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On the Relationship between p-Analytic Functions and Schrödinger Equation

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Abstract. Theory of p-analytic functions was developed by G. N. Polozhy in fourties and later on by many other authors. It has numerous applications in elasticity theory and in hydrodynamics. Its main object of study is a system generalizing the Cauchy-Riemann conditions where a new factor p appears which is assumed to be a positive function. In the present work we show that all p-analytic functions can be obtained from solutions of the Schrödinger equation to which the function $f_0 = \frac{1}{\sqrt{p}}$ is a solution. The inverse is also true. We show that this relationship leads to an explicit correspondence between p_1- and p_2-analytic functions if $\frac{1}{\sqrt{p_1}}$ and $\frac{1}{\sqrt{p_2}}$ are solutions of the same Schrödinger equation. For example, if f_0 is a harmonic function then all p-analytic functions can be obtained from analytic ones. Another simple consequence of our result is that all x^k-analytic functions intensely studied in literature can be obtained directly from the Bessel equation.

Keywords: *p-analytic functions, generalized analytic functions, pseudoanalytic functions, Schrödinger equation*

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1. Introduction

A function $\Phi = u + iv$ of a complex variable $z = x + iy$ is said to be p-analytic in some domain $\Omega \subset \mathbf{C}$ iff

$$u_x = \frac{1}{p} v_y, \quad u_y = -\frac{1}{p} v_x \qquad \text{in } \Omega, \tag{1}$$

where p is a given positive function of x and y which is supposed to be continuously differentiable. In fact we will assume that it is twice differentiable. The theory of p-analytic functions was presented in [12]. p-analytic functions in

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a certain sense represent a subclass of generalized analytic (or pseudoanalytic) functions studied by L. Bers [2, 3] and I. N. Vekua [13], and it should be noticed that this subclass preserves some important properties of usual analytic functions which are not preserved by a too ample class of generalized analytic functions (corresponding details can be found in [12]). p-analytic functions found applications in elasticity theory (see, e.g., [1, 7]) and in axisymmetric problems of hydrodynamics (see, e.g., [5]).

In the present work we show that p-analytic functions are closely related to solutions of the static Schrödinger equation

$$(-\Delta + \nu)f = 0, \tag{2}$$

where for simplicity ν and f are assumed to be real valued functions of x and y (for ν being a complex valued function we are forced to consider p-analytic functions taking values in the algebra of bicomplex numbers, that is u and v are complex valued functions with another imaginary unit j commuting with i that would not require any essential changes in what follows but certainly it would introduce some unnecessary complications in notations and explanations).

The main result consists in the following observation. Let $f_0 = 1/\sqrt{p}$ be a solution of (2), then we show that any p-analytic function can be transformed into a solution of (2) and vice versa. The corresponding transformation is completely explicit and quite simple. As a corollary of this result we obtain that if the functions $1/\sqrt{p_1}$ and $1/\sqrt{p_2}$ are solutions of the Schrödinger equation (2) with the same potential ν, then an explicit correspondence between p_1 and p_2-analytic functions can be proposed. For example, if $p_1 \equiv 1$ and $f_0 = 1/\sqrt{p_2}$ is an arbitrary harmonic function, using our transformation all p_2-analytic functions can be obtained from analytic ones.

Another simple consequence of our result is that all x^k-analytic functions intensely studied in the literature (see, e.g., [4, 8, 12], can be obtained directly from the Bessel equation.

2. Relationship between generalized analytic functions and the Schrödinger equation

Denote $\partial_z = \partial_x - i\partial_y$ and $\partial_{\overline{z}} = \partial_x + i\partial_y$. Usually these operators are introduced with the factor $\frac{1}{2}$, nevertheless here it is somewhat more convenient to consider them without it.

We start with the following result obtained in [11] with the aid of quaternionic analysis methods developed in earlier works (see [10] and the bibliography therein) which can be verified by a direct substitution.

Proposition 1. *For any two particular solutions f_0 and f_1 of* (2)*, where f_0 is assumed to be nonvanishing, the function $w = f_0 \partial_z (f_0^{-1} f_1)$ is a solution of the equation*

$$w_{\overline{z}} = -\frac{\partial_z f_0}{f_0} \overline{w} \tag{3}$$

which is equivalent to the system

$$\partial_x u - \partial_y v = -\frac{\partial_x f_0}{f_0} u + \frac{\partial_y f_0}{f_0} v, \qquad \partial_y u + \partial_x v = \frac{\partial_y f_0}{f_0} u + \frac{\partial_x f_0}{f_0} v. \tag{4}$$

Let us introduce the following operator $P = f_0 \partial_z f_0^{-1} I$, where I is the identity operator. Due to Proposition 1, if f_0 is a nonvanishing solution of (2), the operator P transforms solutions of (2) into solutions of (3).

