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L^p-Theory for Elliptic Operators on $\mathbb{R}^d$ with Singular Coefficients

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Abstract. We study the generation of an analytic semigroup in $L^p(\mathbb{R}^d)$ and the determination of the domain for a class of second order elliptic operators with unbounded coefficients in $\mathbb{R}^d$. We also establish the maximal regularity of type L^q–L^p for the corresponding inhomogeneous parabolic equation. In contrast to the previous literature the coefficients of the second derivatives are not required to be strictly elliptic or bounded. Interior singularities of the lower order terms are also discussed.

Keywords: *Unbounded and degenerate coefficients, analytic semigroups, maximal regularity*

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1. Introduction

Regularity properties of elliptic operators

$$Au(x) = \operatorname{div}(a(x)\nabla u(x)) + F(x) \cdot \nabla u(x) - V(x)u(x), \quad x \in \mathbb{R}^d,$$

(at first defined on the test function space $C_0^\infty(\mathbb{R}^d)$) with unbounded coefficients on $\mathbb{R}^d$ have intensively been investigated in recent years. Besides the traditional applications to Schrödinger equations, this line of research is motivated by the fact that such operators A arise as generators of transition semigroups in stochastic analysis (possibly after some transformations, [9], [25]). In this

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paper we establish L^p–estimates for the elliptic and parabolic problems associated with A. These estimates are closely related to the property that A with the domain

$$D_p = \{u \in W^{2,p}_{loc}(\mathbb{R}^d) : u, \operatorname{div}(a\nabla u), Vu \in L^p(\mathbb{R}^d)\}$$

generates a positive, contractive, and analytic C_0–semigroup on $L^p(\mathbb{R}^d)$, $1 < p < \infty$.

Since we do not assume that the coefficients of A are bounded in $\mathbb{R}^d$, the classical theory of elliptic equations does not apply. Nevertheless, nowadays many generation results are available for elliptic operators with unbounded coefficients in $L^p(\mathbb{R}^d)$, including the enormous literature on Schrödinger operators, corresponding to $a_{ij} = \delta_{ij}$ and $F = 0$. In particular, it is known that an extension of $(A, C_0^\infty(\mathbb{R}^d))$ generates a C_0–semigroup on $L^p(\mathbb{R}^d)$ (which is not necessarily analytic) if the dissipativity condition $pV + \operatorname{div} F \geq 0$ holds. Recent and quite general results in this direction are presented in [23] and [28] using form methods; see also [3] for a different approach based on an approximation procedure.

However, the determination of the domain is a quite different question which requires more assumptions on the coefficients. This problem has been treated supposing that the diffusion coefficients a_{ij} belong to $C_b^1(\mathbb{R}^d)$ and are strictly elliptic. In this case the diffusion part

$$A_0 u(x) = \operatorname{div}(a(x)\nabla u(x)), \quad x \in \mathbb{R}^d,$$

satisfies the classical Calderón–Zygmund estimates, so that $D(A_0) = W^{2,p}(\mathbb{R}^d)$ and A_0 can be controlled by the Laplacian. In this setting the domain of A was computed, e.g., in [4], [5], [7], [9], [25] under similar assumptions as in the present paper; see also the references therein and in particular [8] for further results. In these papers it was also assumed that the lower order coefficients have no singularities inside $\mathbb{R}^d$.

In the case of unbounded and non–strictly elliptic a_{ij}, we are only aware of the domain characterizations given in [18] and [19]. In these papers it was supposed that the coefficients a_{ij} have a special structure. In the present paper we study diffusion coefficients which may be unbounded or degenerate at ∞ without restrictions on their structure, and we also allow for singularities of the lower order coefficients inside $\mathbb{R}^d$. In order to facilitate the understanding, we first treat the case where F and V have no singularities in $\mathbb{R}^d$ in Sections 2 and 3. In contrast to our previous work [25], now the diffusion part A_0 cannot be controlled by the Laplacian anymore. In order to overcome this difficulty, we had to develop various new arguments and to treat the cases $p \leq 2$ and $p > 2$ separately. The case of singular F and V poses further technical problems and requires additional approximation procedures which are presented in the last section.

We first introduce our assumptions for the case without singularities of V and F. We assume that the coefficients of A satisfy the following hypotheses, where $1 < p < \infty$ is given and the scalar products corresponding to the matrices $a(x)$ are denoted by

$$a_0(\xi, \eta) = \sum_{k,l=1}^{d} a_{kl}(x)\xi_k\overline{\eta}_l \quad \text{and} \quad a_0[\xi] := a_0(\xi, \xi) \qquad (x \in \mathbb{R}^d,\ \xi, \eta \in \mathbb{C}^d).$$

(H1) The real-valued functions $a_{kl} \in C^1(\mathbb{R}^d)$ satisfy $a_{kl} = a_{lk}$ for $k, l = 1, \cdots, d$ and

$$a_0[\xi] = \sum_{k,l=1}^{d} a_{kl}(x)\xi_k\xi_l > 0$$

for all x, $\xi \in \mathbb{R}^d$ with $\xi \neq 0$.

(H2) The function $V : \mathbb{R}^d \to \mathbb{R}$ is measurable, and there is a function $U \in C^1(\mathbb{R}^d)$ such that $c_0 \leq U \leq V \leq c_1 U$ and $a_0[\nabla U]^{\frac{1}{2}} \leq \gamma U^{\frac{3}{2}} + C_\gamma$ for some constants $c_1, c_0, \gamma > 0$ and $C_\gamma \geq 0$.

(H3) The function $F \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ satisfies $|F \cdot \xi| \leq \kappa U^{\frac{1}{2}} a_0[\xi]^{\frac{1}{2}}$ for some constant $\kappa > 0$.

(H4) There is a constant $\theta < p$ such that $\theta U + \operatorname{div} F \geq 0$.

Condition (H1) already implies that an extension of $(A_0, C_0^\infty(\mathbb{R}^d))$ in $L^p(\mathbb{R}^d)$ generates a contractive (and analytic) C_0–semigroup on $L^p(\mathbb{R}^d)$, for all $1 < p < \infty$; see, e.g., the form-method approach in [10]. Under further assumptions, the closure of $(A_0, C_0^\infty(\mathbb{R}^d))$ generates a contractive C_0–semigroup on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. The closure is again denoted by A_0. It is not difficult to see that in this case the closure is the only generator extending $(A_0, C_0^\infty(\mathbb{R}^d))$. Moreover, the domain of the closure is given by

$$D(A_0) = \{u \in L^p(\mathbb{R}^d) \cap W^{2,p}_{loc}(\mathbb{R}^d) : \operatorname{div}(a\nabla u) \in L^p(\mathbb{R}^d)\}$$

(see Lemma 2.1 below). In our paper we will assume a rather sharp condition for the property that test functions are a core for A_0 (see Theorem 2.3 and Section 2.b.1 of [15], and the Remark after Lemma 2.1):

(H5) There is a constant $\tau > 0$ such that

$$\sum_{k,l=1}^{d} a_{kl}(x)x_kx_l \leq \tau|x|^4(\log|x|)^2, \quad |x| \geq 1.$$

To state our main results, we introduce the subspace

$$D_p := D(A_0) \cap D(V) = \{u \in W^{2,p}_{loc}(\mathbb{R}^d) : u, A_0u, Vu \in L^p(\mathbb{R}^d)\}$$

for $1 < p < \infty$. Since A_0 and V are closed operators, D_p is a Banach space endowed with the norm

$$\|u\|_{D_p} = \|u\|_p + \|A_0u\|_p + \|Vu\|_p.$$

Let us first suppose that either $F = 0$ (and $1 < p < \infty$) or $1 < p \leq 2$. For these cases we show in Section 2 that (A, D_p) generates an analytic semigroup in $L^p(\mathbb{R}^d)$, under assumptions (H1), (H2), (H3), (H4), (H5), and (2.4).

As mentioned above, (H1) and (H5) take care of the diffusion part, and (H4) implies the dissipativity of A. Hypothesis (H3) allows to control the drift term $F \cdot \nabla u$ by A_0u and Vu, due to the interpolation Lemma 2.5. But we point out that the drift is not a small perturbation of A_0 or $A_0 - V$, cf. Remark 3.6 in [25].

The oscillation condition (H2) (together with the bound (2.4) on γ) plays a the central role in the identification of the domain of A. It was already used in [12] and [13] to show that the domain of the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$ coincides with $W^{2,2}(\mathbb{R}^d) \cap D(V)$ both for smooth and singular potentials. There are counterexamples where this domain characterization fails and (H2) is true with a too large γ, see [11, Note 22], [25, Example 3.7]. The operator $\Delta - V$ was studied in $L^p(\mathbb{R}^d)$ in the papers [26] and [27] also under assumption (H2).

We remark that in (H2) the auxiliary potential U is introduced to obtain more flexible assumptions for V. In Section 7 of [25] we have used this freedom to treat a class of Ornstein–Uhlenbeck type operators on an L^p–space with a weighted measure. Changing the measure to the usual Lebesgue measure, we obtained an operator with nonzero potential V which in fact dominates the resulting new drift term, as required by (H3).

The arguments used in Section 2 are based on variational estimates combined with methods from semigroup theory. In the case $p > 2$, the variational estimates are not sufficient anymore to control the drift term by the diffusion part and the potential as in Lemma 2.5. This problem is solved in Proposition 3.3 with considerable efforts, employing a lengthy localization/covering procedure. Unfortunately, this method requires a stronger version of (H2) and additional estimates which control the growth and the oscillation of the matrix a by means of the potential U:

(H2') The function $V : \mathbb{R}^d \to \mathbb{R}$ is measurable, and there is a function $U \in C^1(\mathbb{R}^d)$ such that, setting

$$\rho(x) = \frac{|a(x)|^{\frac{1}{2}}}{U(x)^{\frac{1}{2}}} \quad \text{and} \quad \nu(x) = \inf_{\xi \in \mathbb{R}^d, |\xi|=1} \sum_{k,l=1}^{d} a_{kl}(x)\xi_k\xi_l \quad \text{for} \quad x \in \mathbb{R}^d,$$

the conditions

$$c_0 \le U \le V \le c_1 U, \quad |a(x)|^{\frac{1}{2}}\,|\nabla U(x)| \le \gamma U(x)^{\frac{3}{2}} + C_\gamma, \tag{1.1}$$

$$|a(x)| \le c_2 U(x), \qquad \sup_{|y-x|\le\rho(x)} |\nabla a(y)| \le c_2\,\nu(x)\,|a(x)|^{-\frac{1}{2}}\,U(x)^{\frac{1}{2}} \tag{1.2}$$

hold for $x \in \mathbb{R}^d$ and some constants $c_j, \gamma, C_\gamma \ge 0$.

