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A General Class of Autoconvolution Equations of the Third Kind

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Dedicated to Prof. Lothar Berg on the occasion of his 75-th birthday

Abstract. Continuing recent investigations by L. Berg and L. v. Wolfersdorf on a model integral equation of autoconvolution type of the third kind, two existence theorems for a general class of such equations are derived. Further, an existence theorem is proved for the model equation with data and solutions of a general logarithmic form. Moreover, a singular perturbation problem for a related integrodifferential equation of first order to the model equation is studied which could serve as a basis for its regularization by the Lavrentiev method.

Keywords: *Quadratic integral equations, autoconvolution equations, singular perturbation of equations*

MSC 2000: 45G10, 45D05, 45J05, 34E15

1. Introduction

Initiated by L. Berg, in the joint paper of him with the second author [2] a class of generalized autoconvolution equations of the third kind has been studied. As remarked in [1] and [8] such equations have infinitely many solutions. But with a suitable ansatz for the solution and after some transformation a theorem of the first author [6] about the iteration method with weighted norms in the Banach space of continuous functions on a closed interval could be used for proving existence of solutions to these equations.

As a complement to the investigations in [2], in the present paper we deal with a general class of such autoconvolution equations of the form

$$k(x)y(x) = \int_0^x m(x,\xi)y(\xi)y(x-\xi)\,d\xi + \int_0^x n(x,\xi)y(\xi)\,d\xi + p(x) \tag{1.1}$$

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for $0 \leq x \leq T$, with given continuous functions k, m, n, p, where $k(0) = 0$. For $m(x, \xi) = a(\xi)$ and $n(x, \xi) = p(x) = 0$ this equation is the *model equation* considered in [2]. The general class of equations (1.1) contains the well-known integral equations of F. Bernstein [3] and F. Bernstein and G. Doetsch [4, 5] for the elliptic theta zero function and for the Mittag-Leffler function, but under our assumptions unfortunately only the latter equation can be treated. Further, following [2] we restrict ourselves to basic existence theorems for solutions of (1.1) with power or logarithm behaviour at $x = 0$. But we expect that also theorems on the smoothness of the solutions for the model equation in [2] could be extended to equation (1.1). Moreover, we add to the existence theorems in [2] a such one for a class of model equations with data k, a and solution y containing general logarithmic terms. Finally, as a new aspect a singular perturbation problem for a related integrodifferential equation of first order to the model equation in the superlinear case of [2] is investigated. The results of this investigation are basic for a regularization of the model integral equation of the third kind by a neighbouring integrodifferential equation (a kind of *Lavrentiev regularization*, cf. [7]).

We remark that with a solution y also the function $e^{Cx}y$, where C is an arbitrary constant, is a solution to equation (1.1) if $n = p = 0$ as in the case of the model equation. In the general case we have to expect a more complex structure of a general solution of (1.1).

The plan of the paper is as follows. After this introduction given in Section 1 we deal with the singular perturbation problem and the general logarithmic case of the model equation in Sections 2 and 3, respectively. The general class of equations (1.1) is then treated in Section 4.

The existence proofs in the paper are based on (a simplified version of) an existence theorem from [6] for operator equations of the form

$$z(x) = f(x) + G[z](x) + L[z, z](x) \tag{1.2}$$

with a linear operator G and a bilinear operator L in $C[0, T]$, $0 < T < \infty$, with the exponentially weighted norms

$$\|z\|_\sigma = \left\|e^{-\sigma x} z(x)\right\| = \max_{0 \leq x \leq T} \left|e^{-\sigma x} z(x)\right| , \quad \sigma > 1 ,$$

where $\|z\| = \|z\|_0$, which we cite here as Lemma 1 for convenience of the reader.

Lemma 1. *Let the linear operator $G : C[0, T] \to C[0, T]$ and the bilinear operator $L : C[0, T] \times C[0, T] \to C[0, T]$ fulfill the inequalities*

$$\|G[z]\|_\sigma \leq M(\sigma)\|z\|_\sigma , \quad \sigma \geq \sigma_0 > 1 \tag{1.3}$$

for any $z \in C[0, T]$ with a continuous function M satisfying $M(\sigma) \to 0$ as $\sigma \to \infty$, and

$$\|L[z_1, z_2]\|_\sigma \leq N\|z_1\|_\sigma\|z_2\|_\sigma , \quad \sigma \geq \sigma_0 > 1 \tag{1.4}$$

with a constant N and

$$\|L[z_1, z_2]\|_\sigma \;\le\; \begin{cases} \nu_1(\sigma)\|z_1\|\,\|z_2\|_\sigma \\ \nu_2(\sigma)\|z_1\|_\sigma\|z_2\| \end{cases} \tag{1.5}$$

with continuous functions ν_k, $k = 1, 2$, satisfying $\nu_k(\sigma) \to 0$ as $\sigma \to \infty$ for any pair $z_1, z_2 \in C[0, T]$. Then equation (1.2) has a uniquely determined solution $z \in C[0, T]$. Moreover, for solutions z_1 and z_2 corresponding to functions $f = f_1$ and $f = f_2$, respectively, the stability estimate

$$\|z_1 - z_2\| \;\le\; \Lambda(Q_1, Q_2)\|f_1 - f_2\| \tag{1.6}$$

holds, where $Q_k = (\|f_k\|, \|G[f_k]\|)$, $k = 1, 2$, and $\Lambda \in C\left(\mathbb{R}^4_+ \to \mathbb{R}\right)$, $\Lambda > 0$ with $\Lambda(x_1, \dots, x_4)$ increasing in $x_1, \dots, x_4$.