Note that the operator ∂_z applied to a real valued function φ can be regarded as a kind of gradient, and if we know that $\partial_z \varphi = \Phi$ in a whole complex plane or in a convex domain, where $\Phi = \Phi_1 + i\Phi_2$ is a given complex valued function such that its real part Φ_1 and imaginary part Φ_2 satisfy the equation

$$\partial_y \Phi_1 + \partial_x \Phi_2 = 0, \tag{5}$$

then we can reconstruct φ up to an arbitrary real constant C in the following way:

$$\varphi(x, y) = \int_{x_0}^{x} \Phi_1(\eta, y) d\eta - \int_{y_0}^{y} \Phi_2(x_0, \xi) d\xi + C, \tag{6}$$

where (x_0, y_0) is an arbitrary fixed point in the domain of interest. By A we denote the integral operator in (6):

$$A[\Phi](x, y) = \int_{x_0}^{x} \Phi_1(\eta, y) d\eta - \int_{y_0}^{y} \Phi_2(x_0, \xi) d\xi + C.$$

Note that formula (6) can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve Γ leading from (x_0, y_0) to (x, y)

$$\varphi(x, y) = \int_{\Gamma} \Phi_1 dx - \Phi_2 dy + C.$$

Thus if Φ satisfies (5), there exists a family of real valued functions φ such that $\partial_z \varphi = \Phi$, given by the formula $\varphi = A[\Phi]$.

Consider the operator $S = f_0 A f_0^{-1} I$. It is clear that $PS = I$.

Proposition 2. *Let f_0 be a nonvanishing particular solution of* (2) *and w be a solution of* (3)*. Then the function $f = Sw$ is a solution of* (2).

Proof. First of all let us check that the function $\Phi = w/f_0$ satisfies (5). Consider

$$\partial_y \Phi_1 + \partial_x \Phi_2 = \frac{1}{f_0}\left((\partial_y u + \partial_x v) - \Big(\frac{\partial_y f_0}{f_0} u + \frac{\partial_x f_0}{f_0} v \Big) \right).$$

From (4) we obtain that this expression is zero. Thus the function $\varphi = A[w/f_0]$ is real valued and satisfies the equation $\partial_z \varphi = w/f_0$. Let us consider the expression

$$\begin{aligned} \partial_{\overline{z}} \partial_z (Sw) &= \partial_{\overline{z}} \left((\partial_z f_0)\, A\Big[\frac{w}{f_0}\Big] + w \right) \\ &= (\Delta f_0)\, A\Big[\frac{w}{f_0}\Big] + (\partial_z f_0)\, \partial_{\overline{z}} A\Big[\frac{w}{f_0}\Big] - \frac{\partial_z f_0}{f_0} \overline{w}. \end{aligned} \tag{7}$$

For the expression $\partial_{\overline{z}} A[\frac{w}{f_0}]$ we have

$$\begin{aligned} \partial_{\overline{z}} A\Big[\frac{w}{f_0}\Big] &= \partial_z A\Big[\frac{w}{f_0}\Big] + 2i \partial_y A\Big[\frac{w}{f_0}\Big] \\ &= \frac{w}{f_0} - 2i \frac{v}{f_0} = \frac{\overline{w}}{f_0}, \end{aligned} \tag{8}$$

where the following observation was used:

$$\begin{aligned} \partial_y A\left[\frac{u+iv}{f_0}\right](x,y) &= \int_{x_0}^{x} \partial_y \left(\frac{u(\eta,y)}{f_0(\eta,y)} \right) d\eta - \frac{v(x_0,y)}{f_0(x_0,y)} = \\ &= -\int_{x_0}^{x} \partial_\eta \left(\frac{v(\eta,y)}{f_0(\eta,y)} \right) d\eta - \frac{v(x_0,y)}{f_0(x_0,y)} = -\frac{v(x,y)}{f_0(x,y)}. \end{aligned}$$

Thus substitution of (8) into (7) gives us the equality

$$\partial_{\overline{z}} \partial_z (Sw) = \nu f_0 A\Big[\frac{w}{f_0}\Big] = \nu S w. \qquad \blacksquare$$

Proposition 3. *Let f be a solution of* (2)*. Then*

$$SPf = f + Cf_0,$$

where C is an arbitrary real constant.

