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Invertibility of Convolution Operators in Problems of Wave Diffraction by a Strip with Reactance and Dirichlet Conditions

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Dedicated to Professor Frank-Olme Speck on the occasion of his 60th birthday

Abstract. This paper deals with the invertibility of convolution type operators that come from a wave diffraction problem with reactance conditions on a strip. The diffraction problem is reformulated as a single convolution type operator on a finite interval. To develop an operator constructive approach, several matrix operator identities are established between this convolution type operator and certain new Wiener-Hopf operators, and certain equivalent properties are obtained between all the related operators. Factorizations are presented for particular semi-almost periodic matrix functions and the corresponding Wiener-Hopf operators. As a result, conditions are obtained to ensure the invertibility of all the convolution type operators associated with the problem. This leads to the well-posedness of the problem including the continuous dependence on the data. In obtaining our results a major role is played by the invertibility of the convolution type operator associated with the wave diffraction by a strip with equal Dirichlet conditions on both sides of the strip, which is obtained through an analytical representation. Both problems and the corresponding operators are considered in the framework of Bessel potential spaces.

Keywords: *Convolution type operator, wave diffraction, Wiener-Hopf operator, equivalence after extension relation*

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1. Introduction

This paper is devoted to the study of convolution type operators

$$\mathcal{W}_{\Phi,\mathcal{I}} = r_{\mathbb{R}\to\mathcal{I}}\mathcal{F}^{-1}\Phi \cdot \mathcal{F}\,, \tag{1.1}$$

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that arise in wave diffraction problems with reactance conditions [5, 6, 14, 15, 17] (and with Dirichlet conditions [10, 16, 17]) on a strip [10]. Here $\mathcal{F}$ denotes the Fourier transform operator, Φ is the Fourier transform of the convolution kernel which, in general, will be a matrix function with piecewise continuous or semi-almost periodic elements, $\mathcal{I} \subset \mathbb{R}$ and $r_{\mathbb{R}\to\mathcal{I}}$ denotes the restriction operator from the real line to $\mathcal{I}$. We will consider the case when $\mathcal{I}$ is a finite interval and study the invertibility of the convolution type operator (1.1) in this case which models the (main) wave diffraction problem with reactance conditions on a finite strip. In particular, conditions are obtained to ensure the invertibility of the convolution type operator (1.1) associated with the main problem. This leads to the well-posedness of the main diffraction problem including the continuous dependence on the data. To this end, extension methods are used to obtain corresponding operators (1.1) with $\mathcal{I} = \mathbb{R}_+$ which fall into the class of Wiener-Hopf operators, so that the theory of Wiener-Hopf operators can be taken into account.

In the study of the Wiener-Hopf operators derived from the main problem an important role is played by the convolution type operator modelling the diffraction problem by a finite strip with Dirichlet conditions. The invertibility of this operator is obtained through extension methods based on operator matrix identities. On the other hand, the convolution type operators derived from the main diffraction problem allow us to choose a convenient Fourier transform of the kernel of one of these operators to work with. To study such a Fourier transform, some factorization procedures for semi-almost periodic matrix functions are proposed.

The theory will be developed in the framework of Bessel potential spaces. A Bessel potential space can be defined as the linear space of distributions, $\phi = r_{\mathbb{R}^n\to\Omega}\,\varphi$, that are obtained by restriction to $\Omega \subset \mathbb{R}^n$ of the elements in the space

$$H^s(\mathbb{R}^n) = \{\varphi \in \mathcal{D}'(\mathbb{R}^n) : \|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}} \cdot \mathcal{F}\varphi\|_{L^2(\mathbb{R}^n)} < +\infty\}$$

($s \in \mathbb{R}$). Moreover, the $H^s(\Omega)$ space endowed with the norm

$$\|\phi\|_{H^s(\Omega)} = \inf\left\{\|\varphi\|_{H^s(\mathbb{R}^n)} : \varphi \in H^s(\mathbb{R}^n), r_{\mathbb{R}^n\to\Omega}\,\varphi = \phi\right\}$$

becomes a Banach space. For $\mathcal{I} \subseteq \mathbb{R}_+$, we will denote by $\widetilde{H}^s(\mathcal{I})$ the closed subspace of $H^s(\mathbb{R})$ defined by the distributions with support contained in $\overline{\mathcal{I}}$. Moreover, in the special case with $s = 0$, we will use the more common notation of $L^2_+(\mathbb{R})$ and $L^2(\mathbb{R}_+)$ for representing $\widetilde{H}^0(\mathbb{R}_+)$ and $H^0(\mathbb{R}_+)$, respectively.