2. Singular perturbation problem

Let us consider the model equation [2]

$$k(x)y(x) \;=\; \int_0^x a(\xi)y(x-\xi)y(\xi)\,d\xi\,. \tag{2.1}$$

If $k(x) \sim Ax$, $A > 0$ and $a(x) \sim 1$ as $x \to 0$ then the continuous solutions y of (2.1) have at $x = 0$ either the value $y(0) = 0$ or the value $y(0) = A$. With interest in the second case, in this section we study the initial value problem for the related integrodifferential equation of the first order

$$\epsilon y_\epsilon'(x) + k(x)y_\epsilon(x) \;=\; \int_0^x a(\xi)y_\epsilon(x-\xi)y_\epsilon(\xi)\,d\xi\,, \quad y_\epsilon(0) \;=\; A \tag{2.2}$$

with $\epsilon \neq 0$. We remark that $y(0) = 0$ for a continuous solution y of (2.1) is only fulfilled for the trivial solution $y(x) \equiv 0$ if, in addition to the above asymptotic relations, there holds $k, a \in C[0, T]$ with $k(x) > 0$ in $(0, T]$ (see the proof of Theorem 4 in [2]).

Theorem 1. *Let $\epsilon \neq 0$, $k \in C[0, T]$ and $a \in L^1(0, T)$. Then problem (2.2) has a unique solution in $C^1[0, T]$.*

Proof. The initial value problem (2.2) is in $C^1[0, T]$ equivalent to the equation

$$y_\epsilon(x) \;=\; L[y_\epsilon, y_\epsilon](x) + f(x) \tag{2.3}$$

where

$$L[z_1, z_2](x) = \int_0^x \frac{1}{\epsilon}\, e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau} \int_0^\eta a(\xi)z_1(\eta-\xi)z_2(\xi)\,d\xi\,d\eta$$
$$f(x) = Ae^{-\frac{1}{\epsilon}\int_0^x k(\tau)\,d\tau}\,.$$

Let us show that (2.3) has a unique solution in $C[0,T]$. We have

$$
\begin{aligned}
&e^{-\sigma x} L[z_1, z_2](x) \\
&\qquad = \int_0^x e^{-\sigma(x-\eta)} \frac{1}{\epsilon}\, e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau} \int_0^\eta a(\xi) e^{-\sigma(\eta-\xi)} z_1(\eta-\xi) e^{-\sigma\xi} z_2(\xi)\, d\xi\, d\eta\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|L[z_1, z_2]\|_\sigma &\le \frac{1}{|\epsilon|} e^{\frac{1}{|\epsilon|}\int_0^T |k(\tau)|\,d\tau} \int_0^T |a(\xi)|\, d\xi \int_0^T e^{-\sigma(T-\eta)}\, d\eta\, \|z_1\|_\sigma \|z_2\|_\sigma \\
&\le \text{Const}\, \frac{1}{\sigma} \|z_1\|_\sigma \|z_2\|_\sigma\,.
\end{aligned}
$$

This estimate shows that the assumptions of Lemma 1 are satisfied for equation (2.3). Consequently, (2.3) has a unique solution y_ϵ in $C[0,T]$. Finally, since the right-hand side of (2.3) is continuously differentiable for $y_\epsilon \in C[0,T]$, we obtain $y_\epsilon \in C^1[0,T]$. Theorem 1 is proved. ∎

Lemma 2. *Let $\epsilon \neq 0$, g be a measurable function such that $|g(x)| \le Cx^{\delta-1}$ with $C \ge 0$, $\delta > 0$, $k \in W^{2,1}(0,T)$ and $A_0 x \le k(x) \le A_1 x$ with $0 < A_0 \le A_1$. Then the function*

$$
y(x) \;=\; k(x) \int_0^x v(\eta)\, d\eta \tag{2.4}
$$

with

$$
v(x) \;=\; \frac{1}{\epsilon k^2(x)} \int_0^x k(\eta) e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau} g(\eta)\, d\eta \tag{2.5}
$$

belongs to $W^{2,1}(0,T)$ and solves the problem

$$
\epsilon y''(x) + k(x) y'(x) - \left[k'(x) + \frac{\epsilon k''(x)}{k(x)} \right] y(x) = g(x), \quad y(0) = y'(0) = 0. \tag{2.6}
$$

Proof. Due to the assumptions of the lemma, the function y, defined in (2.4) with v given by (2.5), belongs to $W^{2,1}(0,T)$. One can immediately check that v is a solution to the equation

$$
\epsilon k(x) v'(x) + [2\epsilon k'(x) + k^2(x)] v(x) \;=\; g(x)\,. \tag{2.7}
$$

Further, from (2.4) we see that $v = \left(\frac{y}{k}\right)'$. Substituting $\left(\frac{y}{k}\right)'$ for v in (2.7) we derive the equation (2.6). Finally, the conditions $y(0) = y'(0) = 0$ follow from (2.4) with (2.5) by the assumptions on k. ∎