Condition (H2') again allows to interpolate the term $F \cdot \nabla u$ between $A_0 u$ and Vu. So we can adopt the arguments from Section 2 in order to establish in Theorem 3.5 that A with domain D_p generates an analytic semigroup also if $p > 2$ provided (H1), (H2'), (H3), (H4), (H5) and (2.4) hold. If a is strictly elliptic, in addition, then the domain D_p is continuously embedded into $W^{2,p}(\mathbb{R}^d)$, $1 < p < \infty$, see Proposition 3.6.

Using certain regularizations of V and F, we can modify our approach to obtain essentially the same theorems if V and F are singular at 0 and satisfy the hypotheses on $\mathbb{R}^d \setminus \{0\}$, see Section 4. However, in the case $p > 2$ the regularization procedure is rather involved, since the regularized coefficients do not satisfy (1.2), in general.

The result that A with domain $D(A) = D_p$ generates an analytic emigroup on $L^p(\mathbb{R}^d)$ has many immediate consequences for the regularity properties of the parabolic problem

$$\begin{cases} \partial_t u(t,x) = Au(t,x) + f(t,x), & t > 0, x \in \mathbb{R}^d \\ \ \ u(0,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

see, e.g., [24]. For instance, the solution u belongs to $W^{1,q}([0,T], L^p(\mathbb{R}^d))$ and $A_0 u, Vu \in L^q([0,T], L^p(\mathbb{R}^d))$ for all $T > 0$, if $f = 0$ and φ belongs to the real interpolation space $(L^p(\mathbb{R}^d), D_p)_{1-1/q,q}$ for some $q \in (1,\infty)$. If $\varphi \in D_p$, then $u \in C^1(\mathbb{R}_+, L^p(\mathbb{R}^d))$ and $A_0 u, Vu \in C(\mathbb{R}_+, L^p(\mathbb{R}^d))$. In Lunardi's monograph [24] one finds plenty of regularity results for u if f is, e.g., Hölder continuous in time. In addition, our results yield *maximal regularity of type* L^q–L^p for A, i.e., for all $f \in L^q([0,T], L^p(\mathbb{R}^d))$ and $\varphi = 0$, we have $u \in W^{1,q}([0,T], L^p(\mathbb{R}^d))$ and $A_0 u, Vu \in L^q([0,T], L^p(\mathbb{R}^d))$, see Theorems 3.7 and 4.4. We refer to [2], [14], and [22] for comprehensive accounts of the theory of maximal regularity, though we will not need the (quite involved) recent theorems presented in [14] and [22].

Finally, let us point out that for $p > 2$ the operators satisfying our hypotheses with bounded V must have bounded coefficients, because of (1.2). For $1 < p \le 2$, we present a simple example of an operator satisfying (H1) – (H5) with unbounded diffusion matrix and drift coefficients but bounded potential:

$$Au(x) = \sum_{k=1}^{N} D_k\left((1 + x_k^2) D_k u(x)\right) + \sum_{k,l=1}^{N} b_{kl} x_l D_k u(x) - cu(x)$$

with real constants b_{kl} and c (where c is sufficiently large).

2. The cases $F = 0$ and $1 < p \leq 2$

In this section we prove our generation theorem in the cases $F = 0$ and $1 < p < \infty$ or for any F and $1 < p \leq 2$. However, some of the auxiliary results will be also valid for $p > 2$ and $F \neq 0$.

We first show that $C_0^\infty(\mathbb{R}^d)$ is a core for the Schrödinger operator $A_0 - V$ if (H1) and (H5) hold. Observe that $V = 0$ is allowed in the next lemma.

Lemma 2.1. *Assume that* (H1) *and* (H5) *hold,* $1 < p < \infty$*, and that* $0 \leq V \in L^\infty_{loc}(\mathbb{R}^d)$*. Then the operator* $A_0 - V$ *defined on* $D(A_0 - V) = \{u \in L^p(\mathbb{R}^d) \cap W^{2,p}_{loc}(\mathbb{R}^d) : (A_0 - V)u \in L^p(\mathbb{R}^d)\}$ *generates a contractive* C_0*–semigroup on* $L^p(\mathbb{R}^d)$*. Moreover,* $C_0^\infty(\mathbb{R}^d)$ *is a core for* $A_0 - V$*.*

Proof. One verifies as in the proof of Theorem 2.3 of [15] that $(A_0 - V, C_0^\infty(\mathbb{R}^d))$ possesses a closure (denoted by $A_0 - V$) which generates a contractive C_0–semigroup on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. (In fact, at this point in [15] it is assumed that $V = 0$ and $1 < p \leq 2$, but the proof can be modified in a straightforward way if $V \neq 0$ and/or $p > 2$.) It is clear that $A_0 - V$ is a restriction of the part A_{max} in $L^p(\mathbb{R}^d)$ of the distributional operator $\operatorname{div} a\nabla - V$. By standard elliptic regularity, see [1], the domain of A_{max} is given by

$$D(A_{max}) = \{u \in L^p(\mathbb{R}^d) \cap W^{2,p}_{loc}(\mathbb{R}^d) : \operatorname{div}(a\nabla u) - Vu \in L^p(\mathbb{R}^d)\}.$$

If $(I - A_{max})u = 0$ for some $u \in D(A_{max})$, then $\langle u, v - \operatorname{div}(a\nabla v) + Vv\rangle = 0$ for all test functions v, where the brackets denote the duality of L^p and $L^{p'}$. Since the closure of $(A_0 - V, C_0^\infty(\mathbb{R}^d))$ also generates a contractive C_0–semigroup on $L^{p'}(\mathbb{R}^d)$, the set of the functions $v - \operatorname{div}(a\nabla v) + Vv$ with $v \in C_0^\infty(\mathbb{R}^d)$ is dense in $L^{p'}(\mathbb{R}^d)$. So we obtain that $u = 0$; hence, $I - A_{max}$ is injective on $D(A_{max})$. This fact implies that $D(A_0 - V) = D(A_{max})$. ∎

Remark. We refer the reader to [21] for more general conditions under which $(A_0 - V, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint in $L^2(\mathbb{R}^d)$. It is not difficult to generalize these results to $1 < p < \infty$. So one obtains weaker conditions than (H5) under which the above lemma holds. However, these more general assumptions require a control of the growth of a through the potential V and reduce to (H5) if V is bounded. Since we need Lemma 2.1 also when $V = 0$, i.e., for the operator A_0, we are forced to retain (H5). We note that (H5) is almost optimal for the case $V = 0$, see [11, Example 3.5] or [15, Section 2.b.1].

We next want to show that A is *regularly dissipative*, that is, for some $\phi \in (0, \pi/2)$ the operators $e^{\pm i\phi}A$ are dissipative. This property is clearly equivalent to the estimate (2.3) below (with $\delta = \cot\phi$).

Proposition 2.2. *Let* $1 < p < \infty$ *and assume that* (H1), (H3), *and* (H4) *are satisfied with* U *replaced by* $V \in L^p_{loc}(\mathbb{R}^d)$. *Then the operator* A *defined on* $C_0^\infty(\mathbb{R}^d)$ *is regularly dissipative in* $L^p(\mathbb{R}^d)$, *with angle* $\phi_p > 0$ *only depending on* p *and the constants in* (H3) *and* (H4).

Proof. Let $u \in C_0^\infty(\mathbb{R}^d)$ and, at first, $2 \le p < \infty$. Set $u^* = \overline{u}\,|u|^{p-2}$. Observe that $\nabla u^* = \overline{u}\,|u|^{p-4}\,((p-1)\operatorname{Re}[\overline{u}\nabla u] - i\operatorname{Im}[\overline{u}\nabla u])$ and $u^*\nabla u = \frac{1}{p}(\nabla|u|^p) + i\operatorname{Im}(\overline{u}\nabla u)|u|^{p-2}$. Integrating by parts and using (H4), we calculate

$$\begin{aligned}
&-\operatorname{Re}\int_{\mathbb{R}^d}(Au)\,u^*\,dx\\
&\qquad= (p-1)\int_{\mathbb{R}^d}|u|^{p-4}\,a_0[\operatorname{Re}\overline{u}\nabla u]\,dx\\
&\qquad\quad+\int_{\mathbb{R}^d}|u|^{p-4}\,a_0[\operatorname{Im}\overline{u}\nabla u]\,dx+\int_{\mathbb{R}^d}(V+\tfrac{1}{p}\operatorname{div}F)|u|^p\,dx\\
&\qquad\ge (p-1)b^2+c^2+(1-\tfrac{\theta}{p})d^2,
\end{aligned}\tag{2.1}$$

where we define $b^2 = \int |u|^{p-4}\,a_0[\operatorname{Re}\overline{u}\nabla u]\,dx$, $c^2 = \int |u|^{p-4}\,a_0[\operatorname{Im}\overline{u}\nabla u]\,dx$, and $d^2 = \int V\,|u|^p\,dx$. Similarly, employing the Cauchy–Schwarz inequality and (H3), we estimate

$$\begin{aligned}
\left|\operatorname{Im}\int_{\mathbb{R}^d}(Au)u^*\,dx\right| &\le |p-2|\int_{\mathbb{R}^d}|u|^{p-4}\,|a_0(\operatorname{Re}(\overline{u}\nabla u),\operatorname{Im}(\overline{u}\nabla u))|\,dx\\
&\quad+\int_{\mathbb{R}^d}|F\cdot\operatorname{Im}(\overline{u}\nabla u)|\,|u|^{p-2}\,dx\\
&\le |p-2|\,bc+\kappa\int_{\mathbb{R}^d}V^{\frac12}|u|^{\frac{p}{2}}\,a_0[\operatorname{Im}\overline{u}\nabla u]^{\frac12}|u|^{\frac{p}{2}-2}\,dx\\
&\le |p-2|\,bc+\kappa cd.
\end{aligned}\tag{2.2}$$

Taking $\delta_p = \delta$ such that $\delta^2 = \frac{|p-2|^2}{4(p-1)} + \frac{\kappa^2}{4(1-\theta/p)}$, we see that

$$\left|\operatorname{Im}\int_{\mathbb{R}^d}(Au)u^*\,dx\right| \le -\delta\operatorname{Re}\int_{\mathbb{R}^d}(Au)u^*\,dx. \tag{2.3}$$

This shows the assertion for $p \ge 2$. If $p \in (1,2)$, we replace $|u|$ by $u_\varepsilon = \sqrt{|u|^2+\varepsilon}$ for $\varepsilon > 0$ in the calculations involving A_0. Passing to the limit as $\varepsilon \to 0$ and using Fatou's lemma, one then establishes (2.1) and (2.2) (in particular, all integrands are integrable). Thus one can deduce (2.3) as above. ∎

Lemma 2.3. *Let* $1 < p < \infty$. *Assume that* (H1)–(H4) *hold with*

$$\frac{\theta}{p} + (p-1)\gamma\left[\frac{\kappa}{p}+\frac{\gamma}{4}\right] < 1. \tag{2.4}$$

Then, for a test function u, we have

$$\int_{\mathbb{R}^d} U^p |u|^p \, dx + \int_{\{u\neq 0\}} U^{p-1}|u|^{p-2} a_0[\nabla u]\, dx \le c\, \|u - Au\|_p^p .$$

The constants c only depend on p and the constants in (H2)–(H4).