Proof. Consider

$$SPf = f_0 A \partial_z \Big[\frac{f}{f_0}\Big] = f_0 \Big(\frac{f}{f_0} + C \Big) = f + Cf_0. \qquad \blacksquare$$

3. p-analytic functions and the Schrödinger equation

Together with equation (3) let us consider the equation

$$W_{\overline{z}} = -\frac{\partial_{\overline{z}} f_0}{f_0}\overline{W}. \tag{9}$$

Notice that the functions

$$F = \frac{1}{f_0} \qquad \text{and} \qquad G = i f_0 \tag{10}$$

are solutions of (9), and $\mathrm{Im}(\overline{F}G) > 0$. Then F and G can be regarded as a Bers generating pair for equation (9).

Solutions of (3) and (9) are closely related to each other. In order to see it we need the concept of the (F,G)-derivative introduced by L. Bers (see, e.g., [2]). For solutions of equation (9) the (F,G)-derivative (denoted by $\dot{W}$) takes the form

$$\dot{W} = W_z + \frac{\partial_z f_0}{f_0}\overline{W}.$$

The following statement can be checked by a direct substitution.

Proposition 4. [11] *Let W be a solution of* (9)*. Then the function $w = i\dot{W}$ is a solution of* (3)*.*

The (F,G)-integral is defined as follows [2]:

$$\int_\Gamma W d_{(F,G)} z = F(z_1)\,\mathrm{Re}\int_\Gamma G^* W dz + G(z_1)\,\mathrm{Re}\int_\Gamma F^* W dz,$$

where Γ is a rectifiable curve leading from z_0 to z_1,

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

For F and G defined by (10) we have

$$F^* = -\frac{i}{f_0}, \qquad G^* = f_0.$$

Due to L. Bers, if $W = uF + vG$ is an (F,G)-pseudoanalytic function where u and v are real valued functions, then

$$\int_{z_0}^{z} \dot{W} d_{(F,G)} z = W(z) - u(z_0)F(z) - v(z_0)G(z), \tag{11}$$

and as $\dot{F} = \dot{G} = 0$, this integral is nothing but the (F,G)-antiderivative of $\dot{W}$.

For F and G defined by (10) we have

$$\int_\Gamma W d_{(F,G)} z = \frac{1}{f_0(z_1)} \operatorname{Re} \int_\Gamma f_0 W dz - i f_0(z_1) \operatorname{Re} \int_\Gamma \frac{i}{f_0} W dz.$$

Moreover, due to Proposition 4 and equation (11) we obtain the following result.

Proposition 5. *Let w be a solution of (3). Then the function*

$$W(z) = -\int_{z_0}^{z} i w d_{(F,G)} z = -\frac{1}{f_0(z)} \operatorname{Re} \int_{z_0}^{z} i f_0 w dz - i f_0(z) \operatorname{Re} \int_{z_0}^{z} \frac{w}{f_0} dz$$

is a solution of (9).

Any (F,G)-pseudoanalytic function [2] can be represented in the form $W = uF + vG$, where u and v are real valued functions satisfying the equation

$$u_{\overline{z}} F + v_{\overline{z}} G = 0.$$

In the case of equation (9) the functions F and G are given by (10) and thus any solution of (9) can be represented in the form $W = \frac{u}{f_0} + i v f_0$, where u and v are real valued functions satisfying the equation

$$u_{\overline{z}} + i f_0^2 v_{\overline{z}} = 0.$$

This equation is equivalent to the system

$$u_x = f_0^2 v_y, \qquad u_y = -f_0^2 v_x$$

which compared to (1) shows us that the function $\Phi = u + iv$ is p-analytic with $p = 1/f_0^2$. Thus we obtain the following result.

Theorem 1. *Let f_0 be a nonvanishing particular solution of (2)and $\Phi = u + iv$ be an $\frac{1}{f_0^2}$-analytic function. Then $W = \frac{u}{f_0} + i v f_0$ is a solution of (9), $w = i\dot{W} = i(W_z + \frac{\partial_z f_0}{f_0}\overline{W})$ is a solution of (3) and $f = Sw$ is a solution of (2).*

We have an inverse result also.

Theorem 2. *Let f_0 be a nonvanishing particular solution of (2) and f be another solution of (2). Then $w = Pf$ is a solution of (3), the function $W(z) = -\int_{z_0}^{z} i w d_{(F,G)} z$ is a solution of (9), and $\Phi = f_0 \operatorname{Re} W + \frac{i}{f_0} \operatorname{Im} W$ is a $\frac{1}{f_0^2}$-analytic function.*

Thus due to Theorem 1 we are able to convert p-analytic functions into solutions of the Schrödinger equation and due to Theorem 2 we have an inverse result.