Lemma 3. *Let $\epsilon > 0$. Then*

$$\max_{0\le x\le T} \int_0^x \eta^\beta e^{-\frac{1}{\epsilon}(x^2-\eta^2)}\, d\eta \le \frac{1}{\beta+1}\,\epsilon^{\frac{\beta+1}{2}} \quad \text{if } -1<\beta\le 1 \tag{2.8}$$

$$\max_{0\le x\le T} \int_0^x \eta^\beta e^{-\frac{1}{\epsilon}(x^2-\eta^2)}\, d\eta \le \frac{T^{\beta-1}}{2}\,\epsilon \quad \text{if } \beta>1\,. \tag{2.9}$$

Proof. Changing the variable of integration $z = \frac{\eta^2}{\epsilon}$ we obtain

$$\int_0^x \eta^\beta e^{-\frac{1}{\epsilon}(x^2-\eta^2)}\, d\eta = \frac{1}{2} e^{\frac{\beta+1}{2}} \int_0^t z^{\frac{\beta-1}{2}} e^{-(t-x)} dz\,, \tag{2.10}$$

where $t = \frac{x^2}{\epsilon}$. Let $-1 < \beta \le 1$. Then in case $t \ge 1$ we have

$$\begin{aligned}\int_0^t z^{\frac{\beta-1}{2}} e^{-(t-x)} dz &= \int_0^1 z^{\frac{\beta-1}{2}} e^{-(t-x)} dz + \int_1^t z^{\frac{\beta-1}{2}} e^{-(t-x)} dz \\ &\le e^{1-t} \int_0^1 z^{\frac{\beta-1}{2}} dz + \int_1^t e^{-(t-z)} dz \\ &= 1 + \left(\frac{2}{\beta+1} - 1\right) e^{1-t}.\end{aligned}$$

Thus,

$$\sup_{1\le t<\infty} \int_0^t z^{\frac{\beta-1}{2}} e^{-(t-x)} dz \le \frac{2}{\beta+1}\,. \tag{2.11}$$

In case $0 \le t < 1$ we obtain

$$\int_0^t z^{\frac{\beta-1}{2}} e^{-(t-x)} dz \le \int_0^t z^{\frac{\beta-1}{2}} dz = \frac{2t^{\frac{\beta+1}{2}}}{\beta+1}\,.$$

This implies

$$\sup_{0\le t<1} \int_0^t z^{\frac{\beta-1}{2}} e^{-(t-x)} dz \le \frac{2}{\beta+1}\,. \tag{2.12}$$

Applying (2.11) and (2.12) in (2.10) we deduce (2.8). If $\beta > 1$ then we obtain $\int_0^x \eta^\beta e^{-\frac{1}{\epsilon}(x^2-\eta^2)}\, d\eta \le T^{\beta-1} \int_0^x \eta e^{-\frac{1}{\epsilon}(x^2-\eta^2)}\, d\eta$. Estimate (2.9) follows using here (2.8) with $\beta = 1$. ∎

Theorem 2. *Let a and k fulfill the assumptions of Theorem 7 in [2], i.e., $k \in C^2[0,T]$, $k > 0$ in $(0,T]$, $a \in C^1[0,T]$, where*

$$\left.\begin{aligned} k(x) &= Ax + Bx^{2+\delta} + o\left(x^{2+\delta}\right) \\ k'(x) &= A + B(2+\delta)x^{1+\delta} + o\left(x^{1+\delta}\right) \\ k''(x) &= B(1+\delta)(2+\delta)x^{\delta} + o\left(x^{\delta}\right) \end{aligned}\right\} \tag{2.13}$$

$$\left.\begin{aligned} a(x) &= 1 + \lambda x^{1+\delta} + o\left(x^{1+\delta}\right) \\ a'(x) &= \lambda(1+\delta)x^{\delta} + o\left(x^{\delta}\right) \end{aligned}\right\} \tag{2.14}$$

as $x \to 0$ with $A, \delta > 0$ and $B, \lambda \in \mathbb{R}$. Further, let y_0 be the solution of equation (2.1) *satisfying $y_0 \in C^1[0,T] \cap C^2(0,T]$ and*

$$y_0(x) = A + Cx^{1+\delta} + o\left(x^{1+\delta}\right), \quad y_0'(x) = C(1+\delta)x^\delta + o\left(x^\delta\right) \tag{2.15}$$

as $x \to 0$ with $C \in \mathbb{R}$ and

$$|y_0''(x)| \le \mathrm{Const}\, x^{\delta-1}. \tag{2.16}$$

Then for any $q \in (1-\delta, 1) \cap (0,1)$ the estimates

$$\left. \begin{array}{l} \max\limits_{0 \le x \le T} x^{q-2}|y_\epsilon(x) - y_0(x)| \\ \max\limits_{0 \le x \le T} x^{q-1}|y_\epsilon'(x) - y_0'(x)| \end{array} \right\} \le M\mu(\epsilon) \tag{2.17}$$

are valid for the solution y_ϵ of problem (2.2) *with $\epsilon > 0$. Here*

$$\mu(\epsilon) = \begin{cases} \epsilon^{\frac{\delta+q-1}{2}} & \text{if} \quad \delta + q \le 3 \\ \epsilon & \text{if} \quad \delta + q > 3, \end{cases} \tag{2.18}$$

and M is a constant depending on $T, k, a,$ and q.

Remark 1. Existence of a solution y_0 of equation (2.1) with properties $y_0 \in C^1[0,T] \cap C^2(0,T]$, (2.15) and (2.16) follows from Theorems 1 and 7 in [2].

