1]. Then it is possible to show (15) exactly as in [13].

`Step 2` *proves the assertion of Proposition* 2.4 *provided that*

$$A^{kl}_{ij}\in C^\infty(Q),\quad G_{ij}\in C^\infty(Q),\quad \operatorname{supp}G_{ij}\subset Q,\quad v_0\in\mathcal{N}(\Omega). \tag{16}$$

The computations in this step require to know that the unique weak solution of (12) is bounded in $L^q(I,W^{1,q}(\Omega))$. This is assured by (16). Indeed, if (16) holds we can construct by the Galerkin method the unique weak solution v of (12) such that $\partial_t v\in L^\infty(I,L^2(\Omega))$, $v\in L^\infty(I,W^{2,2}(\Omega))$, compare for example with [23, proof of Theorem III.3.6].

Now, we add $\gamma_2\Delta v$ to both sides of (12) and after change of the variable $s=\gamma_2 t$, $s\in I^*:=(0,T^*)$, $T^*=\gamma_2 T$ the weak formulation of the problem (12) looks like: for all $\varphi^*\in\mathcal{D}((-\infty,T^*),\mathcal{N}(\Omega))$

$$\int_{I^*}\int_\Omega \partial_s v^*\varphi^* + Dv^*D\varphi^* = \int_{I^*}\int_\Omega \mathbb{B}^*Dv^*D\varphi^* + \frac{1}{\gamma_2}G^*D\varphi^*. \tag{17}$$

We used the notation $v^*(s,x):=v(t,x)$, $G^*(s,x):=G(t,x)$, $(B^*)^{kl}_{ij}(s,x):=\delta_{ik}\delta_{jl}-A^{kl}_{ij}(t,x)/\gamma_2$ for all $s\in I^*$, $x\in\Omega$ and all i,j,k,l. We can use Proposition 2.3 to (17) in the same way as it is done in [13] to get (this is the point, where we need that the L^q norm of Dv is bounded)

$$\|v^*\|_{L^\infty(I^*,L^q(\Omega))} + \|\nabla v^*\|_{q,Q^*} \le \frac{K}{\gamma_1}\big(\|G^*\|_{q,Q^*}+\gamma_2\|v_0\|_{V_0}\big) \tag{18}$$

provided $(2K_r)^{1-2/q}(1-\gamma_1/\gamma_2)<1-\gamma_1/(2\gamma_2)$, which follows from $q\in(2,2+L\gamma_1/\gamma_2)$ if $L>0$ is chosen suitably small. Note also that Lemma 2.2 implies that $u^*\in W^{s,q}(I^*,L^q(\Omega))$ for all $s<1/2$.

Inequality (14) then follows by the backward transformation of time $t=s/\gamma_2$ from (18).

`Step 3` *proves Proposition* 2.4 *assuming that* $A^{kl}_{ij}\in L^\infty(Q)$, $G\in L^q(Q)$ *and* $v_0\in V_0$.

By the convolution of $\tilde{\mathbb{A}}$ ($\tilde{\mathbb{A}}:=\mathbb{A}$ in Q, $\tilde{\mathbb{A}}:=\gamma_1\mathbb{I}$ in $\mathbb{R}^3\backslash Q$) with a smooth convolution kernel we find symmetric matrices $\mathbb{A}^{(n)}\in C^\infty(Q)$ such that $\mathbb{A}^{(n)}(t,x)\to\mathbb{A}(t,x)$ for almost every $(t,x)\in Q$ and for all $D\in\mathbb{S}$, for a.e. $(t,x)\in Q$ it holds $\gamma_1|D|^2\le(\mathbb{A}^{kl}_{ij})^{(n)}(t,x)D_{ij}D_{kl}\le\gamma_2|D|^2$. We can also construct $G^{(n)}\in\mathcal{D}(Q)$ such that $G^{(n)}\to G$ in $L^q(Q)$ and $v_0^{(n)}\in\mathcal{N}(\Omega)$ such that $v_0^{(n)}\to v_0$ in V_0.

Denote by $v^{(n)}$, $\sigma^{(n)}$ the solutions of (12) with the coefficient matrix $\mathbb{A}^{(n)}$, the right-hand side $g^{(n)} = \operatorname{div} G^{(n)}$ and with the initial value $v_0^{(n)}$. For n large enough it follows by Step 2

$$\gamma_2^{-\frac{1}{q}} \left\| v^{(n)} \right\|_{L^\infty(I,L^q(\Omega))} + \left\| Dv^{(n)} \right\|_{q,Q} \le \frac{2K}{\gamma_1} \big(\left\| G \right\|_{q,Q} + \gamma_2^{1-\frac{1}{q}} \left\| v_0 \right\|_{V_0} \big). \tag{19}$$

As the right hand side of (19) does not depend on n we can pass to the limit as $n \to \infty$ in the weak formulation of (12). Since the weak solution of the problem (12) (with $\mathbb{A}$, G, v_0) is unique the statement of the proposition follows from (19) by Korn inequality (for example [17, Section 5.1.1]), (15) and the lower semicontinuity of the norm with respect to the weak convergence. ∎

In the proof of Theorem 1.1 we will also need the L^q theory of stationary systems of the Stokes type with bounded measurable coefficients. We investigate a weak solution $v : \Omega \to \mathbb{R}^n$ of the following stationary problem:

$$-\operatorname{div}(\mathbb{A}Dv) + \nabla\sigma = g, \quad \operatorname{div} v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \tag{20}$$

We assume that the coefficients $\mathbb{A} = (A_{ij}^{kl})_{i,j,k,l=1,\dots,n}$ satisfy the stationary equivalent of (11)

$$\begin{aligned} &\exists \gamma_1, \gamma_2 > 0,\ \forall E \in \mathbb{S},\ \forall x \in \Omega :\ \gamma_1 |E|^2 \le \mathbb{A}_{ij}^{kl}(x) E_{ij} E_{kl} \le \gamma_2 |E|^2 \\ &A_{ij}^{kl} \in L^\infty(\Omega) \ \text{ and } \ A_{ij}^{kl}(x) = A_{kl}^{ij}(x) \quad \forall x \in \Omega\,, \quad i,j,k,l = 1,\dots,n. \end{aligned} \tag{21}$$

A proposition analogous to Proposition 2.4 holds (see [11, Lemma 2.6]).