Remark 1. A considerable part of bibliography dedicated to p-analytic functions consists of studying the case $p = x^k$, where $k \in \mathbb{R}$ (see, e.g., [4], [8], [12]). Let us see what is the form of the corresponding Schrödinger equation. For this we should calculate the potential ν in (2) when $f_0 = 1/\sqrt{p} = x^{-k/2}$ is a particular solution of (2). It is easy to see that

$$\nu = \frac{k^2 - 2k}{4x^2}. \tag{12}$$

The Schrödinger equation with this potential is well studied. Separation of variables leads us to the equation

$$X''(x) + \left(\beta^2 - \frac{4\alpha^2 - 1}{4x^2}\right) X(x) = 0, \tag{13}$$

where β^2 is the separation constant and $\alpha = (k-1)/2$. The function

$$X(x) = \sqrt{x} Z_\alpha(\beta x)$$

is a solution of (13) (see [6, 8.491]), where Z_α denotes any cylindric function of order α (Bessel functions of first or second kind). Thus the study of x^k-analytic functions reduces to the Schrödinger equation (2) with ν defined by (12) which in its turn after having separated variables reduces to a kind of Bessel equation (13).

Remark 2. In the work [9] boundary value problems for p-analytic functions with $p = x/(x^2 + y^2)$ were studied. Considering

$$f_0 = \frac{1}{\sqrt{p}} = \sqrt{\frac{x^2 + y^2}{x}}$$

we see that this function is a solution of the Schrödinger equation (2) with ν having the form $\nu = \frac{3}{4x^2}$, that is we obtain again the potential of the form (12) where $k = 3$ or $k = -1$ and as was shown in the previous Remark the study of corresponding p-analytic functions in a sense reduces to the Bessel equation (13).

Remark 3. Let us indicate another interesting consequence of Theorems 1 and 2. Let p_1 and p_2 be such that $p_1^{-1/2}$ and $p_2^{-1/2}$ are solutions of the Schrödinger equation (2) with the same potential ν. Then our results allow us to transform p_1-analytic functions into p_2-analytic ones and vice versa. Take any p_1-analytic function $\Phi_1 = u_1 + iv_1$. Then according to Theorem 1 the function $W_1 = \sqrt{p_1} u_1 + \frac{iv_1}{\sqrt{p_1}}$ is a solution of (9) where $f_0 = 1/\sqrt{p_1}$, the function $w_1 = i\dot{W}_1$ is a solution of (3) with the same f_0 and $f = Sw_1 = \frac{1}{\sqrt{p_1}} A(\sqrt{p_1} w_1)$ is a solution of (2), where by assumption

$$\nu = \frac{\Delta p_1^{-\frac{1}{2}}}{p_1^{-\frac{1}{2}}} = \frac{\Delta p_2^{-\frac{1}{2}}}{p_2^{-\frac{1}{2}}}.$$

Then due to Theorem 2, $w_2 = Pf = \frac{1}{\sqrt{p_2}} \partial_z (\sqrt{p_2} f)$ is a solution of (3) where $f_0 = 1/\sqrt{p_2}$, the function $W_2(z) = -\int_{z_0}^{z} i w_2 d_{(F,G)} z$, where $F = \sqrt{p_2}$ and $G = i/\sqrt{p_2}$, is a solution of (9), and finally $\Phi_2 = \frac{1}{\sqrt{p_2}} \operatorname{Re} W_2 + i\sqrt{p_2} \operatorname{Im} W_2$ is a p_2-analytic function.

Example 1. Let $p_1 \equiv 1$, that is p_1-analytic functions are simply analytic. Obviously $\nu = 0$ and we are able to obtain representations of p_2-analytic functions in terms of analytic ones for any p_2 such that $p_2^{-1/2}$ is harmonic. For instance, let us choose as p_2 the function

$$p_2 = \frac{1}{(x+a)^2 (y+b)^2}$$

where a and b are positive constants. Then the function $f_0 = 1/\sqrt{p_2} = (x + a)(y + b)$ is harmonic, and we are able to transform analytic functions into p_2-analytic and vice versa. Take an arbitrary analytic function $\Phi_1 = u_1 + iv_1$. As $p_1 \equiv 1$, we have $W_1 = \Phi_1$ and $w_1 = i\dot{W}_1 = i\partial_z \Phi_1$ is analytic also. Consider

$$f = Sw_1 = A\big[i\partial_z \Phi_1\big] = A\big[\partial_y u_1 - \partial_x v_1 + i(\partial_x u_1 + \partial_y v_1)\big].$$