Proof. We assume preliminarily that (H2) is satisfied with $C_\gamma = 0$. Observe that we can fix an $\alpha \in (0,4)$ (depending on p, γ, κ, and θ) such that

$$\frac{\theta}{p} + (p-1)\gamma \left[\frac{\kappa}{p} + \frac{\gamma}{\alpha}\right] < 1. \tag{2.5}$$

We first consider the case $p \ge 2$. For a fixed real $u \in C_0^\infty(\mathbb{R}^d)$ we set

$$f := -Au = -A_0 u - F\cdot \nabla u + Vu. \tag{2.6}$$

If we multiply (2.6) by $U^{p-1}u|u|^{p-2}$ and integrate by parts, we obtain as in the proof of Proposition 2.2 the identity

$$\begin{aligned}\int_{\mathbb{R}^d} (V + \tfrac{1}{p}\operatorname{div} F)\, U^{p-1}|u|^p\, dx + (p-1)\int_{\mathbb{R}^d} U^{p-1}|u|^{p-2} a_0[\nabla u]\, dx \\ = (1-p)\int_{\mathbb{R}^d} u\,|u|^{p-2} U^{p-2} a_0(\nabla u, \nabla U) dx \\ + \Big(\frac{1}{p} - 1\Big)\int_{\mathbb{R}^d} U^{p-2}|u|^p\, F\cdot \nabla U dx + \int_{\mathbb{R}^d} f U^{p-1} u\,|u|^{p-2}\, dx.\end{aligned} \tag{2.7}$$

We introduce the quantities $b^2 = \int U^p|u|^p\, dx$ and $d^2 = \int U^{p-1}|u|^{p-2}a_0[\nabla u]\, dx$. The left hand side of (2.7) is greater than $(1-\frac{\theta}{p})b^2 + (p-1)d^2$ by (H2) and (H4). Employing (H2), (H3), Hölder's and Young's inequalities, we estimate the right hand side of (2.7) by

$$\begin{aligned}&(p-1)\gamma \int_{\mathbb{R}^d} |u|^{p-1} U^{p-2} a_0[\nabla u]^{\frac12} U^{\frac32}\, dx + \Big(1 - \frac{1}{p}\Big)\gamma\kappa \int_{\mathbb{R}^d} U^{p-2}|u|^p U^{\frac12} U^{\frac32}\, dx \\ &\quad + \Big(\int_{\mathbb{R}^d} |f|^p\, dx\Big)^{\frac1p} \Big(\int_{\mathbb{R}^d} U^p|u|^p\, dx\Big)^{1-\frac1p} \\ &\le (p-1)\gamma b d + \Big(1-\frac{1}{p}\Big)\gamma\kappa b^2 + \|f\|_p\, b^{2-\frac{2}{p}} \\ &\le (p-1)\gamma b d + \Big(1-\frac{1}{p}\Big)\gamma\kappa b^2 + \varepsilon b^2 + c_\varepsilon \|f\|_p^p.\end{aligned}$$

Combining these facts, we arrive at

$$\left[1 - \frac{\theta}{p} - \frac{\kappa\gamma(p-1)}{p} - \varepsilon\right] b^2 + (p-1)d^2 \le (p-1)\gamma b d + c_\varepsilon \|f\|_p^p.$$

If we use Young's inequality and (2.5) and take a sufficiently small $\varepsilon > 0$, then we deduce

$$\int_{\mathbb{R}^d} U^p |u|^p \, dx + \int_{\mathbb{R}^d} U^{p-1} |u|^{p-2} a_0[\nabla u] \, dx \le c \, \|f\|_p^p = c \, \|Au\|_p^p \tag{2.8}$$

for some constant $c > 0$. In order to remove the assumption $C_\gamma = 0$, we fix a large λ (depending on γ and C_γ) such that $U + \lambda + 1$ and $V + \lambda + 1$ satisfy (H2) with $C_\gamma = 0$ and apply the previous estimates to the operator $A - \lambda - 1$. Then

$$\begin{aligned} &\int_{\mathbb{R}^d} U^p |u|^p \, dx + \int_{\mathbb{R}^d} U^{p-1} |u|^{p-2} a_0[\nabla u] \, dx \\ &\quad \le c \, \|(\lambda+1)u - Au\|_p^p \le c \left(\|u - Au\|_p + \lambda \|u\|_p \right)^p \le c \, (1+\lambda)^p \, \|u - Au\|_p^p, \end{aligned}$$

by the dissipativity of A.

If $p < 2$, then one can verify as in the proof of Proposition 2.2 that d^2 is a finite number (taking the integral over $\{x \in \mathbb{R}^d : u(x) \neq 0\}$). The claim then follows as for $p \ge 2$. ∎

The above results allow us to treat the case $F = 0$, i.e., the Schrödinger operator $A_0 - V$, on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.

Theorem 2.4. *Let $1 < p < \infty$. Assume that* (H1), (H2) *with $\gamma^2 < 4(p-1)^{-1}$, and* (H5) *hold. Then $A_0 - V$ with domain D_p generates a positive, analytic C_0–semigroup $T(\cdot)$ in $L^p(\mathbb{R}^d)$ such that $\|T(z)\| \le 1$ for $|\arg z| \le \phi_p$ and some $\phi_p > 0$. Test functions are a core of $A_0 - V$, i.e., $C_0^\infty(\mathbb{R}^d)$ is dense in D_p.*

Proof. Lemma 2.3 (with $F = 0$, $\theta = \kappa = 0$) shows that

$$\|Vu\|_p \le c \, \|u - (A_0 - V)u\|_p \tag{2.9}$$

for all test functions u, where the constant c only depends on p and the constants in (H2). Let $u \in D_p = D(A_0) \cap D(V)$. Due to Lemma 2.1 there are test functions approximating u in the graph norm of $A_0 - V$. Thus Proposition 2.2 is valid for $A_0 - V$ defined on D_p. Moreover, (2.9) holds for $u \in D_p$ thanks to Fatou's lemma. We introduce the approximating potentials $U_\varepsilon = \frac{U}{1+\varepsilon U}$ and $V_\varepsilon = \frac{V}{1+\varepsilon V}$, where $\varepsilon > 0$. Then we have

$$\frac{c_0}{1+\varepsilon c_0} \le U_\varepsilon \le V_\varepsilon \le c_1 U_\varepsilon \le \frac{c_1}{\varepsilon} \quad \text{and} \quad a_0[\nabla U_\varepsilon]^{\frac{1}{2}} \le \gamma U_\varepsilon^{\frac{3}{2}} + C_\gamma. \tag{2.10}$$

Let $f \in L^p(\mathbb{R}^d)$. Lemma 2.1 implies that $A_0 - V_\varepsilon$ with domain $D(A_0 - V_\varepsilon) = D(A_0)$ generates a contraction semigroup on $L^p(\mathbb{R}^d)$. Therefore there is a unique $u_\varepsilon \in D(A_0)$ satisfying

$$u_\varepsilon - A_0 u_\varepsilon + V_\varepsilon u_\varepsilon = f, \qquad \|u_\varepsilon\|_p \le \|f\|_p \, , \qquad \|V_\varepsilon u_\varepsilon\|_p \le C \|f\|_p \, .$$

Here the constant C does not depend on ε due to (2.9) and (2.10). Using standard elliptic regularity on balls $B(0,r)$, [17, Theorem 9.11], we see that

$$\|u_\varepsilon\|_{W^{2,p}(B(0,r))} \le C'_{p,r}\,\|f\|_p.$$

Thus there exists a sequence $\varepsilon_n \to 0$ such that the functions (u_{ε_n}) converge weakly to a function $u \in W^{2,p}_{loc}(\mathbb{R}^d)$ as $n \to \infty$. The Rellich–Kondrachov theorem implies that a subsequence of (u_{ε_n}) tends strongly to u in $W^{1,p}_{loc}(\mathbb{R}^d)$; hence we may assume that $u_{\varepsilon_n}(x) \to u(x)$ a.e. in $\mathbb{R}^d$. Fatou's lemma now yields

$$\|u\|_p \le \|f\|_p \qquad \text{and} \qquad \|Vu\|_p \le C\|f\|_p.$$

Let φ be a test function. Then we have

$$\int_{\mathbb{R}^d} f\varphi\,dx = \int_{\mathbb{R}^d} u_{\varepsilon_n}(\varphi - A_0\varphi + V_{\varepsilon_n}\varphi)\,dx \longrightarrow \int_{\mathbb{R}^d} u(\varphi - A_0\varphi + V\varphi)\,dx,$$

as $n \to \infty$, and hence

$$\int_{\mathbb{R}^d} f\varphi\,dx = \int_{\mathbb{R}^d} (u - A_0u + Vu)\varphi\,dx. \tag{2.11}$$

So we derive

$$u - A_0u + Vu = f.$$

This means that $u \in D_p$ since $A_0u = u + Vu - f \in L^p(\mathbb{R}^d)$. As a result, $I-(A_0-V) : D_p \to L^p(\mathbb{R}^d)$ is surjective. The operator A_0-V with domain D_p thus generates a contraction semigroup $T(\cdot)$ on $L^p(\mathbb{R}^d)$ in view of Lemma 2.1 and the Lumer–Phillips theorem. Proposition 2.2 and the Lumer–Phillips theorem then imply that $e^{\pm i\phi_p}A$ also generate contractive C_0–semigroups for some $\phi_p > 0$. Hence $T(\cdot)$ is analytic due to [16, Theorem II.4.9].