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. Then there are constants $K > 0$, $L > 0$ such that if $q \in (2, 2 + L\gamma_1/\gamma_2)$ and $g \in W^{-1,q}(\Omega)$, then the unique weak solution v of (20) with (21) satisfies*

$$\left\| \nabla v \right\|_{q,\Omega} \le \frac{K}{\gamma_1} \left\| g \right\|_{W^{-1,q}(\Omega)} .$$

Now we recall two useful lemmas.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. Assume that $w \in L^\infty(I, C^{0,\alpha}(\overline{\Omega}))$ and $w \in C^{0,\beta}(\overline{I}; L^r(\Omega))$ for some $\alpha, \beta > 0$ and $r > 1$. Then $w \in C^{0,\gamma}(\overline{Q})$ with $\gamma = \min\left\{\alpha, \frac{\alpha\beta r}{\alpha r + n}\right\}$.*

The proof is a slight modification of the proof of [8, Lemma 2.2].

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. If $q_0 > n$ and $u \in W^{1,q_0}(\Omega)$. Then $u \in C(\overline{\Omega})$ and there is $C > 0$ independent of q_0 such that*

$$\sup_{\Omega} |u| \le C \left(\frac{q_0 - 1}{q_0 - n} \right)^{1 - \frac{1}{q_0}} \left\| u \right\|_{1,q_0,\Omega}$$

The assertion follows from the proof of Theorem 2.4.1 in [24], page 58: (2.4.6)–(2.4.9).

3. Proof of Theorem 1.1 for $p = 2$

The proof is divided into several lemmas. In the first lemma it is stated that a unique weak solution to problem (1)–(3) exists for $p = 2$. For our approach it is important that the regularity of $\partial_t u$ (see (22)) is directly obtained.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary, let* (3) *hold with $p = 2$, $f \in L^2(I, (W_0^{1,2}(\Omega))^*)$, $\partial f/\partial t \in L^2(I, (W_{0,\mathrm{div}}^{1,2}(\Omega))^*)$, $f(0) \in L^2(\Omega)$. Let $u_0 \in W^{2,2}(\Omega) \cap W_{0,\mathrm{div}}^{1,2}(\Omega)$. Then there exists a unique weak solution u of the problem* (1), (2) *and a constant $K > 0$ such that*

$$\|u\|_{L^\infty(I,L^2(\Omega))} + \|\nabla u\|_{2,Q} \le K, \quad \|\partial_t u\|_{L^\infty(I,L^2(\Omega))} + \|\nabla \partial_t u\|_{2,Q} \le K. \tag{22}$$

The pressure π can be reconstructed in such a way that it has the zero mean value over Ω at almost every time level $t \in I$, $\pi \in L^2(Q)$ and $\|\pi\|_{2,Q} \le K$.

Proof. For the Navier-Stokes system (i.e. $\mathcal{T}_{ij}(D) = D_{ij}$ for all $D \in \mathbb{S}$) the existence and regularity of u is proved in [23, Theorem III.3.5] via the Galerkin method. The same proof works also for the systems with growth $p = 2$ that we consider here with the difference that we have to use the monotonicity of the operator $\mathcal{T}$ to pass to the limit in the elliptic term.

As $\partial_t u$, $\operatorname{div} \mathcal{T}(Du)$, $\operatorname{div}(u \otimes u)$ and f belong to $L^2(I, W^{-1,2}(\Omega))$, the pressure $\pi(t)$ can be reconstructed by De Rham's theorem at almost every time level $t \in I$ in such a way that

$$\nabla \pi(t) = -\partial_t u(t) + \operatorname{div}(\mathcal{T}(Du(t))) - \operatorname{div}(u \otimes u)(t) + f(t) \quad \text{in } W^{-1,2}(\Omega). \tag{23}$$

Setting up the zero mean value condition for $\pi(t)$ we get for a.e. $t \in I$ an estimate of $\pi(t)$ in $L^2(\Omega)$ via the Nečas Theorem on negative norms, see for example [1], consequently $\pi \in L^2(Q)$ and $\|\pi\|_{2,Q} \le K$. The lemma is proved. ∎

In the next lemma we collect an additional information about the regularity in space of the unique weak solution of problem (1), (2) with (3).

Lemma 3.2. *Let, additionally to the assumptions of Lemma* 3.1, *Ω have C^2 boundary and $f \in L^\infty(I, L^2(\Omega))$. Then the unique weak solution of problem* (1), (2) *with* (3) *satisfies*

$$u \in L^\infty(I, W^{2,2}(\Omega)), \quad \pi \in L^\infty(I, W^{1,2}(\Omega)). \tag{24}$$

Proof. As $\partial_t u \in L^\infty(I, L^2(\Omega))$, see (22), we can move this term to the right hand side of (1). At a.e. time level $t \in I$ we can use the difference technique in the space to get (24). The proof is technically difficult as $\partial\Omega$ is present. The regularity of u cannot be obtained at once, first the interior regularity must be shown and then the regularity near the boundary. We skip the proof and refer the reader to [18, Section 3] where this method is used to get $L^2(I, W^{2,2}(\Omega))$ regularity of a solution to a similar system as we consider here even in 3D. It can be easily modified also for our situation. ∎

Now we improve the regularity of $\partial_t u$ by the linear theory from Section 2. To avoid possible problems in $t = 0$ we employ a cut-off function in time.