Proof. Let $\epsilon > 0$. Denote $y = y_\epsilon - y_0$ and subtract (2.1) from (2.2). We obtain

$$\begin{aligned} \epsilon y'(x) + k(x)y(x) &= \int_0^x [a(x-\eta) + a(\eta)]y_0(x-\eta)y(\eta)\,d\eta \\ &\quad + \int_0^x a(\eta)y(x-\eta)y(\eta)\,d\eta - \epsilon y_0'(x) \\ y(0) &= 0. \end{aligned} \tag{2.19}$$

By Theorem 1 this problem admits a unique solution in $C^1[0,T]$. Due to the assumptions of k, a and the properties of y_0 this solution even belongs to $W^{2,1}(0,T)$ and $y'(0) = 0$. Consequently, differentiating equation (2.19), the equation is equivalent to the problem of the second order

$$\epsilon y''(x) + k(x)y'(x) - \left[k'(x) + \frac{\epsilon k''(x)}{k(x)}\right] y(x) = g[y](x), \; y(0) = y'(0) = 0, \tag{2.20}$$

where

$$
\begin{aligned}
&g[y](x) \\
&= \int_0^x \big[a'(x-\eta)(y_0(x-\eta)-A) + (a(x-\eta)+a(\eta))y_0'(x-\eta)\big]\,y(\eta)\,d\eta \\
&\quad + \int_0^x a(\eta)y'(x-\eta)y(\eta)\,d\eta + A\int_0^x a'(x-\eta)y(\eta)\,d\eta \\
&\quad + \left[A(1+a(x)) - 2k'(x) - \frac{\epsilon k''(x)}{k(x)}\right] y(x) - \epsilon y_0''(x)\,.
\end{aligned}
\tag{2.21}
$$

Let us consider the related equation

$$
v(x) \;=\; \int_0^x \frac{k(\eta)}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau}\, g\Big[k\int_0^{\cdot} v(\xi)\,d\xi\Big](\eta)\,d\eta \tag{2.22}
$$

and define a function y by means of the solution of this equation using as in (2.4) the formula

$$
y(x) \;=\; k(x)\int_0^x v(\eta)\,d\eta\,. \tag{2.23}
$$

It follows from the assumptions on k that there exist $0 < A_0 \le A_1$ such that

$$
A_0 x \;\le\; k(x) \;\le\; A_1 x\,, \quad x \in [0,T]\,. \tag{2.24}
$$

Further, in case the solution v of (2.22) satisfies the conditions

$$
v \in C(0,T]\,, \quad |v(x)| \le \mathrm{Const}\, x^{-q}\,, \tag{2.25}
$$

where $q < 1$ by assumption, then, as we can easily check, the function $g[y] = g\big[k\int_0^{\cdot} v(\xi)\,d\xi\big]$ satisfies the relation $|g(x)| \le \mathrm{Const}\, x^{p-1}$ with $p > 0$. Consequently, by Lemma 2, the function y given by (2.23) belongs to $W^{2,1}(0,T)$ and solves (2.20), hence (2.19). In the following we will show the existence of a solution v with the property (2.25).

Let us define

$$
w(x) \;=\; x^q v(x)\,. \tag{2.26}
$$

The solution v of (2.22) satisfies (2.25) if and only if $w \in C(0,T] \cap L^\infty(0,T)$. The corresponding equation for w writes

$$
w(x) \;=\; G[w](x) + L[w,w](x) + f(x)\,, \tag{2.27}
$$

where

$$
\begin{aligned}
G[w](x) = \int_0^x \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau} \Bigg\{ \int_0^\eta & [a'(\eta-\xi)(y_0(\eta-\xi)-A) \\
& +(a(\eta-\xi)+a(\xi))y_0'(\eta-\xi)]\, k(\xi) \int_0^\xi \tau^{-q} w(\tau)\,d\tau\,d\xi \\
& +A \int_0^\eta a'(\eta-\xi)k(\xi) \int_0^\xi \tau^{-q} w(\tau)\,d\tau\,d\xi \\
& +\big[(A(1+a(\eta))-2k'(\eta))k(\eta)-\epsilon k''(\eta)\big] \int_0^\eta \tau^{-q} w(\tau)\,d\tau \Bigg\}\, d\eta
\end{aligned}
\tag{2.28}
$$

and

$$
\begin{aligned}
L[w_1,w_2](x) = \int_0^x \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau} & \int_0^\eta a(\xi) \Bigg[k'(\eta-\xi) \\
& \times \int_0^{\eta-\xi} \tau^{-q} w_1(\tau)\,d\tau + k(\eta-\xi)(\eta-\xi)^{-q} w_1(\eta-\xi) \Bigg] \\
& \times k(\xi) \int_0^\xi \tau^{-q} w_2(\tau)\,d\tau\,d\xi\,d\eta
\end{aligned}
\tag{2.29}
$$

and

$$
f(x) = -\int_0^x \frac{k(\eta)x^q}{k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau} y_0''(\eta)\,d\eta\,. \tag{2.30}
$$

We will prove that (2.27) has a unique solution in $C[0,T]$ and this solution satisfies a proper estimate implying (2.17).