2. Formulation of the main problem

We will consider the problem of wave diffraction by a finite strip with reactance conditions. The finite strip is denoted here by $\Sigma =]0, a[$ where the dependence on one variable was dropped due to perpendicular wave incidence (which leads us from strips to intervals). The problem can be formulated, in a Bessel potential space setting, as the following boundary-transmission problem for the Helmholtz equation: Find $\varphi \in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}^2_\pm} \in H^1(\mathbb{R}^2_\pm)$, such that

$$\left(\Delta + k^2\right)\varphi = 0 \qquad \text{in} \quad \mathbb{R}^2_\pm \tag{2.1}$$

$$\begin{cases} \varphi_0^+ - \varphi_0^- = h_1 \\ \varphi_1^+ - \varphi_1^- + q\varphi_0^+ = h_2 \end{cases} \qquad \text{on} \quad \Sigma \tag{2.2}$$

$$\begin{cases} \varphi_0^+ - \varphi_0^- = 0 \\ \varphi_1^+ - \varphi_1^- = 0 \end{cases} \qquad \text{on} \quad \mathbb{R} \setminus \overline{\Sigma}\,, \tag{2.3}$$

where $\mathbb{R}^2_\pm$ represents the upper/lower half-plane, $k \in \mathbb{C}$ (with $\Im m\, k > 0$) stands for the wave number, $\varphi_0^\pm = \varphi_{|y=\pm 0}$, $\varphi_1^\pm = (\partial\varphi/\partial y)_{|y=\pm 0}$, $q \in \mathbb{C}$ is the reactance number and the elements $h_1 \in r_{\mathbb{R}\to\Sigma}\widetilde{H}^{1/2}(\Sigma)$, $h_2 \in r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma)$ are arbitrarily given (since the dependence on the data is to be studied for well-posedness). The Bessel potential spaces of order 1 and $\pm 1/2$ are naturally involved due to the energy norm and the *Trace Theorem* [4], respectively.

For the case when Σ is a half-line, the corresponding problem has previously been considered by many authors as a Sommerfeld type problem (see the fundamental survey paper [17], where the corresponding problem for the half-line case was described in the framework of operator theoretical methods). In [17, §5], such a half-line problem was also regarded as a certain class of general screen problems that were analyzed upon the boundary conditions considered. Note that the classical formulation of this problem (in a semi-plane instead of the present strip) usually assumes $h_1 = 0$ due to physical reasons. Here, from the mathematical point of view, we consider the present more general situation which leads also to more general corresponding operators.

The reason to consider the data in the restricted tilde spaces $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{1/2}(\Sigma)$ and $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma)$ is a consequence of the overlapping of the information in (2.2) and (2.3). Such realizations of the data are known as *compatibility conditions* [19] and appear in several different kinds of wave diffraction problems [15]. In fact, the first compatibility condition follows directly from the first equations in (2.2) and (2.3), whilst the second one follows from the second equations in (2.2) and (2.3) and by noting that we have the continuous embedding $H^{1/2}(\Sigma) \hookrightarrow r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma)$.

From an operator-theoretical point of view, the problem (2.1)–(2.3) can be described by the use of a single operator

$$L \quad : \quad D(L) \to \left(r_{\mathbb{R}\to\Sigma} \widetilde{H}^{1/2}(\Sigma) \right) \times \left(r_{\mathbb{R}\to\Sigma} \widetilde{H}^{-1/2}(\Sigma) \right) \tag{2.4}$$

defined as $L\varphi = (h_1, h_2)^T$ if $D(L)$ is defined as the subspace of $H^1(\mathbb{R}^2_+) \times H^1(\mathbb{R}^2_-)$ whose functions satisfy the Helmholtz equation (2.1) and the transmission condition (2.3). The operator L is said to be associated with the reactance problem. In what follows we will analyze if L is a bounded and invertible operator. As was already pointed out above, this will guarantee the well-posedness of the problem including the continuous dependence upon the data.

3. Description of the reactance problem by a single convolution type operator on tilde spaces

In this section we shall explore the structure behind the operator L, defined in (2.4). This will be done within the framework of convolution type operators. To this end, we need to implement some operator extension procedures of the following type.

Definition 3.1.

(i) Two operators W_1 and W_2 (acting between Banach spaces) are said to be *algebraically equivalent after extension* if there exist additional Banach spaces Z_1 and Z_2 and invertible linear operators E and F such that

$$\begin{bmatrix} W_1 & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} W_2 & 0 \\ 0 & I_{Z_2} \end{bmatrix} F\,. \tag{3.1}$$

(ii) If, in addition to (i), the invertible and linear operators E and F in (3.1) are bounded, then we will say that W_1 and W_2 are *topologically equivalent after extension* operators (or simply say that W_1 and W_2 are *equivalent after extension* operators [1]).

(iii) In (ii), if both Z_1 and Z_2 are trivial spaces, W_1 and W_2 are said to be *equivalent* operators.

Remark. The above notion of equivalence after extension coincides with the famous concept of *matricial coupling* between bounded linear operators, as was established for the first time in [1]. In [8] and [10], some differences are discussed between algebraic and topological equivalence after extension relations between convolution type operators.

Let

$$t(\xi) = \left(\xi^2 - k^2\right)^{\frac{1}{2}}, \qquad \xi \in \mathbb{R}$$

denote the branch of the square root that tends to $+\infty$ as $\xi \to +\infty$ with branch cuts along $\pm k \pm i\eta$, $\eta \geq 0$. Then we have the following result on the structure of the operator L.