Lemma 3.3. *Let all assumptions of Lemma* 3.2 *hold. Moreover, let* $\partial_t f \in L^{\tilde{q}}(I, W^{-1,\tilde{q}}(\Omega))$ *for some* $\tilde{q} \in (2,4)$ *and* $\partial\Omega$ *be of the class* $C^{2+\mu}$ *for some* $\mu \in (0,1)$*. Then there exists* $q > 2$ *such that the unique weak solution of* (1), (2) *with* (3) *satisfies for all* $s < 1/2$ *and* $\delta \in (0, \min\{1, T\})$

$$\partial_t u \in L^q((\delta, T), W^{1,q}_{0,\mathrm{div}}(\Omega)) \cap W^{s,q}((\delta, T), L^q(\Omega)). \tag{25}$$

Proof. First we derive an equation for the time derivative of u. We test (1) with $\varphi := \partial_t(\psi\eta)$, where $\psi \in \mathcal{D}((-\infty, T), \mathcal{N}(\Omega))$ and the cut-off function $\eta \in C^\infty(\mathbb{R})$ satisfies $\eta = 0$ on $(-\infty, \delta/2\rangle$, $\eta = 1$ on $\langle\delta, +\infty)$ and $\eta' \in \langle 0, 5/\delta\rangle$ on $\mathbb{R}$. After integration by parts we get that $v := \eta\partial_t u$ satisfies for all $\psi \in \mathcal{D}((-\infty, T), \mathcal{N}(\Omega))$

$$\int_Q -v\partial_t\psi + \partial_{kl}\mathcal{T}_{ij}(Du)D_{kl}vD_{ij}\psi = \langle g, \psi\rangle, \tag{26}$$

where

$$\langle g, \psi\rangle = \int_Q (\eta\partial_t f + \eta'\partial_t u)\psi + (v_i u_j + v_j u_i)D_{ij}\psi. \tag{27}$$

Clearly, $g \in L^{\tilde{q}}(I, (W^{-1,\tilde{q}}(\Omega))$ by (22), (24), the assumption on $\partial_t f$ and the properties of η. Moreover, $\operatorname{div} v = 0$ on Q and $v = 0$ on $(\{0\} \times \Omega) \cup (I \times \partial\Omega)$. Since $A^{kl}_{ij} := \partial_{ij}\mathcal{T}_{kl}(Du)$ on Q satisfies (11) by (3) with $p = 2$, Proposition 2.4 gives us $q \in (2, \tilde{q})$ such that (25) holds. ∎

Note that Lemma 3.3 guarantees that $u \in L^\infty((\delta, T), L^q(\Omega))$ for some $q > 2$ by [21, Corollary 26]. It means that we can move $\partial_t u$ to the right hand side of (1) and for a.e. $t \in (\delta, T)$ use the stationary theory, namely the following theorem proved in [11, Theorem 3.19].

Theorem 3.4. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with* C^2 *boundary. Let* u, π, $\int_\Omega \pi = 0$ *be a weak solution of the problem*

$$u_k \frac{\partial u}{\partial x_k} - \operatorname{div}(\mathcal{T}(Du)) - \nabla\pi = f, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad u = 0 \text{ at } \partial\Omega,$$

where $\mathcal{T}$ *satisfies* (3) *with* $p = 2$, $f \in L^{\tilde{q}}(\Omega)$ *for* $\tilde{q} > 2$*. Then there is a* $q \in (2, \tilde{q})$ *such that* $\|u\|_{2,q,\Omega} + \|\pi\|_{1,q,\Omega} \le K$ *where* K *and* q *may depend only on* Ω, C_1, C_2, $\|f\|_{q,\Omega}$.

Now we are ready to prove the Hölder continuity of ∇u, but to prove the Hölder continuity of π we lack the information about the regularity of π in time. Fortunately, it can be reread from (1) since $\partial_t u \in W^{s,q}((\delta, T), L^q(\Omega))$ for all $s < 1/2$, precisely

Lemma 3.5. *Let $p = 2$ and all assumptions of Theorem* 1.1 *hold. Then there exists a $q > 2$ such that for all $\delta \in (0, \min\{1, T\})$ and $s < 1/2$*

$$\pi \in W^{s,q}((\delta, T), L^q(\Omega)). \tag{28}$$

Proof. Due to our assumptions all Lemmas of this section hold. From (24) and (25) the existence of $q > 2$ follows such that

$$u \in W^{s,q}((\delta, T), W^{1,q}(\Omega)), \quad \partial_t u \in W^{s,q}((\delta, T), L^q(\Omega)). \tag{29}$$

Let $t, t' \in (\delta, T)$ and denote $\Delta^{tt'} u = u(t) - u(t')$ and so on. Subtracting (23) in time t' from the same equation in time t we get for all $\varphi \in W_0^{1,2}(\Omega)$

$$\int_\Omega \Delta^{tt'}\pi \operatorname{div} \varphi = \int_\Omega (\Delta^{tt'}\partial_t u - \Delta^{tt'} f)\varphi - (\Delta^{tt'}(u \otimes u) - \Delta^{tt'}\mathcal{T}(Du))D\varphi. \tag{30}$$

By (8) and (29) it is seen that (30) remains valid also for all $\varphi \in W_0^{1,q'}(\Omega)$. We construct a special test function $\varphi^{tt'}$ as a solution of the problem

$$\begin{aligned} \operatorname{div} \varphi^{tt'} &= \Delta^{tt'}\pi|\Delta^{tt'}\pi|^{q-2} - \frac{1}{|\Omega|}\int_\Omega \Delta^{tt'}\pi|\Delta^{tt'}\pi|^{q-2} \quad \text{in } \Omega \\ \varphi^{tt'} &= 0 \quad \text{at } \partial\Omega. \end{aligned}$$

As the right hand side belongs to $L^{q'}(\Omega)$ for a.e. $(t, t') \in (\delta, I)^2$ by (24) and it has zero mean value over Ω, we get for example by [1, Lemma 3.3] the existence of $\varphi^{tt'}$ and the estimate $\left\|\varphi^{tt'}\right\|_{1,q',\Omega} \le K \left\|\Delta^{tt'}\pi\right\|_{q,\Omega}^{q-1}$.