In view of the assumption $q > 1-\delta$ by $k(x) \geq 0$, $\epsilon > 0$, (2.13) and (2.16) it follows that $f \in C[0,T]$. Further, multiplying by $e^{-\sigma x}$ in (2.28), (2.29) we have

$$
\begin{aligned}
& e^{-\sigma x} G[w](x) \\
& \quad = \int_0^x e^{-\sigma(x-\eta)} \frac{k(\eta)x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\,d\tau} \Bigg\{ \int_0^\eta e^{-\sigma(\eta-\xi)} \\
& \quad \times \big[a'(\eta-\xi)(y_0(\eta-\xi)-A) + (a(\eta-\xi)+a(\xi))y_0'(\eta-\xi)\big] \\
& \quad \times k(\xi) \int_0^\xi e^{-\sigma(\xi-\tau)} \tau^{-q} e^{-\sigma\tau} w(\tau)\,d\tau\,d\xi \\
& \quad + A \int_0^\eta e^{-\sigma(\eta-\xi)} a'(\eta-\xi)k(\xi) \int_0^\xi e^{-\sigma(\xi-\tau)} \tau^{-q} e^{-\sigma\tau} w(\tau)\,d\tau\,d\xi \\
& \quad + \big[(A(1+a(\eta))-2k'(\eta))k(\eta)-\epsilon k''(\eta)\big] \\
& \quad \times \int_0^\eta e^{-\sigma(\eta-\tau)} \tau^{-q} e^{-\sigma\tau} w(\tau)\,d\tau \Bigg\}\, d\eta,
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
&e^{-\sigma x} L[w_1, w_2](x) \\
&\quad = \int_0^x e^{-\sigma(x-\eta)} \frac{k(\eta) x^q}{\epsilon k^2(x)} e^{-\frac{1}{\epsilon}\int_\eta^x k(\tau)\, d\tau} \\
&\quad \times \int_0^\eta a(\xi) \Big[k'(\eta-\xi) \int_0^{\eta-\xi} e^{-\sigma(\eta-\xi-\tau)} \tau^{-q} e^{-\sigma\tau} w_1(\tau)\, d\tau \\
&\quad + k(\eta-\xi)(\eta-\xi)^{-q} e^{-\sigma(\eta-\xi)} w_1(\eta-\xi) \Big] \\
&\quad \times k(\xi) \int_0^\xi e^{-\sigma(\xi-\tau)} \tau^{-q} e^{-\sigma\tau} w_2(\tau)\, d\tau\, d\xi\, d\eta.
\end{aligned}
\tag{2.32}
$$

In the estimations of G and L we apply the inequality

$$
\int_0^x e^{-\sigma(x-\tau)} \tau^{-q}\, d\tau \;=\; \frac{1}{\sigma^{1-q}} \int_0^{\sigma x} z^{-q} e^{-(\sigma x - z)} dz \;\le\; \frac{\text{Const}}{\sigma^{1-q}}
\tag{2.33}
$$

following from Lemma 3.

We now estimate (2.31) making use of the assumptions of the theorem, Lemma 3, (2.24) and (2.33). We obtain

$$
\begin{aligned}
&\|G[w]\|_\sigma \\
&\le \text{Const} \max_{0\le x\le T} \int_0^x \frac{\eta}{\epsilon x^{2-q}} e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)} \\
&\quad \times \bigg\{ \int_0^\eta \big[(\eta-\xi)^{1+2\delta} + (\eta-\xi)^\delta \big] \xi \frac{1}{\sigma^{1-q}}\, d\xi\, \|w\|_\sigma \\
&\quad + \int_0^\eta (\eta-\xi)^\delta \xi \frac{1}{\sigma^{1-q}}\, d\xi\, \|w\|_\sigma + \big(\eta^{1+\delta} + \epsilon\eta^\delta \big) \frac{1}{\sigma^{1-q}} \|w\|_\sigma \bigg\}\, d\eta \\
&\le \text{Const} \max_{0\le x\le T} \bigg\{ \frac{1}{\epsilon} \int_0^x \eta e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)}\, d\eta + \int_0^x e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)}\, d\eta \bigg\} \frac{1}{\sigma^{1-q}} \|w\|_\sigma \\
&\le \frac{\text{Const}}{\sigma^{1-q}} \|w\|_\sigma\,.
\end{aligned}
\tag{2.34}
$$

Similarly, for $L[w_1, w_2]$ in (2.32) we derive

$$
\begin{aligned}
\|L[w_1, w_2]\|_\sigma &\le \text{Const} \max_{0\le x\le T} \int_0^x \frac{\eta}{\epsilon x^{2-q}} e^{-\frac{A_0}{2\epsilon}(x^2-\eta^2)} \\
&\quad \times \int_0^\eta (\eta-\xi)^{1-q} \|w_1\|_\sigma \xi \frac{1}{\sigma^{1-q}} \|w_2\|_\sigma\, d\xi\, d\eta \\
&\le \frac{\text{Const}}{\sigma^{1-q}} \|w_1\|_\sigma \|w_2\|_\sigma\,.
\end{aligned}
\tag{2.35}
$$