The last assertion immediately follows from Lemma 2.1. The positivity of $A_0 - V$ is essentially known: One can argue as in Theorems 3.1 and 3.3 of [3] to obtain an extension $\tilde{A}$ of $(A_0 - V, C_0^\infty(\mathbb{R}^d))$ which generates a positive C_0–semigroup. Since test functions are a core of our generator A, we have $A = \tilde{A}$, and the positivity of $T(t)$ follows. (We note that in [3] it was assumed that the coefficients a_{kl} are uniformly elliptic, but this does not matter in this argument.) ∎

In the case $1 < p \le 2$ the above generation result can be extended to the operator A with $F \ne 0$ using the complete estimate proved in Lemma 2.3. This estimate leads to the following weighted Gagliardo–Nirenberg estimate.

Lemma 2.5. *Let $1 < p \le 2$. Assume that the hypotheses* (H1) *and* (H2) *are satisfied and $\gamma^2 < 4(p-1)^{-1}$ holds. Then for each $\varepsilon > 0$ there is a constant c_ε (depending only on ε, p, and the constants in* (H2)*) such that*

$$\|U^{\frac{1}{2}} a_0 [\nabla u]^{\frac{1}{2}}\|_p \le \varepsilon\|A_0u\|_p + c_\varepsilon\|Vu\|_p$$

for every test function u.

Proof. Again we first suppose that (H2) holds with $C_\gamma = 0$. The estimate (2.8) for the case $F = 0$ shows that

$$\int_{\{u\neq 0\}} U^{p-1}|u|^{p-2}a_0[\nabla u]\,dx \le c\,(\|A_0u\|_p + \|Vu\|_p)^p.$$

Let $u_\varepsilon = \sqrt{|u|^2+\varepsilon}$ for $\varepsilon > 0$ and denote by K the support of u. Assume that $1 < p < 2$ for a moment. Hölder's inequality with the conjugate exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ yields

$$\begin{aligned}\int_{\mathbb{R}^d} U^{\frac{p}{2}}a_0[\nabla u]^{\frac{p}{2}}\,dx &= \int_K (U^{p-1}u_\varepsilon^{p-2}a_0[\nabla u])^{\frac{p}{2}}\,(U^p\,u_\varepsilon^p)^{\frac{2-p}{2}}\,dx\\ &\le \Big(\int_K U^{p-1}u_\varepsilon^{p-2}a_0[\nabla u]\,dx\Big)^{\frac{p}{2}}\,\Big(\int_K U^p\,u_\varepsilon^p\,dx\Big)^{1-\frac{p}{2}}.\end{aligned}$$

Clearly, this estimate also holds for $p = 2$. Using the theorem of dominated convergence, we obtain for $1 < p \le 2$

$$\int_{\mathbb{R}^d} U^{\frac{p}{2}}a_0[\nabla u]^{\frac{p}{2}}\,dx \le \Big(\int_{\{u\neq 0\}} U^{p-1}|u|^{p-2}a_0[\nabla u]\,dx\Big)^{\frac{p}{2}}\,\Big(\int_{\mathbb{R}^d} U^p\,|u|^p\,dx\Big)^{1-\frac{p}{2}}.$$

Combining the above estimates with (H2) and Young's inequality, we deduce

$$\begin{aligned}\|U^{\frac12}a_0[\nabla u]^{\frac12}\|_p &\le c\,(\|A_0u\|_p + \|Vu\|_p)^{\frac{p}{2}}\,\|Vu\|_p^{1-\frac{p}{2}}\\ &\le c'\,(\|A_0u\|_p^{\frac{p}{2}} + \|Vu\|_p^{\frac{p}{2}})\,\|Vu\|_p^{1-\frac{p}{2}}\\ &\le \varepsilon\,\|A_0u\|_p + c_\varepsilon\,\|Vu\|_p\,.\end{aligned}$$

If $C_\gamma \neq 0$, we add a large constant $\lambda > 0$ such that $U + \lambda$ and $V + \lambda$ satisfy (H2) with $C_\gamma = 0$. Then the first part of the proof implies that

$$\begin{aligned}\|U^{\frac12}a_0[\nabla u]^{\frac12}\|_p &\le \|(U+\lambda)^{\frac12}a_0[\nabla u]^{\frac12}\|_p\\ &\le \varepsilon\,\|A_0u\|_p + c_\varepsilon\,\|(\lambda+V)u\|_p \le \varepsilon\,\|A_0u\|_p + c'_\varepsilon\,\|Vu\|_p\end{aligned}$$

due to (H2). ∎

Proposition 2.6. *Let* $1 < p \le 2$. *Assume that the hypotheses* (H1)–(H5) *are satisfied and that* (2.4) *holds. Then there are constants* $C, C' \ge 0$ *depending only on* p *and the constants in* (H2)–(H4) *such that*

$$\|u\|_p + \|A_0u\|_p + \|Vu\|_p \le C\,\|u - Au\|_p \le C'\,(\|u\|_p + \|A_0u\|_p + \|Vu\|_p)$$

for every $u \in D_p$.

Proof. Let $u \in D_p$. Due to Theorem 2.4 there are test functions u_n such that $u_n \to u$, $Vu_n \to Vu$, and $A_0 u_n \to A_0 u$ in $L^p(\mathbb{R}^d)$; thus we may suppose that $\nabla u_n \to \nabla u$ a.e.. Lemma 2.5 and (H3) then imply that $F \cdot \nabla u_n \to F \cdot \nabla u$ in $L^p(\mathbb{R}^d)$. So it suffices to show the proposition for a test function u. The second asserted estimate follows directly from Lemma 2.5 and (H3). To prove the other one, we first suppose that $C_\gamma = 0$ in (H2). We denote by c a generic constant only depending on p and the constants in (H2)–(H4). We have $\|Vu\|_p \le c\,\|Au\|_p$ due to (2.8) and (H2). Moreover, assumption (H3) and Lemma 2.5 yield

$$\|F\cdot\nabla u\|_p \le \kappa\,(\varepsilon\,\|A_0 u\|_p + c\,\|Vu\|_p) \le \kappa\varepsilon\,(\|Au\|_p + \|F\cdot\nabla u\|_p + \|Vu\|_p) + c\,\|Vu\|_p\,,$$

where $\varepsilon := (2\kappa)^{-1}$. As a consequence, we have

$$\|F\cdot\nabla u\|_p \le c\,(\|Au\|_p + \|Vu\|_p) \le c\,\|Au\|_p\,.$$

These inequalities further imply that

$$\|A_0 u\|_p = \|Au - F\cdot\nabla u + Vu\|_p \le c\,\|Au\|_p\,,$$

so that $\|A_0 u\| + \|Vu\|_p \le \tilde{C}\,\|Au\|_p$ in this case. Finally, in the general case we find again $\lambda > 0$ such that $U + \lambda + 1$ and $V + \lambda + 1$ satisfy (H2) with $C_\gamma = 0$. Then we obtain

$$\begin{aligned}\|u\|_p + \|A_0 u\| + \|Vu\|_p &\le \|u\|_p + \tilde{C}\,\|(1+\lambda)u - Au\|_p \\ &\le (1+\lambda\tilde{C})\|u\|_p + \tilde{C}\,\|u - Au\|_p \\ &\le C\,\|u - Au\|_p\end{aligned}$$

by the dissipativity of A. ∎

Theorem 2.7. *Let $1 < p \le 2$. Assume that the hypotheses* (H1)–(H5) *are satisfied and that* (2.4) *holds. Then A with domain D_p generates a positive, analytic C_0–semigroup $T(\cdot)$ in $L^p(\mathbb{R}^d)$ such that $\|T(z)\| \le 1$ for $|\arg z| \le \phi_p$ and some $\phi_p > 0$. Test functions are a core of A.*

Proof. For $t \in [0,1]$ and $u \in D_p$ we set $L_t u := A_0 u + tF\cdot\nabla u - Vu$. Note that these operators satisfy (H1)–(H5) with the same constants. Proposition 2.6 thus shows that

$$\|u\|_{D_p} \le C\,\|u - L_t u\|_p$$

for every $u \in D_p$, with C independent of $t \in [0,1]$. We have $1 \in \rho(L_0)$ due to Theorem 2.4. Therefore $1 \in \rho(L_1) = \rho(A)$ by a continuity argument, see, e.g., [17, Theorem 5.2]. As in the proof of Proposition 2.6 we extend Proposition 2.2 to the operator A with domain D_p. Thus the Lumer–Phillips theorem implies that the operators A and $e^{\pm i\phi_p}A$ generate contractive C_0–semigroups for some $\phi_p > 0$. So the semigroup generated by A is analytic by [16, Theorem II.4.9]. The last assertion follows from Theorem 2.4. The positivity of $T(t)$ can be seen as in Theorem 2.4. ∎

3. The case $p > 2$

In the case $p > 2$ the elementary proof of Lemma 2.5 does not work anymore. Thus we need a different approach to control the drift term by the diffusion part and the potential. As in [4], [5], [25], we employ localization techniques. To that purpose we change our hypothesis (H2) to the stronger assumption (H2').

The following version of the Besicovitch covering theorem follows from the proof of Lemma 2.2 in [25].

Lemma 3.1. *Let $k > 1$ and $\{B(x, r(x)) : x \in \mathbb{R}^d\}$ be a collection of balls such that the radii $r(x)$ are uniformly bounded and $\sigma r(y) \le r(x) \le \frac{1}{\sigma} r(y)$ for a constant $\sigma > 0$ if two balls $B(x, r(x))$ and $B(y, r(y))$ overlap. Then there exists a natural number N (depending only on d, k, σ) and a countable covering $\{B(x_n, \frac{1}{k} r(x_n))\}$ of $\mathbb{R}^d$ such that each $y \in \mathbb{R}^d$ is contained in at most N of the balls $B(x_n, r(x_n))$.*

From the proof of Proposition 3.3 we separate a lemma dealing with local perturbations of the Calderón–Zygmund estimate. We denote the norm of $L^p(B(x, r))$ by $\| \cdot \|_{p,r}$.