Testing (30) with $\varphi^{tt'}$ we get on the left hand side (the second term vanishes as $\int_\Omega \Delta^{tt'}\pi = 0$) $I_\pi := \int_\Omega \Delta^{tt'}\pi \operatorname{div} \varphi^{tt'} = \left\|\Delta^{tt'}\pi\right\|_{q,\Omega}^q$. The right hand side can be estimated with a help of Hölder inequality, (3) and (24) by $I_\pi/2 + K(\|\Delta^{tt'}\partial_t u\|_{q,\Omega}^q + \|\Delta^{tt'} f\|_{-1,q,\Omega}^q + \|\Delta^{tt'}\nabla u\|_{q,\Omega}^q)$. We divide the both sides by $|t - t'|^{1-sq}$ and after the integration of the so obtained inequality twice over (δ, T) we get by (8) and (29)

$$\|\pi\|_{W^{s,q}((\delta,T),L^q(\Omega))}^q = \int_\delta^T \int_\delta^T \frac{\left\|\Delta^{tt'}\pi\right\|_{q,\Omega}^q}{|t - t'|^{1+sq}} \,\mathrm{d}t\,\mathrm{d}t' \le K.$$

The lemma is proved. ∎

Now we are prepared to prove Theorem 1.1 in the case $p = 2$. Let (5), (6) hold. Then (7) follows by Lemmas 3.1, 3.2. Let, moreover, (8) hold. Combining the results from Lemmas 3.1, 3.2, 3.3 with Theorem 3.4 we have got the existence of $q > 2$ such that the unique weak solution u, π of (1), (2) with (3) satisfies (24), (25) and

$$u \in L^\infty((\delta, T), W^{2,q}(\Omega)), \quad \pi \in L^\infty((\delta, T), W^{1,q}(\Omega)). \tag{31}$$

As $W^{1,q}((\delta,T),L^q(\Omega)) \hookrightarrow C^{1-1/q}(\langle\delta,T\rangle,L^q(\Omega))$ and $W^{1,q}(\Omega) \hookrightarrow C^{0,1-2/q}(\overline{\Omega})$ in two dimensions, it follows from (25) and (31) by Lemma 2.6 that $\nabla u \in C^{0,\alpha}(\overline{(\delta,T)\times\Omega})$ for a certain $\alpha > 0$. Moreover, (28) holds by Lemma 3.5 and by embedding [21, Corollary 26] $\pi \in C^{s-1/q}(\langle\delta,T\rangle,L^q(\Omega))$ for all $s \in (1/q,1/2)$. This together with (31) gives by Lemma 2.6 the Hölder continuity of π on $\overline{(\delta,T)\times\Omega}$ and concludes the proof of Theorem 1.1 if $p = 2$.

4. Proof of Theorem 1.1 for $p \in (2,4)$

Following [18] we introduce quadratic approximations $\mathcal{T}^A$ of $\mathcal{T}$. For $A > 1$ we define an approximative potential

$$F^A(s) = \begin{cases} F(s) & \text{for } s \in (0,A^2\rangle, \\ as + b\sqrt{s} + c & \text{for } s > A^2 \end{cases}$$

in such a way that $F^A \in C^2(\langle 0,+\infty))$ and, trivially, $F^A(0) = 0$. Defining $\mathcal{T}^A_{ij}(D) := \partial_{ij}F^A(|D|^2)$ for all $D \in \mathbb{S}$ and i,j, there is a constant $C(A) > 0$ such that for all $D,E \in \mathbb{S}$ it is $C_1|E|^2 \le \partial_{kl}\mathcal{T}^A_{ij}(D)E_{ij}E_{kl} \le C(A)|E|^2$. It means that $\mathcal{T}^A$ satisfies (3) with $p = 2$ and by the previous section the problems

$$\begin{aligned} &\partial_t u - \operatorname{div}\mathcal{T}^A(Du) = f - \nabla\pi - \operatorname{div}(u\otimes u), \qquad \operatorname{div} u = 0 \ \text{ in } Q \\ &\int_\Omega \pi(t) = 0 \ \text{ for a.e. } t \in I, \quad u = 0 \ \text{ in } I\times\partial\Omega, \quad u(0) = u_0 \text{ in } \Omega \end{aligned} \tag{32}$$

possess the unique weak solutions u^A, π^A satisfying (7) and (9). Although we do not a priori know that u^A, π^A are bounded in the norms of the spaces from (7) and (9) uniformly in A, the regularity of u^A, π^A is very important as it allows us to compute with these norms since they are for fixed $A > 1$ finite.