Theorem 3.2. *The operator L is equivalent after extension to the convolution type operator*

$$\widetilde{\mathcal{W}}_{\Phi,\Sigma} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi \cdot \mathcal{F} : \widetilde{H}^{-1/2}(\Sigma) \to r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma), \tag{3.2}$$

where

$$\Phi = 1 - \frac{1}{2}\, q\, t^{-1}. \tag{3.3}$$

Proof. Note first the well-known fact [17] that a function $\varphi \in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}^2_\pm} \in H^1(\mathbb{R}^2_\pm)$, satisfies the Helmholtz equation (2.1) if and only if it can be expressed as

$$\varphi(x,y) = \mathcal{F}^{-1}_{\xi\mapsto x} e^{-t(\xi)y}\mathcal{F}_{x\mapsto\xi}\varphi_0^+(x)\chi_{\mathbb{R}_+}(y) + \mathcal{F}^{-1}_{\xi\mapsto x} e^{t(\xi)y}\mathcal{F}_{x\mapsto\xi}\varphi_0^-(x)\chi_{\mathbb{R}_-}(y) \tag{3.4}$$

for $(x,y) \in \mathbb{R}^2$, where $\mathcal{F}_{x\mapsto\xi}\varphi(x,y) \int_{\mathbb{R}} \varphi(x,y)e^{i\xi x}dx$, and $\chi_{\mathbb{R}_+}$ and $\chi_{\mathbb{R}_-}$ denote the characteristic functions of the positive and negative half-line, respectively.

Define the space

$$Z = \Big\{(\phi,\psi) \in \left[H^{1/2}(\mathbb{R})\right]^2 : \phi - \psi \in \widetilde{H}^{1/2}(\Sigma), \quad \mathcal{F}^{-1}t\cdot\mathcal{F}(\phi+\psi) \in \widetilde{H}^{-1/2}(\Sigma)\Big\}.$$

Then the trace operator $T_0 : D(L) \to Z$ defined by

$$T_0\varphi = \varphi_0 := \begin{bmatrix} \varphi_0^+ \\ \varphi_0^- \end{bmatrix}$$

is an invertible operator. In fact, such a trace operator is continuously invertible with the inverse operator $K : \varphi_0 \mapsto \varphi$ defined by the representation formula (3.4). Moreover, with the help of the operators T_0 and K the operator L can be rewritten in the form of an operator matrix composition depending on $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$ (and which can be checked by direct computation):

$$L = \begin{bmatrix} 0 & r_{\mathbb{R}\to\Sigma} \\ I_{r_{\mathbb{R}\to\Sigma}\widetilde{H}^{1/2}(\Sigma)} & \frac{q}{2}\, r_{\mathbb{R}\to\Sigma} \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{W}}_{\Phi,\Sigma} & 0 \\ 0 & I_{\widetilde{H}^{1/2}(\Sigma)} \end{bmatrix} \mathcal{W}_{\Phi_1,\mathbb{R}}\, T_0, \tag{3.5}$$

where $\mathcal{W}_{\Phi_1,\mathbb{R}}$ is the convolution operator on the whole line

$$\mathcal{W}_{\Phi_1,\mathbb{R}} = \mathcal{F}^{-1}\Phi_1 \cdot \mathcal{F} \quad : \quad Z \to \widetilde{H}^{-1/2}(\Sigma) \times \widetilde{H}^{1/2}(\Sigma),$$

with

$$\Phi_1 = \begin{bmatrix} -t & -t \\ 1 & -1 \end{bmatrix}.$$

Now it can be easily verified that the matrix operator

$$\begin{bmatrix} 0 & r_{\mathbb{R}\to\Sigma} \\ I_{r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma)} & \frac{q}{2}\, r_{\mathbb{R}\to\Sigma} \end{bmatrix},$$

which maps $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma) \times \widetilde{H}^{1/2}(\Sigma)$ into $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{1/2}(\Sigma) \times r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma)$, is a bounded, invertible operator with the inverse

$$\begin{bmatrix} -\frac{q}{2} I_{r_{\mathbb{R}\to\Sigma}\widetilde{H}^{1/2}(\Sigma)} & I_{r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma)} \\ l_{0|r_{\mathbb{R}\to\Sigma}\widetilde{H}^{1/2}(\Sigma)} & 0 \end{bmatrix}.$$

It is also easy to see that $\mathcal{W}_{\Phi_1,\mathbb{R}} T_0$ is continuously invertible with the inverse operator

$$K\,\mathcal{W}^{-1}_{\Phi_1,\mathbb{R}} = K\,\mathcal{W}_{\Phi_1^{-1},\mathbb{R}} \,:\, \widetilde{H}^{-1/2}(\Sigma) \times \widetilde{H}^{1/2}(\Sigma) \to D(L)\,.$$

Therefore, (3.5) represents an equivalence after extension relation between L and the convolution type operator on a finite interval, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, defined in (3.2). ∎

4. Operator extensions concerning space orders and supports

In this section we are interested in studying the invertibility of the operator $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$. To this end, we choose to work with operators *connected* with this one but having a *better* structure. In particular, we will make use of Wiener-Hopf operators (having therefore standard Bessel potential spaces as their image spaces instead of the above restricted tilde spaces, see (3.2)). We do this by first considering an auxiliary problem of wave diffraction by a finite strip with Dirichlet boundary conditions (which leads to a bounded, invertible convolution type operator) and then extending the operator $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$ by use of the above auxiliary operator, which allows us to work in a L^2-space setting.