Lemma 3.2. *Let $x \in \mathbb{R}^d$, $r > 0$, $1 < p < \infty$ and $q \in C^1(B(x, r), \mathbb{R}^{d \times d})$ such that $q(y) = q(y)^T > 0$ for $y \in B(x, r)$, and $0 < \nu I \le q(x) \le \Lambda I$ for some numbers $\Lambda, \nu > 0$. Set $\omega = \sup\{|q(y) - q(x)|\,;\ y \in B(x, r)\}$. Then there are constants $c, \eta > 0$ only depending on d and p such that if $\frac{\omega}{\nu} \le \eta$ and $u \in C_0^\infty(\mathbb{R}^d)$ then*

$$\|D^2 u\|_{p,\frac{r}{2}} \le \frac{c}{\nu} \Big(\| \operatorname{tr}(q D^2 u)\|_{p,r} + \frac{1}{r} \|q \nabla u\|_{p,r} + \frac{\omega}{r} \|\nabla u\|_{p,r} + \frac{\Lambda}{r^2} \|u\|_{p,r} \Big).$$

Proof. Throughout the proof, $x \in \mathbb{R}^d$ is fixed and we write c for a generic constant only depending on d and p. Let $\nu \le \lambda_1^2 \le \cdots \le \lambda_d^2 \le \Lambda$ be the eigenvalues of $q(x)$, where $\lambda_k > 0$. By the Calderón–Zygmund estimate, see [17, Theorem 9.9], we have $\|D^2 v\|_p \le c \|\Delta v\|_p$ for every test function v. Using the change of variables $y \mapsto y' = (\lambda_1^{-1} y_1, \cdots, \lambda_d^{-1} y_d)$, we deduce

$$\nu \|\partial_{ij} v\|_p \le \lambda_i \lambda_j \|\partial_{ij} v\|_p \le c \Big\| \sum\nolimits_k \lambda_k^2 \, \partial_{kk} v \Big\|_p$$

for $i, j \in \{1, \cdots, d\}$. There is an orthogonal matrix J such that $J^{-1} q(x) J = \operatorname{diag}(\lambda_k^2)$. Thus another change of variables implies

$$\nu \|D^2 v\|_p \le c \| \operatorname{tr}(q(x) D^2 v)\|_p .$$

We fix $u \in C_0^\infty(\mathbb{R}^d)$ and a smooth cut off function χ supported in $B(x, r)$ such that $\chi = 1$ on $B(x, r/2)$, $|\nabla \chi| \le c/r$, and $|D^2 \chi| \le c/r^2$. Then the above estimate (with $v = \chi u$) yields

$$\begin{aligned}\|D^2u\|_{p,\frac{r}{2}} &\le \|D^2(\chi u)\|_p \\ &\le c\nu^{-1}\,\|\operatorname{tr}(q(x)D^2(\chi u))\|_p \\ &\le \frac{c}{\nu}\Big(\|\operatorname{tr}(q(x)D^2u)\|_{p,r} + \frac{1}{r}\,\|q(x)\nabla u\|_{p,r} + \frac{|q(x)|}{r^2}\,\|u\|_{p,r}\Big) \\ &\le \frac{c}{\nu}\Big(\|\operatorname{tr}(qD^2u)\|_{p,r} + \omega\|D^2u\|_{p,r} + \frac{1}{r}\,\|q\nabla u\|_{p,r} \\ &\quad + \frac{\omega}{r}\,\|\nabla u\|_{p,r} + \frac{\Lambda}{r^2}\,\|u\|_{p,r}\Big).\end{aligned} \tag{3.1}$$

To get rid of $\|D^2u\|_{p,r}$ on the right hand side of (3.1), we shall derive an analogous estimate in the whole space and then use a covering argument.

Observe that $|q(y)| \le |q(x)| + \omega \le \Lambda(1+\omega/\nu)$ and $(q(y)\xi|\xi) \ge \nu - \omega$ for $y \in B(x,r)$ and $|\xi| = 1$. So if $\omega \le \nu/2$, we have $\frac{\nu}{2}I \le q(y) \le 2\Lambda I$ for $y \in B(x,r)$. We extend q to $\mathbb{R}^d$ setting $q(y) = q(x + r(y-x)/|y-x|)$ if $|y-x| \ge r$ so that $\frac{\nu}{2}I \le q(x') \le 2\Lambda I$ and $|q(y) - q(x')| \le 2\omega$ for all $y, x' \in \mathbb{R}^d$. As a result (3.1) holds for all centers $x' \in \mathbb{R}^d$. By Lemma 3.1 there exists a countable covering $\{B(x_n, r/2)\}$ of $\mathbb{R}^d$ such that at each point $y \in \mathbb{R}^d$ at most N of the balls $B(x_n, r)$ overlap. We then raise the estimates (3.1) with $x = x_n$ to the pth power and sum over n. Taking the pth root, we arrive at

$$\|D^2u\|_p \le \frac{C'}{\nu}\Big(\|\operatorname{tr}(qD^2u)\|_p + \omega\|D^2u\|_p + \frac{1}{r}\,\|q\nabla u\|_p + \frac{\omega}{r}\,\|\nabla u\|_p + \frac{\Lambda}{r^2}\,\|u\|_p\Big)$$

for a constant $C' > 0$ only depending on d and p. If $\omega/\nu \le \eta := (2C')^{-1}$, we can eliminate the term $\|D^2u\|_p$ on the right hand side thus obtaining

$$\|D^2u\|_p \le \frac{2C'}{\nu}\Big(\|\operatorname{tr}(qD^2u)\|_p + \frac{1}{r}\,\|q\nabla u\|_p + \frac{\omega}{r}\,\|\nabla u\|_p + \frac{\Lambda}{r^2}\,\|u\|_p\Big).$$

Now the assertion follows as in (3.1) using the same cut–off function. ∎

Proposition 3.3. *Assume that* (H1) *and* (H2') *hold and that* $1 < p < \infty$. *Then, for each* $\varepsilon > 0$, *there exists a constant* c_ε *only depending on* p *and the constants in* (H2') *such that*

$$\|U^{\frac{1}{2}}\,a_0[\nabla u]^{\frac{1}{2}}\|_p \le \varepsilon\|A_0u\|_p + c_\varepsilon\|Vu\|_p$$

for every test function u.

Proof. `Step (1):` We recall that $\rho(x) = |a(x)|^{\frac{1}{2}}\,U(x)^{-\frac{1}{2}}$. For $x \in \mathbb{R}^d$ and $r > 0$ we set

$$\omega(x,r) = \sup\{|a(y) - a(x)|\,;\ y \in B(x,r)\}.$$

Let $\delta \in (0,1)$. Assumption (H2') implies that

$$\frac{\omega(x, \delta\rho(x))}{\nu(x)} \le c_2\,\delta \tag{3.2}$$

for all x. So we can fix a number $\delta_1 \in (0,1)$ such that

$$\frac{\omega(x, \delta\rho(x))}{\nu(x)} \le \eta \tag{3.3}$$

for $0 < \delta \le \delta_1$, $x \in \mathbb{R}^d$, and the constant η from Lemma 3.2. Moreover, the radii $r = r(x) := \delta\rho(x)$ and the quotients $\omega(x,r)/\nu(x)$ are uniformly bounded for $x \in \mathbb{R}^d$ and $0 < \delta \le \delta_1$ by (H2') and (3.2). Replacing U and V by $\mu + U$ and $\mu + V$ for sufficiently large $\mu = \mu(\gamma, C_\gamma) > 0$, we can assume that (H2') holds with $C_\gamma = 0$. This implies that

$$|\nabla U^{-\frac{1}{2}}(x)| \le \frac{\gamma}{2}\,|a(x)|^{-\frac{1}{2}}\,, \qquad x \in \mathbb{R}^d.$$

For $y \in B(x, \delta\rho(x))$, $x \in \mathbb{R}^d$, and a suitable point z on the line segment between x and y, we thus obtain

$$\begin{aligned}
|U(x)^{-\frac{1}{2}} - U(y)^{-\frac{1}{2}}| &\le \frac{\delta\gamma}{2}\,\frac{|a(x)|^{\frac{1}{2}}}{|a(z)|^{\frac{1}{2}}}\,U(x)^{-\frac{1}{2}} \\
&\le \frac{\delta\gamma}{2}\Big(\frac{|a(x)|}{|a(x)| - \omega(x,r)}\Big)^{\frac{1}{2}}\,U(x)^{-\frac{1}{2}} \\
&\le \frac{\delta\gamma}{2}\Big(1 - \frac{\omega(x,r)}{\nu(x)}\Big)^{-\frac{1}{2}}\,U(x)^{-\frac{1}{2}} \\
&\le \frac{1}{2}\,U(x)^{-\frac{1}{2}}\,.
\end{aligned} \tag{3.4}$$

In the last step we have used (3.2) and we take $\delta \in (0, \delta_2]$ for a sufficiently small $\delta_2 \in (0, \delta_1]$. This estimate yields

$$\tfrac{1}{4}\,U(y) \le U(x) \le \tfrac{9}{4}\,U(y), \qquad y \in B(x,r),\ x \in \mathbb{R}^d. \tag{3.5}$$

Inequality (3.2) further implies that

$$|((a(x) - a(y))\xi|\xi)| \le \omega(x,r) \le \frac{\omega(x,r)}{\nu(x)}\,(a(x)\xi|\xi) \le \frac{1}{2}\,(a(x)\xi|\xi) \tag{3.6}$$

for $|\xi| = 1$, $\delta \in (0, \delta_3]$, and some $\delta_3 \in (0, \delta_2]$. From (3.5) and (3.6) we infer

$$\beta_1\,U(y)a(y) \le U(x)a(x) \le \beta_2\,U(y)a(y), \qquad y \in B(x,r),\ x \in \mathbb{R}^d, \tag{3.7}$$

in the sense of quadratic forms, for some constants $0 < \beta_1 \le \beta_2$. Now fix $\delta = \delta_3/2$ and the corresponding radii $r = r(x) = \delta\rho(x)$. Due to (3.4) and (3.6), there are constants $0 < \beta_1' \le \beta_2'$ such that $\beta_1' r(y) \le r(x) \le \beta_2' r(y)$ whenever the balls $B(x, r(x))$ and $B(y, r(y))$ overlap. Thus we can apply Lemma 3.1 to the balls $B(x, r(x))$. We finally observe that

$$|\nabla a(y)\xi|^2 \le c_2^2\, U(x)\nu(x)\, |\xi|^2 \le c_2^2 \beta_2\, U(y)\, (a(y)\xi|\xi) \tag{3.8}$$

for $y \in B(x, r)$ and $\xi \in \mathbb{R}^d$ due to (H2') and (3.7).