The estimates (2.34) and (2.35) imply the assumptions (1.3) – (1.5) of Lemma 1. Thus, by Lemma 1, equation (2.27) has a unique solution w in $C[0, T]$. In particular, the equation (2.27) has a unique solution $w = 0$ in $C[0,T]$ if $f = 0$. Consequently, the stability estimate (1.6) in Lemma 1 with $z_1 = w$, $z_2 = 0$ and $f_1 = f$, $f_2 = 0$ yields $\|w\| \le \text{Const}\,\|f\|$. Further, estimating (2.30) by means of the assumption $q > 1-\delta$, (2.16), (2.24) and Lemma 3 we have $\|f\| \le \text{Const}\,\mu(\epsilon)$ with $\mu(\epsilon)$ defined in (2.18). Thus,

$$\|w\| \;\le\; \text{Const}\,\mu(\epsilon)\,. \tag{2.36}$$

Finally, by (2.23) and (2.26) we have the formula for $y = y_\epsilon - y_0$ in terms of w

$$y(x) \;=\; k(x)\int_0^x \xi^{-q} w(\xi)\,d\xi\,. \tag{2.37}$$

From (2.13), (2.36) and (2.37) we obtain the first estimate in (2.17). Using the differentiated formula (2.37) also the second estimate in (2.17) follows. Theorem 2 is proved. ∎

The assertion (2.17) of Theorem 2 implies the following corollary.

Corollary 1. *Under the assumptions of Theorem 2 the uniform convergence*

$$y_\epsilon \to y_0\,,\;\; y'_\epsilon \to y'_0 \quad \textit{in} \quad C[0,T]$$

as $\epsilon \to 0^+$ holds.

Remark 2. It is expected that the convergence $y_\epsilon \to y_0$ in $C[0,T]$ already holds if $k \in C^1[0,T]$ and $a \in C[0,T]$ with the corresponding asymptotics as $x \to 0$. But this has not been proved.

3. General logarithmic case

In the following we study the existence of solutions for two types of generalized autoconvolution equations. We start with equation (2.1) where

$$k(x) \;=\; Ax + x^2\sum_{n=0}^{N} B_n \ln^n x + C(x) \qquad (A>0,\; B_n \in \mathbb{R}) \tag{3.1}$$

with $C(x) = o(x^2)$ as $x \to 0$ and $\int_0^T \frac{|C(x)|}{x^3}dx < \infty$,

$$a(x) \;=\; 1 + x\sum_{n=0}^{N} \beta_n \ln^n x + \gamma(x) \qquad (\beta_n \in \mathbb{R}) \tag{3.2}$$

with $\gamma(x) = o(x)$ as $x \to 0$ and $\int_0^T \frac{|\gamma(x)|}{x^2}dx < \infty$.

Theorem 3. *Let k with $1/k \in C(0,T]$ and $a \in C[0,T]$ have the finite asymptotic expansions* (3.1) *and* (3.2)*, respectively. Then equation* (2.1) *has a solution $y \in C[0,T]$ of the form*

$$y(x) \;=\; A + x\sum_{n=1}^{N+1} \mu_n \ln^n x + xz(x) \tag{3.3}$$

with $z \in C[0,T]$ and $z(0) = 0$, where the $\mu_n, n = 0,1,\ldots,N+1$, are the solutions of the equations

$$(-1)^n \frac{2^n}{n!} \sum_{j=n+1}^{N+1} (-1)^j \frac{j!}{2^j}\mu_j \;=\; B_n - A(-1)^n \frac{2^{n-1}}{n!} \sum_{j=n}^{N} (-1)^j \frac{j!}{2^j}\beta_j \tag{3.4}$$

for $n = 0.\ldots,N$. This solution is unique in the class of functions of type (3.3).

Proof. Inserting the ansatz (3.3) into equation (2.1) we get the equation for z

$$z(x) \;=\; f_0(x) + G_0[z](x) + L_0[z,z](x)\,, \tag{3.5}$$

where

$$\begin{aligned} f_0(x) = \frac{1}{xk(x)} \Bigg\{ & A^2 \int_0^x a(\xi)\,d\xi + A\int_0^x \xi\,[a(\xi) + a(x-\xi)] \sum_{n=1}^{N+1} \mu_n \ln^n \xi\, d\xi \\ & -k(x)\Bigg[A + x\sum_{n=1}^{N+1} \mu_n \ln^n x\Bigg] \\ & + \int_0^x \xi(x-\xi)a(\xi) \sum_{n=1}^{N+1} \mu_n \ln^n \xi \sum_{m=1}^{N+1} \mu_m \ln^m (x-\xi)\,d\xi \Bigg\} \end{aligned}$$

and

$$\begin{aligned} G_0[z](x) &= \frac{1}{xk(x)} \int_0^x \xi\,[a(\xi) + a(x-\xi)] \\ &\qquad \times \Bigg[A + (x-\xi)\sum_{n=1}^{N+1} \mu_n \ln^n (x-\xi)\Bigg] z(\xi)\,d\xi \\ L_0[z_1,z_2](x) &= \frac{1}{xk(x)} \int_0^x \xi(x-\xi)a(\xi)z_1(\xi)z_2(x-\xi)\,d\xi\,. \end{aligned}$$