4.1. The Dirichlet problem of wave diffraction by a finite strip. We now consider the (auxiliary) problem of wave diffraction by a finite strip with Dirichlet boundary conditions. This problem leads to a bounded, invertible convolution type operator which plays an important role in proving the invertibility of the convolution type operator $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$.

Dirichlet Problem: Find $\varphi \in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}^2_\pm} \in H^1(\mathbb{R}^2_\pm)$, such that

$$\begin{array}{rcll} \left(\Delta + k^2\right)\varphi & = & 0 & \text{in } \mathbb{R}^2_\pm \\ \left\{\begin{array}{l}\varphi_0^+ \\ \varphi_0^-\end{array}\right. & \begin{array}{c}= \\ =\end{array} & \begin{array}{l}h \\ h\end{array} & \text{on } \Sigma \\ \left\{\begin{array}{l}\varphi_0^+ - \varphi_0^- \\ \varphi_1^+ - \varphi_1^-\end{array}\right. & \begin{array}{c}= \\ =\end{array} & \begin{array}{l}0 \\ 0\end{array} & \text{on } \mathbb{R}\setminus\overline{\Sigma}\,, \end{array} \tag{4.1}$$

where $k \in \mathbb{C}$ (with $\Im m\, k > 0$) is the wave number and $h \in H^{1/2}(\Sigma)$ is a given function.

Reasoning similarly as in §2 and §3, this problem leads to the associated convolution type operator

$$\mathcal{W}_{t^{-1},\Sigma} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}t^{-1}\cdot\mathcal{F} : \widetilde{H}^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)\,. \tag{4.2}$$

Remark. Imposing the same boundary data h in both equations of (4.1) incorporates already some compatibility between the boundary data (and corresponds also to physically the most important case). If one allows different given data in these two equations, say $h_1 \neq h_2$, we will need to add some compatibility conditions between them, which will lead as before to restricted tilde image spaces (precisely, we need to have $h_1 - h_2 \in r_{\mathbb{R}\to\Sigma}\widetilde{H}^{1/2}(\Sigma)$).

Theorem 4.1. *The convolution type operator $\mathcal{W}_{t^{-1},\Sigma}$, which is defined in* (4.2) *and associated to the Dirichlet problem, is equivalent after extension to the Wiener-Hopf operator*

$$\mathcal{W}_{\Upsilon,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon\cdot\mathcal{F} \quad : \quad \left[L^2_+(\mathbb{R})\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2,$$

with

$$\Upsilon = \begin{bmatrix} \zeta^{-\frac{1}{2}}\tau_{-a} & 0 \\ \lambda_-^{\frac{1}{2}}t^{-1}\lambda_+^{\frac{1}{2}} & \zeta^{\frac{1}{2}}\tau_a \end{bmatrix}, \tag{4.3}$$

where $\zeta = \lambda_-/\lambda_+$ and $\lambda_\pm(\xi) = \xi \pm k$, $\tau_b(\xi) = \exp[i\xi b]$, for $\xi \in \mathbb{R}$.

Thus there are bounded, invertible linear operators E_1 and F_1 and Banach spaces X and Y such that

$$\begin{bmatrix} \mathcal{W}_{t^{-1},\Sigma} & 0 \\ 0 & I_X \end{bmatrix} = E_1 \begin{bmatrix} \mathcal{W}_{\Upsilon,\mathbb{R}_+} & 0 \\ 0 & I_Y \end{bmatrix} F_1\,. \tag{4.4}$$

Proof. First it follows from [12, Theorem 2.1] (see also [7, 13] for some generalizations) that $\mathcal{W}_{t^{-1},\Sigma}$ is algebraically equivalent after extension to the following Wiener-Hopf operator

$$\mathcal{W}_{\Upsilon_1,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_1 \cdot \mathcal{F} : \widetilde{H}^{-1/2}(\mathbb{R}_+)\times\widetilde{H}^{1/2}(\mathbb{R}_+) \to H^{-1/2}(\mathbb{R}_+)\times H^{1/2}(\mathbb{R}_+)\,,$$

with

$$\Upsilon_1 = \begin{bmatrix} \tau_{-a} & 0 \\ t^{-1} & \tau_a \end{bmatrix}.$$

Thus (3.1) holds with W_1 and W_2 being replaced by $\mathcal{W}_{t^{-1},\Sigma}$ and $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$, respectively, and for some linear invertible (not necessarily bounded) operators E and F.