In order to prove Theorem 1.1 we estimate u^A, π^A in suitable spaces uniformly with respect to A and pass to the limit as $A \to \infty$ in (32). Since we want to show the uniform estimates of u^A, π^A we need uniform estimates of $\mathcal{T}^A$. For this purpose we define $\theta_A(D) = (1 + \min(|D|^2, A^2))^{1/2}$ for all $D \in \mathbb{S}$. In [18, Lemma 2.22] the existence of $C_3, C_4 > 0$ is proved such that for all $A > 1$, $D,E \in \mathbb{S}$

$$C_3\theta_A^{p-2}(D)|E|^2 \le \partial_{kl}\mathcal{T}^A_{ij}(D)E_{ij}E_{kl} \le C_4\theta_A^{p-2}(D)|E|^2 \tag{33}$$

$$C_3|D|^2 \le \mathcal{T}^A_{ij}(D)D_{ij}. \tag{34}$$

From (33) it follows by the Mean Value Theorem that there exists a $K > 0$ such that for all $A > 1$, $D \in \mathbb{S}$

$$|\mathcal{T}^A_{ij}(D)| \le K\theta_A^{p-2}(D)|D|, \quad \theta_A^{p-2}(D)|D|^2 \le K\mathcal{T}^A_{ij}(D)D_{ij}. \tag{35}$$

To illustrate the technique we prove the last statement. Since $\mathcal{T}^A(0) = 0$ we have

$$\mathcal{T}^A_{ij}(D)D_{ij} = \int_0^1 \partial_{kl}\mathcal{T}^A_{ij}(sD)\mathrm{d}sD_{kl}D_{ij} \geq C_3 \int_0^1 \theta_A^{p-2}(sD)\mathrm{d}s|D|^2$$

for $D \in \mathbb{S}$. It remains to show $I(A,D,p) := \int_0^1 \theta_A^{p-2}(sD)\mathrm{d}s \geq K\theta_A^{p-2}(D)$. Note that the function $s \to \theta_A^{p-2}(sD)$ is nondecreasing on $\langle 0,1\rangle$, i.e., $I(A,D,p) \geq \theta_A^{p-2}(D/2)/2$. This gives for $|D|/2 < A$ that $I(A,D,p) \geq (1+|D|^2/4)^{(p-2)/2}/2$ and consequently $I(A,D,p) \geq 1/2(1/4)^{(p-2)/2}\theta_A^{p-2}(D)$. As for $|D|/2 \geq A$ we get $I(A,D,p) \geq (1+|A|^2)^{(p-2)/2}/2 = \theta_A^{p-2}(D)/2$ the claim is proved.

As we want to prove a global result we need the following description of the boundary of Ω. Let us choose some $\alpha > 0$ small. As $\partial\Omega$ is of the class $C^{2+\mu}$, it can be described in the local coordinates by $C^{2+\mu}$ maps $a_\ell : (-\alpha,\alpha) \to \mathbb{R}$, $\ell \in \{1,\dots,k\}$,

$$a'_\ell(0) = 0, \qquad \sup_{(-\alpha,\alpha)} |a'_\ell| \leq \alpha, \tag{36}$$

i.e., for all $x_0 \in \partial\Omega$ there exist $B(x_0)$ and $\ell \in \{1,\dots,k\}$ such that in the corresponding local coordinates $(x_1,x_2) \in \partial\Omega \cap B(x_0)$ if and only if $x_2 = a_\ell(x_1)$ for $x_1 \in (-\alpha,\alpha)$.

Denoting

$$\begin{aligned}
U_\ell^+ &:= \{(x_1,x_2) : x_1 \in (-\alpha,\alpha), x_2 \in (a_\ell(x_1), a_\ell(x_1)+\alpha)\}\\
U_\ell^- &:= \{(x_1,x_2) : x_1 \in (-\alpha,\alpha), x_2 \in (a_\ell(x_1)-\alpha, a_\ell(x_1))\}\\
U_\ell &:= \{(x_1,x_2) : x_1 \in (-\alpha,\alpha), x_2 \in (a_\ell(x_1)-\alpha, a_\ell(x_1)+\alpha)\}
\end{aligned}$$

we may assume that $U_\ell^+ \subset \Omega$, $U_\ell^- \subset \mathbb{R}^2 \setminus \overline{\Omega}$ and choose an open smooth set $U_0 \subset \overline{U_0} \subset \Omega$ so that $\bigcup_{\ell=0}^k U_\ell \supset \overline{\Omega}$. Let ξ_ℓ ($\ell = 0,1,\dots,k$) be a partition of the unity corresponding to the covering $U_0, U_1, \dots, U_k$ of Ω. In the coordinate system corresponding to $(U_\ell, \xi_\ell, a_\ell)$, $\ell = 1,\dots,k$, we can define a tangential derivative $\frac{\partial u}{\partial \tau} := \frac{\partial u}{\partial x_1} + a'_\ell \frac{\partial u}{\partial x_2}$.

Let us start with the uniform estimates recalling that all constants $K > 0$ are always independent of A. We will follow the scheme of Section 3, but now we have the whole scale of problems (32) for all $A > 1$ and we show the estimates independent of A. An easy consequence of the fact that $C_3|E|^2 \leq \partial_{ij}\mathcal{T}_{kl}(D)E_{ij}E_{kl}$ for all $D, E \in \mathbb{S}$ is the following lemma.

Lemma 4.1. *Let $A > 1$, $p > 2$ and $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{2+\mu}$ boundary, $\mu \in (0,1)$. Let u^A, π^A be a weak solution of* (32) *with f satisfying* (5) *and u_0 satisfying* (6)*. Then there exists a constant $K > 0$ such that*

$$\begin{aligned}
\left\|u^A\right\|_{L^\infty(I,L^2(\Omega))} + \left\|\nabla u^A\right\|_{2,Q} &\leq K\\
\left\|\partial_t u^A\right\|^2_{L^\infty(I,L^2(\Omega))} + \left\|\nabla\partial_t u^A\right\|_{2,Q} &\leq K.
\end{aligned} \tag{37}$$

Proof. As the estimates of $\mathcal{T}^A$ from below does not depend on A, the proof is essentially the same as the proof of Lemma 3.1. Recall that it is based on the Galerkin method. The only point where the proofs differ is when estimating L^2 norm of $\partial_t u_m^A(0)$, where u_m^A is a Galerkin approximation. Since a uniform estimate of $\operatorname{div} \mathcal{T}^A(Du_0)$ in $L^2(\Omega)$ is needed there, we assume (6) and the estimate then follows from (33) as Du_0 is bounded in Ω. ∎

As in the previous section we now move $\partial_t u$ in (32) to the right hand side and get better uniform estimates of u^A in space. The presence of $\partial\Omega$ causes that we can prove them only for $p \in (2,4)$ and in comparison with Lemma 3.2 we get (38) below only for $s \in (1,2)$.