In view of assumptions (3.1) and (3.2) we have

$$\begin{aligned} xk(x)f_0(x) = {} & A^2x + A^2\sum_{n=0}^{N} \beta_n \int_0^x \xi \ln^n \xi\,d\xi + 2A\sum_{n=1}^{N+1} \mu_n \int_0^x \xi \ln^n \xi\,d\xi \\ & - \Bigg[Ax + x^2\sum_{m=0}^{N} B_m \ln^m x\Bigg]\Bigg[A + x\sum_{n=1}^{N+1} \mu_n \ln^n x\Bigg] + F_0(x)\,, \end{aligned} \tag{3.6}$$

where $F_0 \in C[0,T]$ with $F_0(x) = o(x^2)$ as $x \to 0$ and $\int_0^T \frac{|F_0(x)|}{x^3} dx < \infty$. Calculating the coefficients of the functions $x^2 \ln^n x$, $n = 0, 1, \ldots, N, N+1$, in the right-hand side of (3.6), we see that the coefficient of the highest term $x^2 \ln^{N+1} x$ automatically vanishes as well as the coefficient of x. Putting the $N+1$ coefficients of $x^2 \ln^n x$, $n = 0, \ldots, N$ equal to zero, we obtain the linear system (3.4) for the $N+1$ parameters μ_n, $n = 1, \ldots, N+1$. This system is regular since it has upper triangular matrix with nonvanishing elements in the main diagonal. So, (3.4) has a unique solution $(\mu_1, \ldots, \mu_{N+1})$, and for these parameters μ_n we have the relation $xk(x)f_0(x) = F_1(x)$, where F_1 has the same properties mentioned above as F_0. Therefore, by (3.1) and $1/k \in C(0,T]$ then $f_0 \in C[0,T]$ with $f_0(0) = 0$ and $\int_0^T \frac{|f_0(x)|}{x} dx < \infty$ holds.

We decompose

$$G_0[z](x) \;=\; \frac{2}{x^2} \int_0^x \xi z(\xi)\, d\xi + G_1[z](x)$$

where

$$\begin{aligned} &G_1[z](x) \\ &\quad= \frac{2(Ax - k(x))}{x^2 k(x)} \int_0^x \xi\, z(\xi)\, d\xi \\ &\qquad + \frac{A}{xk(x)} \int_0^x \xi\, [a(\xi) + a(x-\xi) - 2]\, z(\xi)\, d\xi \\ &\qquad + \frac{A}{xk(x)} \int_0^x \xi(x-\xi)[a(\xi) - a(x-\xi)] \sum_{n=1}^{N+1} \mu_n \ln^n(x-\xi)\, z(\xi)\, d\xi \end{aligned} \tag{3.7}$$

and write equation (3.5) in the form

$$z(x) - \frac{2}{x^2} \int_0^x \xi\, z(\xi)\, d\xi \;=\; g(x)\,,$$

where $g(x) \;=\; f_0(x) + G_1[z](x) + L_0[z,z](x)$. On account of (3.1) and (3.2) we obtain the estimates

$$\begin{aligned} |L_0[z,z](x)| &\le \mathrm{Const}\, x \|z\|^2 \\ |G_1[z](x)| &\le \mathrm{Const}\, x \left[1 + |\ln x|^{N+1}\right] \|z\| \end{aligned}$$

which imply $L_0[z,z], G_1[z] \in C[0,T]$ for any $z \in C[0,T]$ with $L_0[z,z](0) = G_1[z](0) = 0$ and $\int_0^T \frac{|L_0[z,z](x)|}{x} dx < \infty$, $\int_0^T \frac{|G_1[z](x)|}{x} dx < \infty$. Observing the above relations for f_0 we therefore obtain that also $g \in C[0,T]$ with $g(0) = 0$ and $\int_0^T \frac{|g(x)|}{x} dx < \infty$.

Solving (3.7), equation (3.5) with $z(0) = 0$ is reduced to the equation

$$z(x) \;=\; f(x) + G[z](x) + L[z,z](x)\,, \tag{3.8}$$

where

$$f(x) \;=\; f_0(x) + 2\int_0^x \frac{f_0(\xi)}{\xi}\, d\xi \;\in\; C[0,T]$$

with $f(0) = 0$ and

$$G[z](x) = G_1[z](x) + 2\int_0^x \frac{G_1[z](\xi)}{\xi}\, d\xi \tag{3.9}$$

$$L[z_1,z_2](x) = \; L_0[z_1,z_2](x) + 2\int_0^x \frac{L_0[z_1,z_2](\xi)}{\xi}\, d\xi\,.$$

Again, for any $z \in C[0,T]$ we have $G[z] \in C[0,T]$ with $G[z](0) = 0$ and for any pair $z_1, z_2 \in C[0,T]$ also $L[z_1,z_2] \in C[0,T]$ with $L[z_1,z_2](0) = 0$. Hence $z(0) = f(0) = 0$ for the solution z of (3.8).

Applying Lemma 1 we have to verify inequalities (1.3) – (1.5) for $G[z]$ and $L[z_1,z_2]$. At first by (3.1) and (3.2) we estimate in (3.7) and get from (3.9)

$$\|G[z]\|_\sigma \;\le\; \text{Const}\,\frac{1}{\sigma}\left[1 + \ln^{N+1}\sigma\right]\|z\|_\sigma \qquad (\sigma > 1)$$

which proves (1.3). Further, as in the proof of Theorem 3 in [2] we have the estimates $\|L[z_1,z_2]\|_\sigma \;\le\; \text{Const}\,\|z_1\|_\sigma\|z_2\|_\sigma$ and $\|L[z_1,z_2]\|_\sigma \;\le\; \text{Const}\,\frac{1}{\sigma}\|z_1\|\|z_2\|_\sigma$ and analogously with z_1 and z_2 interchanged. This shows (1.4) and (1.5). Theorem 3 is proved. ∎

Remark 3. The special case of the theorem for $N = 0$ was proved in [2].