Next, we show that the Wiener-Hopf operator $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ is equivalent to $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$. Here, the operator equivalence in question is constructed in an explicit way and can be directly obtained by computing the following operator composition:

$$\mathcal{W}_{\Upsilon_1,\mathbb{R}_+} = E_2\, \mathcal{W}_{\Upsilon,\mathbb{R}_+}\, F_2\,, \tag{4.5}$$

where E_2 and F_2 are defined by

$$E_2 = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1} \begin{bmatrix} \lambda_-^{\frac{1}{2}} & 0 \\ 0 & \lambda_-^{-\frac{1}{2}} \end{bmatrix} \cdot \mathcal{F} l_0 : \left[L^2(\mathbb{R}_+)\right]^2 \to H^{-1/2}(\mathbb{R}_+)\times H^{1/2}(\mathbb{R}_+)$$

$$F_2 = l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1} \begin{bmatrix} \lambda_+^{-\frac{1}{2}} & 0 \\ 0 & \lambda_+^{\frac{1}{2}} \end{bmatrix} \cdot \mathcal{F} : \widetilde{H}^{-1/2}(\mathbb{R}_+)\times\widetilde{H}^{1/2}(\mathbb{R}_+) \to \left[L^2_+(\mathbb{R})\right]^2,$$

with $l_0 : \left[L^2(\mathbb{R}_+)\right]^2 \to \left[L^2_+(\mathbb{R})\right]^2$ being the zero extension operator. In fact, the bounded operators E_2 and F_2 are invertible with

$$E_2^{-1} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1} \begin{bmatrix} \lambda_-^{-\frac{1}{2}} & 0 \\ 0 & \lambda_-^{\frac{1}{2}} \end{bmatrix} \cdot \mathcal{F} l_0$$

$$F_2^{-1} = l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1} \begin{bmatrix} \lambda_+^{\frac{1}{2}} & 0 \\ 0 & \lambda_+^{-\frac{1}{2}} \end{bmatrix} \cdot \mathcal{F}$$

(see [22, §2.10.3]). In view of the structure of the Fourier symbols of E_2 and F_2 [22], it follows that the right hand-side of (4.5) can be rewritten in the form of an unique Wiener-Hopf operator with Υ_1 as its Fourier symbol.

We now study the Fredholm property of the Wiener-Hopf operator $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$. The Fourier symbol Υ (see (4.3)) of the Wiener-Hopf operator $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ belongs to the C^*-algebra of the semi-almost periodic (SAP) two by two matrix functions on the real line (see [3, 21]). This means that Υ belongs to the smallest closed subalgebra of $[L^\infty(\mathbb{R})]^{2\times 2}$ which contains the (classical) algebra of (two by two) *almost periodic elements* and the (two by two) continuous matrices with possible jumps at infinity. Additionally, the element in the second row and first column of Υ (that is, the lifted Fourier symbol of $\mathcal{W}_{t^{-1},\Sigma}$) is 1. Thus, and by the criteria for the Fredholm property of such operators (see [2]), we conclude that $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ is a Fredholm operator with index zero.

Now, since $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ is equivalent to $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ and algebraically equivalent after extension to $\mathcal{W}_{t^{-1},\Sigma}$ (through the operator identity (4.5)), and by noting the structure of the identity (4.5), it follows that the operators $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ and $\mathcal{W}_{t^{-1},\Sigma}$ are also Fredholm operators with index zero. Moreover, from the operator identities provided by both the equivalence relation and the algebraic equivalence after extension relation, we have equal dimensions for the corresponding defect spaces of all the three operators $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$, $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ and $\mathcal{W}_{t^{-1},\Sigma}$. From this, and since by [1, Theorem 3] we have that Fredholm operators in Banach spaces are equivalent after extension if and only if their corresponding defect spaces have equal dimensions, the last statement of Theorem 4.1 follows. ∎

Remark. Note that the smoothness orders of the spaces in the definition of the operator $\mathcal{W}_{t^{-1},\Sigma}$ are the so-called *critical orders* [10]. For spaces with such smoothness orders the method of constructing equivalence after extension relations proposed in [10] does not work.

We now factorize the Fourier symbol Υ in such a way that the influence of the oscillating behavior (at infinity) of the elements will be removed. To this end, we use a technique due to Novokshenov [20] and propose the following factorization of Υ (which will lead to the inverse of the corresponding Wiener-Hopf operator):

$$\begin{aligned}
\Upsilon &= \begin{bmatrix} \tau_{-a}\rho & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \tau_a\left(\zeta^{\frac{1}{2}}-\rho\right) \\ \tau_{-a}\left(\rho-\zeta^{-\frac{1}{2}}\right) & \rho\left(\zeta^{\frac{1}{2}}+\zeta^{-\frac{1}{2}}-\rho\right) \end{bmatrix} \begin{bmatrix} 1 & \tau_a\rho \\ 0 & 1 \end{bmatrix} \\
&= \left(\begin{bmatrix} \tau_{-a}\rho & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau_{-a}\left(\rho-\zeta^{-\frac{1}{2}}\right) & 1 \end{bmatrix} \right) \\
&\quad \times \left(\begin{bmatrix} 1 & \tau_a\left(\zeta^{\frac{1}{2}}-\rho\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tau_a\rho \\ 0 & 1 \end{bmatrix} \right) \\
&= \Upsilon_- \ \Upsilon_+ .
\end{aligned} \tag{4.6}$$