Due to the fact that the pair u_1 and v_1 satisfies the Cauchy-Riemann conditions we have

$$\partial_y u_1 - \partial_x v_1 + i(\partial_x u_1 + \partial_y v_1) = -2(\partial_x v_1 - i\partial_y v_1) = -2\partial_z v_1.$$

Thus

$$f = -2A[\partial_z v_1] = -2(v_1 + C),$$

where C is an arbitrary real constant. It is obvious that this function is harmonic. Now in order to proceed with the example, let us choose a concrete harmonic function f, for example, $f = x$. Let us construct the corresponding p_2-analytic function. We have

$$\begin{aligned} w_2 = Pf &= \frac{1}{\sqrt{p_2}} \partial_z (\sqrt{p_2} f) \\ &= (x+a)(y+b)\partial_z \left(\frac{x}{(x+a)(y+b)} \right) = \frac{a}{(x+a)} + \frac{ix}{(y+b)}. \end{aligned}$$

Now consider the function $W_2(z) = -\int_{z_0}^{z} iw_2 d_{(F,G)}z$. In order to simplify our calculations let us take $z_0 = 0$. Then by definition of the (F,G)-integral we obtain

$$\begin{aligned}
&W_2(z)\\
&= \frac{1}{(x+a)(y+b)} \operatorname{Re} \int_0^z (x'+a)(y'+b) \left(\frac{x'}{(y'+b)} - \frac{ia}{(x'+a)} \right) dz' \\
&\quad - i(x+a)(y+b) \operatorname{Re} \int_0^z \frac{i}{(x'+a)(y'+b)} \left(\frac{x'}{(y'+b)} - \frac{ia}{(x'+a)} \right) dz' \\
&= \frac{1}{(x+a)(y+b)} \operatorname{Re} \int_0^1 \left(xt(xt+a) - ia(yt+b)\right)(x+iy)dt \\
&\quad - i(x+a)(y+b) \operatorname{Re} \int_0^1 \left(\frac{a}{(xt+a)^2(yt+b)} + \frac{ixt}{(xt+a)(yt+b)^2} \right)(x+iy)dt \\
&= \frac{1}{(x+a)(y+b)} \int_0^1 \left(x^2t(xt+a) + ay(yt+b)\right) dt \\
&\quad - i(x+a)(y+b) \int_0^1 \left(\frac{ax}{(xt+a)^2(yt+b)} - \frac{xyt}{(xt+a)(yt+b)^2} \right) dt.
\end{aligned}$$

Now calculating the integrals

$$\int_0^1 \frac{dt}{(xt+a)^2(yt+b)} = -\frac{x}{a(x+a)(ay-bx)} + \frac{y}{(ay-bx)^2} \ln \left| \frac{a(y+b)}{b(x+a)} \right|$$

$$\int_0^1 \frac{tdt}{(xt+a)(yt+b)^2} = -\frac{1}{(ay-bx)(y+b)} - \frac{a}{(ay-bx)^2} \ln \left| \frac{b(x+a)}{a(y+b)} \right|$$

we obtain

$$\begin{aligned}
W_2 &= \frac{x^3}{3(x+a)(y+b)} + \frac{a(x^2+y^2)}{2(x+a)(y+b)} + \frac{aby}{(x+a)(y+b)} \\
&\quad - i(x+a)(y+b) \left(-\frac{x^2}{(x+a)(ay-bx)} + \frac{xy}{(ay-bx)(y+b)} \right) \\
&= \frac{x^3}{3(x+a)(y+b)} + \frac{a\left(x^2+y^2+2by\right)}{2(x+a)(y+b)} - ix.
\end{aligned}$$

Finally we have that

$$\Phi_2 = \frac{1}{\sqrt{p_2}} \operatorname{Re} W_2 + i\sqrt{p_2} \operatorname{Im} W_2 = \frac{a}{2}(x^2+y^2+2by) + \frac{x^3}{3} - \frac{ix}{(x+a)(y+b)}$$

is the p_2-analytic function corresponding to the harmonic function $f = x$.

It is clear that the results presented in this work represent only some first corollaries of the revealed relation between p-analytic functions and solutions of the Schrödinger equation. Both fields are so rich that without doubts this new bridge will serve for further advances in the two theories.

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