`Step (2)`: In the remainder of the proof, c denotes a generic constant only depending on p and the constants in (H2'). Fix $x \in \mathbb{R}^d$. Let $\nu(x) = \lambda_1^2 \le \cdots \le \lambda_d^2 = |a(x)|$ be the eigenvalues of $a(x)$, where $\lambda_k > 0$. Recall that

$$\|\nabla v\|_p^2 \le c\, \|v\|_p\, \|\Delta v\|_p$$

for each test function v. As in the proof of Lemma 3.1 we deduce by two changes of variables that

$$\|\lambda_j\, \partial_j v\|_p^2 \le c\, \|v\|_p \Big\| \sum\nolimits_k \lambda_k^2\, \partial_{kk} v \Big\|_p .$$

Changing variables again, we obtain

$$\|U(x)^{\frac12} a(x)^{\frac12} \nabla v\|_p^2 \le c\, \|U(x) v\|_p\, \|\operatorname{tr}(a(x) D^2 v)\|_p . \tag{3.9}$$

We now take a smooth function χ supported in $B(x, r/2)$ and satisfying $\chi = 1$ on $B(x, r/4)$, $|\nabla\chi| \le c/r$, and $|D^2\chi| \le c/r^2$. Then (3.9) for $v = \chi u$, the definition of r, and standard manipulations with positive-definite matrices imply that

$$\begin{aligned}
&\|U(x)^{\frac12} a(x)^{\frac12} \nabla u\|_{p,\frac r4}^2 \\
&\quad \le \|U(x)^{\frac12} a(x)^{\frac12} \nabla(\chi u)\|_{p,\frac r2}^2 \\
&\quad \le c\, \|U(x) u\|_{p,\frac r2} \Big[\|\operatorname{tr}(a(x) D^2 u)\|_{p,\frac r2} + \frac{1}{r}\, \|a(x)\nabla u\|_{p,\frac r2} + \frac{|a(x)|}{r^2}\, \|u\|_{p,\frac r2} \Big] \\
&\quad \le c\, \|U(x) u\|_{p,\frac r2} \Big[\|\operatorname{tr}(a(x) D^2 u)\|_{p,\frac r2} + U(x)^{\frac12}\, \|a(x)^{\frac12} \nabla u\|_{p,\frac r2} + U(x)\, \|u\|_{p,\frac r2} \Big].
\end{aligned}$$

Next we employ (3.7) and (3.5) to estimate

$$\begin{aligned}
\|U^{\frac12} a_0 [\nabla u]^{\frac12}\|_{p,\frac r4}^2 \le c\, \|Uu\|_{p,\frac r2} \Big[&\|\operatorname{tr}(a D^2 u)\|_{p,\frac r2} + \omega(x, r)\, \|D^2 u\|_{p,\frac r2} \\
&+ \|U^{\frac12} a_0 [\nabla u]^{\frac12}\|_{p,\frac r2} + \|Uu\|_{p,\frac r2} \Big].
\end{aligned}$$

At this point, because of (3.3), we can apply Lemma 3.2 to the restriction of a

to $B(x,r/2)$. Consequently,

$$\begin{aligned}\|U^{\frac12}a_0[\nabla u]^{\frac12}\|^2_{p,\frac r4} \le c\,\|Uu\|_{p,\frac r2}\Big[\|\operatorname{tr}(aD^2u)\|_{p,\frac r2} + \|U^{\frac12}a_0[\nabla u]^{\frac12}\|_{p,\frac r2} + \|Uu\|_{p,\frac r2}\Big]\\ + \frac{c\,\omega(x,r)}{\nu(x)}\,\|Uu\|_{p,\frac r2}\Big[\|\operatorname{tr}(aD^2u)\|_{p,r} + \frac1r\,\|a\nabla u\|_{p,r}\\ + \frac{\omega(x,r)}{r}\,\|\nabla u\|_{p,r} + \frac{|a(x)|}{r^2}\,\|u\|_{p,r}\Big].\end{aligned}$$

Observe that the definition of r and (3.3) yield

$$\frac{\omega(x,r)}{r} = \frac{\omega(x,r)}{\delta\,|a(x)|^{\frac12}}\,U(x)^{\frac12} \le \delta^{-1}\eta\,\frac{\nu(x)}{|a(x)|^{\frac12}}\,U(x)^{\frac12} \le \delta^{-1}\eta\,\nu(x)^{\frac12}\,U(x)^{\frac12}.$$

Using these facts and again (3.7), (3.5), we arrive at

$$\|U^{\frac12}a_0[\nabla u]^{\frac12}\|^2_{p,\frac r4} \le c\,\|Uu\|_{p,r}\Big[\|\operatorname{tr}(aD^2u)\|_{p,r} + \|U^{\frac12}a_0[\nabla u]^{\frac12}\|_{p,r} + \|Uu\|_{p,r}\Big].$$

So (3.8) yields

$$\|U^{\frac12}a_0[\nabla u]^{\frac12}\|^2_{p,\frac r4} \le c\,\|Uu\|_{p,r}\Big[\|Au\|_{p,r} + \|U^{\frac12}a_0[\nabla u]^{\frac12}\|_{p,r} + \|Uu\|_{p,r}\Big].$$

By a standard application of Young's inequality, we deduce

$$\|U^{\frac12}a_0[\nabla u]^{\frac12}\|^p_{p,\frac r4} \le \varepsilon'\,\|Au\|^p_{p,r} + \varepsilon'\,\|U^{\frac12}a_0[\nabla u]^{\frac12}\|^p_{p,r} + c_{\varepsilon'}\,\|Uu\|^p_{p,r} \tag{3.10}$$

for $\varepsilon' > 0$. Due to Lemma 3.1 there is a countable covering $B(x_n, r(x_n)/4)$ such that at each point $y \in \mathbb{R}^d$ at most N of the balls $B(x_n, r(x_n))$ overlap. We now sum the inequalities (3.10) for $x = x_n$ over n. This yields

$$\|U^{\frac12}a_0[\nabla u]^{\frac12}\|^p_p \le c\varepsilon'\,\|Au\|^p_p + c\varepsilon'\,\|U^{\frac12}a_0[\nabla u]^{\frac12}\|^p_p + c'_{\varepsilon'}\,\|Uu\|^p_p\,.$$

We conclude the proof by choosing a small $\varepsilon' > 0$ and then taking the pth root. ∎

We can now establish the following two results exactly as Proposition 2.6 and Theorem 2.7 employing Proposition 3.3 instead of Lemma 2.5.

Proposition 3.4. *Let* $2 < p < \infty$. *Assume that the hypotheses* (H1), (H2'), (H3), (H4), *and* (H5) *are satisfied and that* (2.4) *holds. Then there are constants* $C, C' \ge 0$ *depending only on* p *and the constants in* (H2'), (H3), (H4) *such that*

$$\|u\|_p + \|A_0u\|_p + \|Vu\|_p \le C\,\|u - Au\|_p \le C'\,(\|u\|_p + \|A_0u\|_p + \|Vu\|_p)$$

for every $u \in D_p$.

Theorem 3.5. *Let $2 < p < \infty$. Assume that the hypotheses* (H1), (H2'), (H3), (H4), *and* (H5) *are satisfied and that* (2.4) *holds. Then A with $D(A) = D_p$ generates a positive, analytic C_0–semigroup $T(\cdot)$ in $L^p(\mathbb{R}^d)$ such that $\|T(z)\| \le 1$ for $|\arg z| \le \phi_p$ and some $\phi_p > 0$. Test functions are a core of A.*

The above approach also shows that the graph norm of A is stronger than the norm of $W^{2,p}(\mathbb{R}^d)$ for $1 < p < \infty$, provided that a is strictly elliptic.

Proposition 3.6. *Let $1 < p < \infty$ and assume that the hypotheses* (H1), (H2'), (H3), (H4), *and* (H5) *are satisfied and that* (2.4) *holds. We further suppose that a is strictly elliptic, i.e., $\nu I \le a$ for a constant $\nu > 0$. Then D_p is continuously embedded in $W^{2,p}(\mathbb{R}^d)$.*

Proof. We keep the notation introduced in the proof of Proposition 3.3 retaining the same choices of δ and $r(x) = \delta\rho(x)$. Let $u \in C_0^\infty(\mathbb{R}^d)$. Employing Lemma 3.2 and proceeding as in the proof of Proposition 3.3, we obtain

$$\begin{aligned}\|D^2u\|_{p,\frac{r}{2}} &\le \frac{c}{\nu}\Big(\|\operatorname{tr}(aD^2u)\|_{p,r} + \frac{1}{r}\,\|a\,\nabla u\|_{p,r} + \frac{\omega(x,r)}{r}\,\|\nabla u\|_{p,r} + \frac{|a(x)|}{r^2}\,\|u\|_{p,r}\Big)\\ &\le \frac{c}{\nu}\Big[\|Au\|_{p,r} + \|U^{\frac12}a_0[\nabla u]^{\frac12}\|_{p,r} + \|Uu\|_{p,r}\Big].\end{aligned}$$

The covering argument then yields

$$\|D^2u\|_p \le c\Big[\|Au\|_p + \|U^{\frac12}a_0[\nabla u]^{\frac12}\|_p + \|Uu\|_p\Big] \le c\Big[\|Au\|_p + \|u\|_p\Big],$$

where we use Propositions 3.3 and 3.4 in the second inequality. ∎

We further want to show that A has maximal regularity of type L^q–L^p. For that purpose we suppose that the assumptions of Theorem 2.7 hold for $p = 2$ and the assumptions of Theorem 2.7 or 3.5 hold for some $p = r \le 2$ or $p = r > 2$, respectively. Then the same assumptions are valid for some $\rho \in (1,p)$ or $\rho \in (p,\infty)$, respectively. Observe that the semigroups generated by A on $L^2(\mathbb{R}^d)$, $L^r(\mathbb{R}^d)$, and $L^\rho(\mathbb{R}^d)$ are *consistent*, i.e., they coincide on the intersection of these spaces. (In fact, since test functions are a core for A on each space, the resolvents of A are consistent, which implies that the semigroups are consistent.)

By rescaling, we may assume that the spectrum of A is contained in the open left half plane. It is known that the operator A has maximal regularity of type L^q on $L^r(\mathbb{R}^d)$ if its imaginary powers satisfy $\|(-A)^{is}\|_r \le Me^{a|s|}$ for some $a \in [0,\pi/2)$ and all $s \in \mathbb{R}$ thanks to the Dore–Venni theorem, see, e.g., [2, Theorem II.4.10.7]. If an operator B is maximal dissipative and invertible on a Hilbert space, then $\|(-B)^{is}\| \le Me^{\pi|s|/2}$ by a result due to Kato, [20, Theorem 5]. Hence, $\|(-A)^{is}\|_2 \le Me^{a|s|}$ for $a = \pi/2 - \phi$ and some $\phi \in (0,\pi/2]$, since A is regularly dissipative. Moreover, A generates a positive contraction semigroup on $L^\rho(\mathbb{R}^N)$, so that $\|(-A)^{is}\|_\rho \le M_\varepsilon \exp((\varepsilon+\pi/2)|s|)$ for each $\varepsilon > 0$ and $s \in \mathbb{R}$ because of the Coifman–Weiss transference principle, see [6, Theorem 5.8]. If we combine these facts with the Riesz-Thorin interpolation theorem, we obtain the following result.