Lemma 4.2. *Let $p \in (2,4)$ and all assumptions of Lemma 4.1 be satisfied. Then for every $s \in (1,2)$ there exists a constant $K > 0$ such that for every $A > 1$ the unique weak solution u^A, π^A of (32) satisfies*

$$\left\|\pi^A\right\|_{L^\infty(I,W^{1,s}(\Omega))} + \left\|u^A\right\|_{L^\infty(I,W^{2,s}(\Omega))} \le K. \tag{38}$$

Proof. As all assumptions of Lemma 4.1 hold we have $\|\partial_t u\|_{L^\infty(I,L^2(\Omega))} < K$. Consequently, we move $\partial_t u$ to the right hand side of (32) and at almost every time level $t \in I$ use the stationary theory developed in [11], where the authors study similar stationary systems for $p < 2$. As we can proceed similarly here we just mention the main steps of the proof.

First of all the interior regularity is shown. We can even show that there is a $K > 0$ such that $\|\xi_0 u^A{}_{L^\infty(I,W^{2,2}(\Omega))}\| < K$. The proof is based on testing (32) with $\varphi_k := \operatorname{rot}(\xi_0^{2k} \operatorname{curl} u)$ for $k \in \mathbb{N}$ large enough, compare [11, Lemma 4.6, Step 1]. The next step is to get the estimates of $\partial u^A/\partial\tau$ in $W^{1,2}$ norm. Similarly as in [11, Lemma 5.1] these estimates depend on ∇u^A. Nevertheless, if $p \in (2,4)$ this information is sufficient to reread estimate (38) from (32) with help of (33), (34) and (35), compare [11, Lemma 5.6]. ∎

In the next lemma we show the uniform estimates that correspond to the results in Lemma 3.3 and Theorem 3.4.

Lemma 4.3. *Let $p \in (2,4)$ and all assumptions of Theorem 1.1 hold. There exists a $q > 2$ such that for every $2\epsilon \in (0, \min\{1,T\})$ there is a $K > 0$ such that for all $A > 1$*

$$\left\|\partial_t u^A\right\|_{L^\infty((2\epsilon,T),L^q(\Omega))} + \left\|\nabla^2 u^A\right\|_{L^\infty((2\epsilon,T),L^q(\Omega))} \le K. \tag{39}$$

Proof. Due to the assumptions, Lemmas 4.1 and 4.2 hold. Next we gain an information on $\partial_t u^A$. By the same procedure as in Lemma 3.3 we derive (26) for $v := \eta\partial_t u^A$ and $\sigma := \eta\partial_t \pi^A$, η being a cut-off function with properties $\eta = 0$ on $(-\infty, \epsilon + \delta/2)$, $\eta = 1$ on $(\epsilon + \delta, +\infty)$ and $\eta' \in (0, 5/\delta)$ on $\mathbb{R}$ for some

$\delta \in (0, \min\{1, T\} - \epsilon)$. In the following all constants $K > 0$ are independent of δ. Without loss of the generality we may assume that $\tilde{q} < 4$ in (8), so by (8), (37) and (38) we see that the functional g defined by (27) satisfies $\|g\|_{L^{\tilde{q}}(I, W^{-1,\tilde{q}}(\Omega))} \leq K/\delta$.

By (33) it holds for all $t \in (\epsilon + \delta/2, T)$, $x \in \Omega$ and $E \in \mathbb{S}$ that

$$C_3|E|^2 \leq \partial_{ij}\mathcal{T}^A_{kl}(Du^A(t,x))E_{ij}E_{kl} \leq C_4 V^{p-2}_{A,\delta/2}|E|^2, \tag{40}$$

where $V_{A,\delta/2} = \sup_{(\epsilon+\delta/2,T)\times\Omega} |\theta_A(Du^A)|$. Let us note, that we a priori know that $V_{A,0} < +\infty$. Consequently, Proposition 2.4 gives us $K > 0$, $L > 0$ such that for all $q \in (2, 2 + LV^{2-p}_{A,\delta/2}\rangle$

$$\left\|\nabla \partial_t u^A\right\|_{q,(\epsilon+\delta,T)\times\Omega} + V^{(2-p)/q}_{A,\delta/2} \left\|\partial_t u^A\right\|_{L^\infty((\epsilon+\delta,T),L^q(\Omega))} \leq \frac{K}{\delta}, \tag{41}$$

as we may assume that $L < \tilde{q} - 2$ also $q_0 := 2 + LV^{2-p}_{A,\delta/2} < \tilde{q}$. From (37) and (41) we have $\|\partial_t u^A\|_{L^\infty((\epsilon+\delta,T),L^2(\Omega))} \leq K$ and $\|\partial_t u^A\|_{L^\infty((\epsilon+\delta,T),L^{q_0}(\Omega))} \leq KV^{(p-2)/q_0}_{A,\delta/2}/\delta$. Interpolating these results we get for $q \in (2, q_0)$ and $b \in (0,1)$ satisfying $1/q = b/q_0 + (1-b)/2$ the estimate

$$\left\|\partial_t u^A\right\|_{L^\infty((\epsilon+\delta,T),L^q(\Omega))} \leq \frac{K}{\delta^b} V^E_{A,\delta/2} \quad \text{with } E := \frac{b(p-2)}{q_0}. \tag{42}$$

Note that $E \to 0$, $q \to 2$ as $b \to 0$.