Here we use the normalized sine function

$$\rho(\xi) = \frac{2}{\pi} \int_0^{\xi} \frac{\sin y}{y}\, dy$$

which has the following useful behavior at infinity:

$$\rho(\xi) = \operatorname{sign} \xi + \mathcal{O}\big(|\xi|^{-1}\big)\ .$$

Note also that

$$\tau_{\pm a}\rho \in H^{\infty}_{\pm}\,, \tag{4.7}$$

that is, $\tau_{\pm a}\rho$ are functions bounded and holomorphic in the upper/lower half-planes.

Theorem 4.2. *The Wiener-Hopf operator $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ with SAP Fourier symbol Υ is invertible with its inverse being given by the formula*

$$\mathcal{W}^{-1}_{\Upsilon,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_+^{-1}\cdot\mathcal{F}l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_-^{-1}\cdot\mathcal{F}\,. \tag{4.8}$$

Proof. The result is a direct consequence of the structure of Υ_- and Υ_+ (particularly because of (4.7)) and of the corresponding factorization (4.6). ∎

Corollary 4.3. *The convolution type operator $\mathcal{W}_{t^{-1},\Sigma}$ defined in* (4.2) *is bounded and invertible with its inverse*

$$\mathcal{W}^{-1}_{t^{-1},\Sigma} = B_{11}\,,$$

where B_{11} is the operator in the first block (with respect to the natural space decomposition) of the operator matrix

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = F_1^{-1} \begin{bmatrix} r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_+^{-1}\cdot\mathcal{F}l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_-^{-1}\cdot\mathcal{F} & 0 \\ 0 & I_Y \end{bmatrix} E_1^{-1}\,,$$

and E_1 and F_1 are the same as in Theorem 4.1.

Proof. The result follows directly from Theorems 4.1 and 4.2 in conjunction with (4.4) and (4.8). ∎

4.2. Extended operators for the reactance problem. Considering now the composition of the operators $\mathcal{W}_{t^{-1},\Sigma}$ and $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, we have the following result.

Corollary 4.4. *The operator $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$ is equivalent to*

$$\mathcal{W}_{\Phi,\Sigma,1/2} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi\cdot\mathcal{F}l : H^{1/2}(\Sigma) \to H^{1/2}(\Sigma)\,,$$

where $l : H^{1/2}(\Sigma) \to H^{1/2}(\mathbb{R})$ is an extension operator (the particular choice of which does not change the definition of $\mathcal{W}_{\Phi,\Sigma,1/2}$).

Proof. From Theorem 4.1 and the special form of Φ (see (3.3)), we have the following equivalent equations:

$$\begin{aligned}\widetilde{\mathcal{W}}_{\Phi,\Sigma} f &= g\\ \mathcal{W}_{t^{-1},\Sigma} l_0 \widetilde{\mathcal{W}}_{\Phi,\Sigma} f &= \mathcal{W}_{t^{-1},\Sigma} l_0 g\\ r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi\cdot\mathcal{F} l \mathcal{W}_{t^{-1},\Sigma} f &= \mathcal{W}_{t^{-1},\Sigma} l_0 g \qquad (4.9)\end{aligned}$$

for $f \in \widetilde{H}^{-1/2}(\Sigma)$ and $g \in r_{\mathbb{R}\to\Sigma}\widetilde{H}^{-1/2}(\Sigma)$, where l is an operator of extension whose particular form does not change the left hand-side of (4.9). In fact, the equation (4.9), which involves the action of the operator $\mathcal{W}_{\Phi,\Sigma,1/2}$, can be written in the form

$$\varphi(\xi) - \frac{q}{2}\int_0^a \mathcal{F}^{-1}t^{-1}(\xi - x)\,\varphi(x)\,dx = \psi(\xi)\,, \qquad \xi \in \Sigma\,.$$

It is clear that the above equation is dependent on $\varphi \in H^{1/2}(\Sigma)$ and independent of the remaining part of the extension $l\varphi = l\mathcal{W}_{t^{-1},\Sigma} f \in H^{1/2}(\mathbb{R})$. ∎

Remark. Based on a *transmission property*, in [9] are discussed different possibilities of improving the space smoothness of convolution type operators on a finite interval.