Theorem 3.7. *Let the assumptions of Theorem* 2.7 *hold for* $p = 2$ *and let the assumptions of Theorem* 2.7 *or* 3.5 *hold for some* $p = r \le 2$ *or* $p = r > 2$, *respectively. Let* $f \in L^q([0,T], L^r(\mathbb{R}^d))$ *and* $\varphi = 0$, *for some* $q \in (1,\infty)$ *and* $T > 0$. *Then the solution* u *of* (1.3) *belongs* $W^{1,q}([0,T], L^r(\mathbb{R}^d))$ *and* $A_0u, Vu \in L^q([0,T], L^r(\mathbb{R}^d))$.

The same conclusion holds in the setting of Theorem 2.4 if either $\gamma < 2$ and $1 < r \le 2$ or if $\gamma^2 < 4/(r-1)$ and $r > 2$.

4. Interior singularities

We now consider singularities of the lower order coefficients, again assuming that (H1) and (H5) hold and that $1 < p < \infty$. For simplicity we suppose that F, V, and U satisfy (H2), (H3), and (H4) on $\mathbb{R}^d \setminus \{0\}$ and that (2.4) is true. Then $D_p = D(A_0) \cap D(V)$ is still dense in $L^p(\mathbb{R}^d)$ since D_p contains $C_0^\infty(\mathbb{R}^d \setminus \{0\})$. In addition, we require that

$$U(x) \to \infty \qquad \text{as } \; x \to 0. \tag{4.1}$$

As before, we may assume without loss of generality that (H2) holds with $C_\gamma = 0$. Since a is uniformly elliptic in a neighborhood of 0, we can rewrite (H2) as $|\nabla U^{-\frac{1}{2}}| \le \gamma_1$ in a neighborhood of the origin and for a suitable $\gamma_1 > 0$. Then (4.1) yields

$$U(x) \ge (\gamma_1|x|)^{-2} \qquad \text{as } \; x \to 0. \tag{4.2}$$

So our methods only apply to strongly singular potentials. Of course, some weaker singularities can easily be handled by perturbation arguments based on Sobolev embeddings. However, the whole picture seems to be quite complicated even for Schrödinger operators in $L^2(\mathbb{R}^d)$, see [12], [13]. We further note that for sufficiently small γ_1 in (4.2) and $a_{ij} = \delta_{ij}$ the space $C_0^\infty(\mathbb{R}^d \setminus \{0\})$ is a core for $A_0 - V = \Delta - V$ due to Theorem 4.1 in [27]. Since the resulting upper bound for γ differs from our smallness condition (2.4), we do not invoke the results from [27]. Therefore, $C_0^\infty(\mathbb{R}^d \setminus \{0\})$ could be not a core of the operators studied below, in general.

As in the proof of Theorem 2.4 we introduce the approximating potentials $V_\varepsilon = \frac{V}{1+\varepsilon V}$ and $U_\varepsilon = \frac{U}{1+\varepsilon U}$, where $\varepsilon \in (0,1]$. Observe that both functions can be extended continuously by setting $V(0) = U(0) = 1/\varepsilon$. Moreover, U_ε belongs to $C^1(\mathbb{R}^d)$ with $\nabla U_\varepsilon(0) = 0$ due to (4.2). One can check as in (2.10) that (H2) holds for U_ε and V_ε with constants independent of $\varepsilon \in (0,1]$. We further define $F_\varepsilon = (1+\varepsilon U)^{-3/2}F$ and $F_\varepsilon(0) = 0$. Because of

$$|F_\varepsilon \cdot \xi| = (1+\varepsilon U)^{-\frac{3}{2}}\,|F\cdot\xi| \le \kappa\,(1+\varepsilon U)^{-\frac{3}{2}}\,U^{\frac12}\,a_0[\xi]^{\frac12} \le \kappa\,U_\varepsilon^{\frac12}\,a_0[\xi]^{\frac12}$$

for $\xi \in \mathbb{R}^d$, F_ε and U_ε satisfy hypothesis (H3) with the same constant. This estimate also shows that the function F_ε belongs to $C^1(\mathbb{R}^d,\mathbb{R}^d)$. Then we obtain

$$\begin{aligned}\operatorname{div} F_\varepsilon &= (1+\varepsilon U)^{-\frac32}\operatorname{div} F - \tfrac{3\varepsilon}{2}\,(1+\varepsilon U)^{-\frac52}\,\nabla U\cdot F\\ &\ge -\theta U(1+\varepsilon U)^{-1} - \tfrac32\kappa\gamma\varepsilon U^2(1+\varepsilon U)^{-2}\\ &\ge -(\theta+\tfrac32\kappa\gamma)U_\varepsilon\,.\end{aligned}$$

Hence F_ε and U_ε fulfill (H4) with a uniform constant if in addition

$$\theta + \tfrac32\kappa\gamma < p. \tag{4.3}$$

In view of (2.4) this condition holds automatically if $p > 5/2$.

As a consequence, Lemmas 2.3 and 2.5 and Propositions 2.2 and 2.6 hold for the operators

$$A^{(\varepsilon)} = A_0 + F_\varepsilon\cdot\nabla - V_\varepsilon, \qquad 0<\varepsilon\le 1,$$

with uniform constants. Observe that in this case $D_p = D(A_0)$ since V_ε is bounded.

We first consider the Schrödinger case $F = F_\varepsilon = 0$. By approximation, Proposition 2.2 is true for $A_0 - V_\varepsilon$ defined on $D(A_0)$. Let $u \in D(A_0)\cap D(V)$. Then $V_\varepsilon u \to Vu$ in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$ by monotone convergence, and thus Proposition 2.2 holds for $A_0 - V$ defined on $D(A_0)\cap D(V)$. Arguing as in Theorem 2.4, we then establish the following result. (In (2.11) one has to use $\varphi \in C_0^\infty(\mathbb{R}^d\setminus\{0\})$.)

Theorem 4.1. *Let $1<p<\infty$. Assume that* (H1) *and* (H5) *hold and that U and V satisfy* (H2) *with $\gamma^2 < 4(p-1)^{-1}$ on $\mathbb{R}^d\setminus\{0\}$. Moreover, let* (4.1) *and* (4.3) *be true. Then $A_0 - V$ with domain D_p generates an analytic C_0–semigroup $T(\cdot)$ in $L^p(\mathbb{R}^d)$ such that $\|T(z)\|\le 1$ for $|\arg z|\le\phi_p$ and some $\phi_p>0$.*

In a second step we treat the full operator A for the case $1<p\le 2$. For $u\in D(A_0)$ there are $u_n \in C_0^\infty(\mathbb{R}^d)$ such that $u_n \to u$ and $A_0u_n \to A_0 u$ in $L^p(\mathbb{R}^d)$, by Lemma 2.1. Using Lemma 2.5 and (H3) for U_ε and F_ε, we see that $F_\varepsilon\cdot\nabla u_n \to F_\varepsilon\cdot\nabla u$ in $L^p(\mathbb{R}^d)$ as $n\to\infty$. Thus Proposition 2.2 holds for $A^{(\varepsilon)}$ defined on $D(A_0)$. Next, we take $u\in D(A_0)\cap D(V)$. Then $V_\varepsilon u \to Vu$ in

$L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$. Further, $F_\varepsilon \cdot \nabla u$ converges to $F \cdot \nabla u$ pointwise for $x \neq 0$, and $|F_\varepsilon \cdot \nabla u| \leq |F \cdot \nabla u|$. Fatou's Lemma, (H3), and Lemma 2.5 show that

$$\begin{aligned} \|F \cdot \nabla u\|_p &\leq \liminf_{\varepsilon \to 0} \|F_\varepsilon \cdot \nabla u\|_p \\ &= \liminf_{\varepsilon \to 0} \lim_{n \to \infty} \|F_\varepsilon \cdot \nabla u_n\|_p \\ &\leq c \liminf_{\varepsilon \to 0} \lim_{n \to \infty} (\|A_0 u_n\|_p + \|V_\varepsilon u_n\|_p) \\ &= c\,(\|A_0 u\|_p + \|V u\|_p). \end{aligned}$$

As a consequence, $F_\varepsilon \cdot \nabla u \to F \cdot \nabla u$ in $L^p(\mathbb{R}^d)$ by dominated convergence. Combining these facts, we see that the conclusions of Propositions 2.2 and 2.6 are valid for A defined on $D(A_0) \cap D(V)$. Now one can proceed as in the proof of Theorem 2.7 to derive the next result.