From now on all constants $K > 0$ are independent of $t \in I$. For a.e. $t \in (\epsilon + \delta, T)$ it is $\|\partial_t u^A\|_{q,\Omega} + \|\pi^A\|_{q,\Omega} \leq KV^E_{A,\delta/2}/\delta^b$. We fix one of these $t \in (\epsilon + \delta, T)$ and improve an information about $\nabla^2 u^A$ at time level t. We use the similar process as in the proof of Lemma 4.2. The estimates of $\nabla^2 u^A$ in the interior of Ω should be shown first and then the more difficult estimates near $\partial\Omega$. We show only the latter estimates. Fix $q \in (2, q_0)$ and also corresponding $b \in (0,1)$, $\ell \in \{1, \ldots, k\}$, see (36).

On the coordinate system $(a, \xi, U) := (a_\ell, \xi_\ell, U_\ell)$ the tangential derivative is defined and we set $v := \xi\frac{\partial u^A}{\partial\tau}(t) + \tilde{u}$, where $\tilde{u}$ solves

$$\operatorname{div}\tilde{u} = -\frac{\partial u^A}{\partial\tau}\nabla\xi - a''\frac{\partial u^A_1}{\partial x_2} \text{ in } U^+, \quad \tilde{u} = 0 \text{ at } \partial U^+.$$

Since the compatibility condition $\int_{U^+} -\frac{\partial u^A}{\partial\tau}\nabla\xi - a''\frac{\partial u^A_1}{\partial x_2} = \int_{U^+} \operatorname{div}(-\xi\frac{\partial u^A}{\partial\tau}(t)) = 0 = \int_{\partial U^+} \tilde{u}$ holds, the solution of this problem exists, see for example [1, Lemma 3.3], and as $\xi \in C^\infty(U)$, $a \in C^2(-\alpha, \alpha)$ there exists a $K > 0$ such that $\|\tilde{u}\|_{1,q,U^+} \leq K\|\nabla u^A\|_{q,\Omega}$. Testing (32) with $\phi = -\frac{\partial\varphi}{\partial\tau}\xi$ for $\varphi \in W^{1,q'}_{0,\operatorname{div}}(\Omega)$ we get after a long but elementary calculation that v (of course $v \equiv 0$ in $\Omega \setminus U^+$) solves

$$\int_\Omega \partial_{ij}\mathcal{T}^A_{kl}(Du^A(t))D_{kl}v(t)D_{ij}\varphi = \langle g(t),\varphi\rangle \quad \forall\varphi \in W^{1,q'}_{0,\mathrm{div}}(\Omega), \qquad \operatorname{div} v = 0 \ \text{ on } \Omega, \quad v = 0 \ \text{ at } \partial\Omega, \tag{43}$$

with

$$\begin{aligned}\langle g(t),\varphi\rangle = &\int_\Omega \Big(\partial_t u^A_i + u^A_j \frac{\partial u^A_i}{\partial x_j} - f_i\Big)\frac{\partial \varphi_i}{\partial \tau}\xi + \pi^A\Big(\operatorname{div}\varphi\frac{\partial\xi}{\partial\tau} - a''\frac{\partial\varphi_1}{\partial x_2}\xi - \frac{\partial\varphi_i}{\partial\tau}\frac{\partial\xi}{\partial x_i}\Big)\\ &+ \partial_{kl}\mathcal{T}^A_{ij}(Du^A)\Big(D_{kl}\tilde{u} + \frac{\partial u^A_k}{\partial\tau}\frac{\partial\xi}{\partial x_l} + \frac{\partial u^A_k}{\partial x_2}\frac{\partial a'}{\partial x_l}\xi\Big)D_{ij}\varphi\\ &+ \mathcal{T}^A_{ij}(Du^A)\Big(-\frac{\partial\xi}{\partial\tau}D_{ij}\varphi + \xi\frac{\partial a'}{\partial x_i}\frac{\partial\varphi_j}{\partial x_2} + \frac{\partial\xi}{\partial x_j}\frac{\partial\varphi_i}{\partial\tau}\Big)\end{aligned}$$

for all $\varphi \in W^{1,q'}_0(\Omega)$. The functional g satisfies by (8), (38) and (42) the estimates $\|g(t)\|_{-1,q,\Omega} \le KV^E_{A,\delta/2}/\delta^b$. Due to (40) we can apply Proposition 2.5 to (43) and get

$$\Big\|\xi\nabla\frac{\partial u^A}{\partial\tau}(t)\Big\|_{q,\Omega} < \frac{K}{\delta^b}V^E_{A,\delta/2}. \tag{44}$$

Note that below (41) $L > 0$ can be chosen so that (44) holds for all $q \in (2, q_0)$.

We reconstruct the whole $\nabla^2 u^A$ as in [11, Lemma 5.6]. Defining $G := \frac{\partial}{\partial x_2}(\mathcal{T}^A_{12}(Du^A)(t))$ it follows from the equality (we suppress the arguments to shorten the record)

$$\partial_{12}\mathcal{T}^A_{12}(\,)\Big(D_{12}\Big(\frac{\partial u^A}{\partial x_2}\Big)\Big) = \frac{1}{2}\Big(G - \partial_{11}\mathcal{T}^A_{12}(\,)D_{11}\Big(\frac{\partial u^A}{\partial x_2}\Big) - \partial_{22}\mathcal{T}^A_{12}(\,)D_{22}\Big(\frac{\partial u^A}{\partial x_2}\Big)\Big)$$

by (33) and especially $\partial_{12}\mathcal{T}_{12}(\,) \ge C_3\theta_A^{p-2}$, $\operatorname{div} u^A = 0$ and the definition of $\partial u^A/\partial\tau$ that $\Theta^A_t := \theta_A^{p-2}(Du^A(t))|\nabla^2 u^A(t)|$, $\Xi^A_t := \theta_A^{p-2}(Du^A(t))|\nabla\frac{\partial u^A}{\partial\tau}(t)|$ satisfy

$$\big\|\xi\Theta^A_t\big\|_{q,\Omega} \le K_1\Big(\|\xi G(t)\|_{q,\Omega} + \big\|\xi\Xi^A_t\big\|_{q,\Omega} + \sup|a'|\,\big\|\xi\Theta^A_t\big\|_{q,\Omega}\Big), \tag{45}$$

where $K_1 > 0$ is an absolute constant. Proceeding similarly to [11, Theorem 3.19, Step 3] and using (8), (38) and (42) we reread from (32)

$$\|\xi G(t)\|_{q,\Omega} \le K_2\Big(\sup|a'|\,\big\|\xi\Theta^A_t\big\|_{q,\Omega} + K\,\big\|\xi\Xi^A_t\big\|_{q,\Omega} + KV^E_{A,\delta/2}/\delta^b\Big).$$