Now, instead of studying $\mathcal{W}_{\Phi,\Sigma,1/2}$ directly, we consider the following image and domain extension of $\mathcal{W}_{\Phi,\Sigma,1/2}$:

$$\mathcal{W}_{\Phi,\Sigma,0} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi\cdot\mathcal{F} l_0 : L^2(\Sigma) \to L^2(\Sigma),$$

which is a linear and bounded operator. Note that $\dim\operatorname{coker}\mathcal{W}_{\Phi,\Sigma,1/2} = \dim\operatorname{coker}\mathcal{W}_{\Phi,\Sigma,0}$ and that $\dim\ker\mathcal{W}_{\Phi,\Sigma,1/2} = \dim\ker\mathcal{W}_{\Phi,\Sigma,0}$. This is a consequence of the structure of Φ (which can be presented in terms of operators as the *identity plus additional smoothing*) and in terms of the space embedding $H^{1/2}(\Sigma) \hookrightarrow L^2(\Sigma)$. Moreover, if we have the knowledge of $\mathcal{W}^{-1}_{\Phi,\Sigma,0}$ (the inverse of $\mathcal{W}_{\Phi,\Sigma,0}$), then a representation of the inverse of $\mathcal{W}_{\Phi,\Sigma,1/2}$ can be derived from $\mathcal{W}^{-1}_{\Phi,\Sigma,0}$ by use of the corresponding space restrictions.

Theorem 4.5. *The convolution type operator $\mathcal{W}_{\Phi,\Sigma,0}$ is equivalent after extension to the following Fredholm operator with vanishing analytical index:*

$$\mathcal{W}_{\Psi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi\cdot\mathcal{F} l_0 : \left[L^2(\mathbb{R}_+)\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2,$$

where

$$\Psi = \begin{bmatrix} \tau_{-a} & 0 \\ \Phi & \tau_a \end{bmatrix}$$

and $\tau_a(\xi) = \exp[i\xi a]$ *for* $\xi \in \mathbb{R}$.

Proof. Since $\Phi(\pm\infty) = 1$ and $\Phi(\xi) \neq 0$, for all $\xi \in \mathbb{R}$, the operator $\mathcal{W}_{\Psi,\mathbb{R}_+}$ has the Fredholm property (see, for example, [2, Theorem 4.1]). Further, $\mathcal{W}_{\Psi,\mathbb{R}_+}$ has zero Fredholm index since the continuous function on the real line Φ has no jumps at infinity (cf., e.g., the Fredholm index formula (2.14) in [11, Theorem 2.10]). The theorem thus follows by arguing similarly as in the proof of Theorem 4.1. ∎

From Theorem 4.5 and the equivalence relations between the operators L, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$ the following corollary follows easily.

Corollary 4.6. *The operators L, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$ are Fredholm operators with zero index.*

5. Analysis of Fourier symbol Ψ and invertibility of related operators

We are now in a position to prove the invertibility of all our main convolution type operators. In doing this, we need a new operator factorization scheme provided with the help of an auxiliary invertible Wiener-Hopf operator.

Let $Q = \left\{c \in \mathbb{C} : 1 - \frac{1}{2}\, c\, t^{-1}(\xi) \neq 0 \text{ for } \xi \in \mathbb{R}\right\}$. Then we have the following result.

Lemma 5.1. *If $q \in Q$, then the Wiener-Hopf operator*

$$\mathcal{W}_{\Phi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi\cdot\mathcal{F}l_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

is invertible with the inverse $\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi_+^{-1}\cdot\mathcal{F}l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi_-^{-1}\cdot\mathcal{F}l_0$, where

$$\Phi_\pm = \exp\left\{\frac{1}{2}(I \pm S_{\mathbb{R}})\ \log\left(1 - \frac{q}{2}\, t^{-1}\right)\right\} \tag{5.1}$$

and $S_{\mathbb{R}}$ is the Cauchy integral operator on $\mathbb{R}$.

Proof. Note that for $q \in Q$ the Fourier symbol $\Phi = 1 - \frac{1}{2}\, q\, t^{-1}$ is a non-vanishing continuous function on the real line with the same nonzero limits at $\pm\infty$. Thus, by use of the well-known Fredholm criterium for Wiener-Hopf operators with continuous Fourier symbols (see, e.g., [3, Theorem 2.15]), it follows that $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is a Fredholm operator. Further, noting that, as ξ moves from $-\infty$ to $+\infty$, the point $\Phi(\xi)$ traces out a continuous oriented curve in $\mathbb{C}\backslash\{0\}$ having zero windings around the origin, it follows that $\mathcal{W}_{\Phi,\mathbb{R}_+}$ has a zero Fredholm index.

On other hand, since $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is a scalar Wiener-Hopf operator with a nonzero Fourier symbol, the *Coburn Theorem* (see [3, Theorem 2.5]) can be applied to derive that $\ker \mathcal{W}_{\Phi,\mathbb{R}_+} = \{0\}$ or the range of $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is dense in $L^2(\mathbb{R}_+)$.