Theorem 4.2. *Let $1 < p \leq 2$. Assume that* (H1) *and* (H5) *hold and that U, V, and F satisfy* (H2), (H3), *and* (H4) *with* (2.4) *on $\mathbb{R}^d \setminus \{0\}$. Moreover, let* (4.1) *and* (4.3) *be true. Then A with $D(A) = D_p$ generates an analytic C_0–semigroup $T(\cdot)$ in $L^p(\mathbb{R}^d)$ such that $\|T(z)\| \leq 1$ for $|\arg z| \leq \phi_p$ and some $\phi_p > 0$.*

Finally we deal with the complete operator A for $p > 2$, now assuming (H2') on $\mathbb{R}^d \setminus \{0\}$ instead of (H2). Again (1.1) (and thus (H2)) are satisfied by U_ε, V_ε, and a_{kl} with uniform constants. But (1.2) is false for U_ε if the diffusion coefficients are unbounded. So it is not clear a priori whether we can extend Proposition 3.3 to U_ε with uniform constants. However, we can almost prove this fact by additional arguments, see (4.5). We write c (c_η) for a generic constant only depending on p and the constants in (H2'), (H3), and (H4) (and on $\eta > 0$). Take $u \in D(A_0) \cap D(V)$ and a smooth function χ with support in $B(0,2)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $B(0,1)$. Then we have

$$\begin{aligned} &\|U_\varepsilon^{\frac{1}{2}} a_0[\nabla u]^{\frac{1}{2}}\|_p \\ &\quad = \|U_\varepsilon^{\frac{1}{2}} \big\{a_0[\nabla(\chi u)] + a_0[\nabla((1-\chi)u)] + 2a_0(\nabla(\chi u), \nabla((1-\chi)u))\big\}^{\frac{1}{2}}\|_p \\ &\quad \leq c\,\|U_\varepsilon^{\frac{1}{2}} a_0[\nabla(\chi u)]^{\frac{1}{2}}\|_p + c\,\|U_\varepsilon^{\frac{1}{2}} a_0[\nabla((1-\chi)u)]^{\frac{1}{2}}\|_p \\ &\quad \leq c\,\|U_\varepsilon^{\frac{1}{2}} a_0[\nabla(\chi u)]^{\frac{1}{2}}\|_p + c\,\|U^{\frac{1}{2}} a_0[\nabla((1-\chi)u)]^{\frac{1}{2}}\|_p\,. \end{aligned}$$

To estimate the first summand in the last line, we extend the diffusion coefficients a_{kl} from $B(0,2)$ to $\tilde{a}_{kl} \in C_b^1(\mathbb{R}^d)$ such that $\tilde{a}_{kl} = \tilde{a}_{lk}$ are strictly elliptic. Now we can apply Proposition 2.3 of [25] and obtain, for each $\eta > 0$,

$$\|U_\varepsilon^{\frac{1}{2}} a_0[\nabla(\chi u)]^{\frac{1}{2}}\|_p \leq \eta\,\|A_0(\chi u)\|_p + c_\eta\,\|U_\varepsilon(\chi u)\|_p$$

(In fact, Proposition 2.3 of [25] is stated for test functions, but by approximation it also holds for χu since U_ε is bounded.) For the second summand, we extend

U from $\mathbb{R}^d \setminus B(0,1)$ to $\tilde{U} \in C^1(\mathbb{R}^d)$ such that $\tilde{U}$ is strictly positive. Then we are in a position to use Proposition 3.3 (which can be extended to $D(A_0) \cap D(\tilde{U})$ by approximation, using Theorem 2.4), and obtain

$$\|U^{\frac{1}{2}} a_0[\nabla((1-\chi)u)]^{\frac{1}{2}}\|_p \leq \eta \, \|A_0((1-\chi)u)\|_p + c_\eta \, \|U((1-\chi)u)\|_p$$

for each $\eta > 0$. Thus we deduce

$$\begin{aligned}\|U_\varepsilon^{\frac{1}{2}} a_0[\nabla u]^{\frac{1}{2}}\|_p &\leq c\eta \, [\|A_0 u\|_p + \|a_0(\chi, \nabla u)\|_p + \|a_0(1-\chi, \nabla u)\|_p] \\ &\quad + c\,\|u\|_p + c_\eta \, \|U_\varepsilon \chi u\|_p + c_\eta \, \|U(1-\chi)u\|_p \\ &\leq c\eta \|A_0 u\|_p + c\eta \|U_\varepsilon^{\frac{1}{2}} a_0[\nabla u]^{\frac{1}{2}}\|_p + c_\eta \|U_\varepsilon \chi u\|_p + c_\eta \|U(1-\chi)u\|_p,\end{aligned}$$

where we have used that U_ε is uniformly bounded from below for $\varepsilon \in (0,1]$. So we arrive at the desired estimate

$$\begin{aligned}\|U_\varepsilon^{\frac{1}{2}} a_0[\nabla u]^{\frac{1}{2}}\|_p &\leq c\eta \, \|A_0 u\|_p + c_\eta \, \|U(1-\chi)u\|_p + c_\eta \, \|U_\varepsilon \chi u\|_p && (4.4)\\ &\leq c\eta \, \|A_0 u\|_p + c_\eta \, \|Uu\|_p && (4.5)\end{aligned}$$

for sufficiently small $\eta > 0$ and all $0 < \varepsilon \leq 1$ and $u \in D(A_0) \cap D(V)$. (Note that we have U, and not U_ε, on the right hand side.) Combined with (H3), this inequality yields

$$\|F_\varepsilon \cdot \nabla u\|_p \leq \kappa \, \|U_\varepsilon^{\frac{1}{2}} a_0[\nabla u]^{\frac{1}{2}}\|_p \leq c\kappa\eta \, \|A_0 u\|_p + c_\eta \kappa \, \|Uu\|_p \tag{4.6}$$

for $u \in D(A_0) \cap D(V)$, $0 < \varepsilon \leq 1$, and small $\eta > 0$. We further define

$$f_\varepsilon := u - A_0 u - F_\varepsilon \cdot \nabla u + Vu.$$

Let $0 < \varepsilon' \leq \varepsilon \leq 1$. Because of $U_{\varepsilon'} \geq U_\varepsilon$, the coefficients a_{kl}, $U_{\varepsilon'}$, $V_{\varepsilon'}$, and F_ε satisfy (H1), (H2), (H3), (H4), and (H5) with uniform constants. Lemma 2.3 now yields

$$\|V_{\varepsilon'} u\|_p \leq c \, \|u - A_0 u - F_\varepsilon \cdot \nabla u + V_{\varepsilon'} u\|_p \tag{4.7}$$

for test functions u and a constant not depending on ε and ε'. Let $u \in D(A_0)$ such that $V(1-\chi)u \in L^p(\mathbb{R}^d)$ (where χ and $\tilde{U}$ are chosen as above). By Theorem 2.4 applied to $\tilde{U}$, there are test functions u_n such that $A_0 u_n \to A_0 u$, $U(1-\chi)u_n \to U(1-\chi)u$, and $u_n \to u$ in $L^p(\mathbb{R}^d)$. Estimate (4.4) and (H3) thus show (4.7) for such u. Letting $\varepsilon' \to 0$, we then derive $\|Vu\|_p \leq c\,\|f_\varepsilon\|_p$ for $u \in D(A_0) \cap D(V)$. So we conclude that

$$\|A_0 u\|_p \leq \|f_\varepsilon\|_p + \|u\|_p + \|F_\varepsilon \cdot \nabla u\|_p + \|Vu\|_p \leq c\,\|f_\varepsilon\|_p + \tfrac{1}{2}\,\|A_0 u\|_p$$

taking a sufficiently small η in (4.6). As a consequence,

$$\|u\|_p + \|A_0 u\|_p + \|Vu\|_p \leq C\,\|f_\varepsilon\|_p \leq C'\,(\|u\|_p + \|A_0 u\|_p + \|Vu\|_p) \tag{4.8}$$

for constants independent of $\varepsilon \in (0,1]$ and $u \in D(A_0) \cap D(V)$. Since $F_\varepsilon \cdot \nabla u \to F \cdot \nabla u$ pointwise, Fatou's lemma and (4.6) imply

$$\|F \cdot \nabla u\|_p \leq \varliminf_{\varepsilon \to 0} \|F_\varepsilon \cdot \nabla u\|_p \leq c\kappa\eta \, \|A_0 u\|_p + c_\eta \kappa \, \|Uu\|_p \,. \tag{4.9}$$

Hence, $F_\varepsilon \cdot \nabla u \to F \cdot \nabla u$ in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$ by the theorem of dominated convergence. Thus we can let $\varepsilon \to 0$ in (4.8) and obtain the assertion of Proposition 2.6 in the present situation. We can extend Proposition 2.2 for $A^{(\varepsilon)}$ to $u \in D(A_0) \cap D(V)$ as we extended (4.7). Since $F_\varepsilon \cdot \nabla u \to F \cdot \nabla u$ and $V_\varepsilon u \to Vu$ in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$, Proposition 2.2 is then also valid for A defined on $D(A_0) \cap D(V)$. Now the next theorem can be shown exactly as Theorem 2.7.

Theorem 4.3. *Let $p > 2$. Assume that* (H1) *and* (H5) *hold and that U, V, and F satisfy* (H2'), (H3), *and* (H4) *with* (2.4) *on $\mathbb{R}^d \setminus \{0\}$. Moreover, let* (4.1) *and* (4.3) *be true. Then A with $D(A) = D_p$ generates an analytic C_0–semigroup $T(\cdot)$ in $L^p(\mathbb{R}^d)$ such that $\|T_p(z)\| \leq 1$ for $|\arg z| \leq \phi_p$ and some $\phi_p > 0$.*

In order to extend Theorem 3.7, let the assumptions of Theorem 4.2 hold for $p = 2$ and let the assumptions of Theorem 4.2 or 4.3 hold for some $p = r \leq 2$ or $p = r > 2$, respectively. In view of the proof of Theorem 3.7, we have to show that A generates positive and consistent semigroups on $L^2(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$. This is true for $r \leq 2$, since then the theory of Section 2 applies to $A^{(\varepsilon)}$ and $A^{(\varepsilon)}u \to Au$ as $\varepsilon \to 0$ for $u \in D(A_0) \cap D(V)$. Thus positivity and consistency follows from the Trotter–Kato theorem, [16, Theorem III.4.8]. The same argument works in the case $F = 0$ for all $r \in (1, \infty)$. If $r > 2$, we still have $T(t) \geq 0$ on $L^2(\mathbb{R}^d)$, so that it remains to verify that the semigroups on $L^r(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ coincide for $f \in L^2(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$. Consider again the operator $L_t u = A_0 u - Vu + tF \cdot \nabla u$, $0 \leq t \leq 1$. Note that the resolvent of $A_0 - V = L_0$ is consistent. For small $t > 0$ and large $\lambda \in \rho(L_0)$, we have

$$R(\lambda, L_t) = R(\lambda, L_0) \sum_{k=0}^{\infty} [tF \cdot \nabla R(\lambda, L_0)]^n.$$

Due to (4.9), this expansion implies that the resolvent of L_t is consistent. By finitely many iterations of this argument, we derive the consistency of the resolvent of $L_1 = A$, whence the consistency of the semigroups follows.

Theorem 4.4. *Let the assumptions of Theorem* 4.2 *hold for $p = 2$ and let the assumptions of Theorem* 4.2 *or* 4.3 *hold for some $p = r \leq 2$ or $p = r > 2$, respectively. Then the semigroups $T(t)$ are positive. Let $f \in L^q([0,T], L^r(\mathbb{R}^d))$ and $\varphi = 0$, for some $q \in (1, \infty)$ and $T > 0$. Then the solution u of* (1.3) *belongs $W^{1,q}([0,T], L^r(\mathbb{R}^d))$ and $A_0 u, Vu \in L^q([0,T], L^r(\mathbb{R}^d))$.*

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