Also here $K_2 > 0$ is an absolute constant and we may assume that the description on $\partial\Omega$ is such that $(K_1 + K_2)\sup_{(-\alpha,\alpha)}|a'| < 1/2$, compare (36). Consequently, we get from (45) by (44) that $\|\xi\Theta^A_t\|_{q,\Omega} \le KV^{p-2}_{A,\delta}V^E_{A,\delta/2}/\delta^b$. Summing this over all coordinate systems gives us together with (38) that

$$\big\|\theta_A^{p-1}(Du^A(t))\big\|_{1,q,\Omega} \le \big\|\theta_A^{p-1}(Du^A(t))\big\|_{q,\Omega} + \big\|\Theta^A_t\big\|_{q,\Omega} \le \frac{K}{\delta^b}V^{p-2}_{A,\delta}V^E_{A,\delta/2}. \tag{46}$$

We want to use an embedding theorem on the left hand side of (46). Note that as q_0, and consequently also q, depend on $V_{A,\delta/2}$ and may tend to 2 as $A \to +\infty$ we need to know the precise dependence of the embedding constant on q. Since $q_0 < \tilde{q}$ and $q - 2 = (q_0 - 2)bq/q_0 \geq 2bLV_{A,\delta/2}^{2-p}/\tilde{q}$ it follows by means of the embedding theorem recalled in Lemma 2.7 that

$$\sup_{\Omega} \theta_A^{p-1}(Du^A(t)) \leq \frac{K}{\delta^b} b^{\frac{1}{q}-1} V_{A,\delta}^{p-2} V_{A,\delta/2}^{(1-1/q)(p-2)+E}. \tag{47}$$

Now recall that $K > 0$ is independent of t and that (47) holds for a.e. $t \in (\epsilon+\delta, T)$. Taking the essential supremum of (47) over this interval and dividing by $V_{A,\delta}^{p-2}$ we come to

$$V_{A,\delta} \leq \frac{K}{\delta^b} b^{\frac{1}{q}-1} V_{A,\delta/2}^{(p-2)(1-1/q)+E}. \tag{48}$$

The constant $b \in (0,1)$ can be fixed for $p \in (2,4)$ independently of A so small that $\kappa := E + (p-2)(1-\frac{1}{q}) \in (0,1)$. Then $\lambda := \frac{1}{1-\kappa} > 0$ and we rewrite (48) in the form

$$\delta^{b\lambda} V_{A,\delta} \leq K \left(\left(\frac{\delta}{2}\right)^{b\lambda} V_{A,\delta/2} \right)^{\kappa}. \tag{49}$$

As $\sup_{\delta\in(0,\min\{1,T\}-\epsilon)} \delta^{b\lambda} V_{A,\delta} < +\infty$ we can take supremum of (49) over $\delta \in (0, \min\{1,T\} - \epsilon)$ and get (39) from (42) and (46). ∎

Let us now prove Theorem 1.1 if $p \in (2,4)$. If (3), (5) and (6) hold we get for the approximations u^A the estimates (38). Hence, there are $u \in L^{\tilde{r}}(I, W^{2,r}(\Omega))$, $\pi \in L^{\tilde{r}}(I, W^{1,r}(\Omega))$ for all $\tilde{r} > 1$, $r \in (1,2)$, such that up to a subsequence $u^A \rightharpoonup u$ and $\pi^A \rightharpoonup \pi$ weakly in the corresponding spaces. Moreover, we can assume by (37) that $\partial_t u \in L^{\tilde{r}}(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$ for all $\tilde{r} > 1$ and $\partial_t u^A \rightharpoonup \partial_t u$ weakly in these spaces and by the Leray-Lions Theorem also $\nabla u^A \to \nabla u$ strongly in $L^{\tilde{r}}(Q)$ for all $\tilde{r} > 1$. Particularly, we may assume $\nabla u^A \to \nabla u$ a.e. in Q. This information allows us to pass to the limit in the weak formulation of (32) as $A \to \infty$, compare [17, Section 5.5.3], and get (7) from (37) and (38). If, moreover, (8) holds, it follows from Lemma 4.3 that for every $\epsilon \in (0, \min\{1,T\})$ the function Du is bounded on $(\epsilon, T) \times \Omega$ and that there is a $t_0 \in (\epsilon, 2\epsilon)$ such that $\|u(t_0)\|_{2,2,\Omega}$ is finite. Consequently, if $A > \text{ess-sup}\{|Du(t,x)| : t \in (\epsilon, T), x \in \Omega\}$, the functions u, π are the unique weak solution of the problem

$$\partial_t u - \operatorname{div} \mathcal{T}^A(Du) = f - \nabla\pi - \operatorname{div}(u \otimes u), \quad \operatorname{div} u = 0 \text{ on } (t_0, T) \times \Omega,$$
$$\int_\Omega \pi(t) = 0 \text{ for a.e. } t \in I, \quad u = 0 \text{ on } (t_0, T) \times \partial\Omega, \quad u(t_0) \in W^{2,2}(\Omega)$$

and (9) follows by Theorem 1.1 with $p = 2$. Theorem 1.1 is proved.

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