Consequently, $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is invertible. By the factorization theory of continuous functions (see [18, Chapter III, §5]) we obtain the representation (5.1) for the construction of the inverse operator. ∎

Theorem 5.2. *Let $q \in Q$. The Wiener-Hopf operator $\mathcal{W}_{\Psi,\mathbb{R}_+} : [L^2(\mathbb{R}_+)]^2 \to [L^2(\mathbb{R}_+)]^2$ is a bounded and invertible operator.*

Proof. That $\mathcal{W}_{\Psi,\mathbb{R}_+}$ is a bounded operator is clear since Ψ is an essentially bounded function.

We now prove the invertibility of $\mathcal{W}_{\Psi,\mathbb{R}_+}$. We first factorize $\mathcal{W}_{\Psi,\mathbb{R}_+}$ in the form

$$\begin{aligned}\mathcal{W}_{\Psi,\mathbb{R}_+} &= \begin{bmatrix} I & r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}l_0\,\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & \mathcal{W}_{\Phi,\mathbb{R}_+} \end{bmatrix} \\ &\quad\times \begin{bmatrix} 0 & -I \\ I & \mathcal{W}^{-1}_{\Phi,\mathbb{R}_+} r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0 \end{bmatrix},\end{aligned} \tag{5.2}$$

where

$$\mathcal{C} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}l_0\,\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}\, r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)\,,$$

and by Lemma 5.1 the inverse of $\mathcal{W}_{\Phi,\mathbb{R}_+}$ exists since $q \in Q$.

From (5.2) and Lemma 5.1 it can be seen that $\mathcal{W}_{\Psi,\mathbb{R}_+}$ is invertible if and only if $\mathcal{C}$ is invertible. By (5.2) and Theorem 4.5 we conclude that $\mathcal{C}$ is a Fredholm operator with a vanishing analytical index. Thus, to derive the invertibility of $\mathcal{W}_{\Psi,\mathbb{R}_+}$ it is enough to show that $\mathcal{C}$ is an injective operator, that is, $\langle\mathcal{C}\varphi,\varphi\rangle_{L^2(\mathbb{R}_+)} = 0$ implies $\varphi = 0$. Now for $\varphi \in L^2(\mathbb{R}_+)$,

$$\langle\mathcal{C}\varphi,\varphi\rangle_{L^2(\mathbb{R}_+)} = \langle\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}\, r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0\varphi, r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0\varphi\rangle_{L^2(\mathbb{R}_+)}\,.$$

Thus it is enough to show that

$$\langle\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}\,\phi,\phi\rangle_{L^2(\mathbb{R}_+)} = 0 \qquad \text{implies} \qquad \phi = 0 \tag{5.3}$$

since the right a-shift operator $r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is obviously injective.

Let $\psi = \mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}\,\phi$. Then (5.3) is equivalent to

$$\langle\psi,\mathcal{W}_{\Phi,\mathbb{R}_+}\,\psi\rangle_{L^2(\mathbb{R}_+)} = 0 \qquad \text{implies} \qquad \psi = 0\,. \tag{5.4}$$

Since $\Phi = 1 - \frac{1}{2}\,q\,t^{-1}$ and

$$\langle\psi,\mathcal{W}_{\Phi,\mathbb{R}_+}\,\psi\rangle_{L^2(\mathbb{R}_+)} = \langle\mathcal{F}l_0\psi,\Phi\cdot\mathcal{F}l_0\,\psi\rangle_{L^2(\mathbb{R})}\,,$$

it follows that (5.4) is true. ∎

From Theorem 5.2 and the equivalence relations between the operators L, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$ we have the following corollary.

Corollary 5.3. *Let $q \in Q$. The operators L, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$ are all invertible.*

As pointed out at the end of Section 2, this corollary directly yields the following result on the well-posedness of the main diffraction problem.

Corollary 5.4. *Let $q \in Q$. There is a unique solution $\varphi \in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}^2_\pm} \in H^1(\mathbb{R}^2_\pm)$, to the reactance diffraction problem* (2.1)–(2.3) *which is continuously dependent on the data with respect to the indicated space topologies.*

We conclude this paper with some final remarks:

`(i)` Our approach depends on the particular structure of the Fourier symbols of the convolution type operators. In particular, in establishing the equivalence between the equations in (4.9), a predominant role is played by the particular symbol t^{-1}. Thus, for different classes of operators, new techniques are needed to obtain, for example, certain commutative properties of the corresponding composition operators. Partial results into this direction may be found in [10].

`(ii)` It is expected to generalize the present method to other interesting classes of boundary transmission problems such as those involving *third kind boundary conditions* [17] on a strip. In such cases, one of the difficulties is the invertibility of certain matrix Wiener-Hopf operators that should be used in the place of $\mathcal{W}_{\Phi,\mathbb{R}_+}$ in the present section.

`(iii)` The present approach (reduction by matrix factorization) applies to wider classes of operators compared with the idea of perturbing the matrix operator

$$\begin{bmatrix} r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}l_0 & 0 \\ I & r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0 \end{bmatrix} : \left[L^2(\mathbb{R}_+)\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2,$$

by replacing I with a strongly elliptic operator. This is because our approach allows consideration of invertible operators which may not be strongly elliptic.

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