4. General equation

Now we deal with equation (1.1) under the assumptions that $1/k \in C(0,T]$ and

$$k(x) \;=\; Ax + Bx^{1+\alpha} + C(x) \quad (A > 0,\; B \in \mathbb{R})\,, \tag{4.1}$$

where $\alpha > 0$, $C(x) = o\left(x^{1+\alpha}\right)$ as $x \to 0$ with $\int_0^T \frac{|C(x)|}{x^{2+\alpha}}dx < \infty$, $m \in C([0,T] \times [0,T])$ and

$$m(x,\xi) \;=\; 1 + M_1 x^\alpha + M_2 \xi^\alpha + \gamma(x,\xi) \quad (M_j \in \mathbb{R})\,, \tag{4.2}$$

where $\gamma(x,\xi) = o\left(x^\alpha + \xi^\alpha\right)$ as $x^2 + \xi^2 \to 0$ with $\int_0^T \frac{1}{x^{2+\alpha}} \int_0^x |\gamma(x,\xi)|\, d\xi dx < \infty$, $n \in C([0,T] \times [0,T])$ and

$$n(x,\xi) \;=\; N_0 + N_1 x^\alpha + N_2 \xi^\alpha + \delta(x,\xi) \quad (N_j \in \mathbb{R})\,, \tag{4.3}$$

where $\delta(x,\xi) = o\left(x^\alpha + \xi^\alpha\right)$ as $x^2 + \xi^2 \to 0$ with $\int_0^T \frac{1}{x^{2+\alpha}} \int_0^x |\delta(x,\xi)|\, d\xi dx < \infty$, $p \in C[0,T]$ and

$$p(x) \;=\; cx + dx^{1+\alpha} + \epsilon(x) \quad (c,d \in \mathbb{R})\,, \tag{4.4}$$

where $\epsilon(x) = o\left(x^{1+\alpha}\right)$ as $x \to 0$ with $\int_0^T \frac{|\epsilon(x)|}{x^{2+\alpha}} dx < \infty$.

At first we are looking for solutions to (1.1) of the form

$$y(x) \;=\; \lambda + \sum_{j=1}^{\nu} \mu_j x^{\kappa_j} + x^{\alpha} z(x)\,, \quad z \in C[0,T]\,, \tag{4.5}$$

where $\lambda \in \mathbb{R}$, $\nu \in \{1,2,\ldots\}$, $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_\nu < \alpha$ and without loss of generality $\mu_j \neq 0$, $j = 1,\ldots,\nu$. Plugging the ansatz (4.5) and the asymptotic expansions (4.1) – (4.4) into equation (1.1) and comparing the coefficients of x, we obtain the possible values for λ

$$\lambda_{1,2} \;=\; \frac{1}{2}\left[A - N_0 \pm \sqrt{(A-N_0)^2 - 4c}\,\right]\,. \tag{4.6}$$

We remark that for $c = 0$ as in the model equation (2.1) we have the values $A - N_0$ and zero for λ. In case $p(x) = 0$ as in equation (2.1) the value zero of λ yields the trivial solution $y = 0$ of the equation. In dealing with real solutions of (1.1) only, we assume the inequality

$$4c \;\leq\; (A-N_0)^2 \tag{4.7}$$

in the following.

In view of (4.5) equation (1.1) reduces to the following equation for z

$$z(x) \;=\; f_0(x) + G_0[z](x) + L_0[z,z](x)\,, \tag{4.8}$$

where

$$\begin{aligned} f_0(x) = \frac{1}{x^{\alpha}k(x)} \Bigg\{ & p(x) - \left[\lambda + \sum_{j=1}^{\nu} \mu_j x^{\kappa_j}\right] k(x) + \lambda \int_0^x n(x,\xi)\,d\xi \\ & + \sum_{j=1}^{\nu} \mu_j \int_0^x n(x,\xi)\xi^{\kappa_j}\,d\xi + \lambda^2 \int_0^x m(x,\xi)\,d\xi \\ & + \lambda \sum_{j=1}^{\nu} \mu_j \int_0^x m(x,\xi)\left[\xi^{\kappa_j} + (x-\xi)^{\kappa_j}\right]d\xi \\ & + \sum_{j=1}^{\nu}\sum_{i=1}^{\nu} \mu_j\mu_i \int_0^x m(x,\xi)\xi^{\kappa_j}(x-\xi)^{\kappa_i}\,d\xi \Bigg\} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & G_0[z](x) \\ & = \frac{1}{x^{\alpha}k(x)} \int_0^x \Bigg\{ n(x,\xi)\xi^{\alpha}z(\xi) + \lambda m(x,\xi)\left[\xi^{\alpha}z(\xi) + (x-\xi)^{\alpha}z(x-\xi)\right] \\ & \quad + m(x,\xi)\Big[\sum_{j=1}^{\nu} \mu_j\Big(\xi^{\kappa_j}(x-\xi)^{\alpha}z(x-\xi) + (x-\xi)^{\kappa_j}\xi^{\alpha}z(\xi)\Big)\Big]\Bigg\}\,d\xi \end{aligned} \tag{4.10}$$

and

$$L_0[z_1, z_2](x) = \frac{1}{x^\alpha k(x)} \int_0^x m(x,\xi)\xi^\alpha (x-\xi)^\alpha z_1(\xi) z_2(x-\xi)\, d\xi\,. \tag{4.11}$$

Since $k(x) \sim Ax$ as $x \to 0$, for obtaining $f_0 \in C[0,T]$ in (4.9) we have to put the coefficients of the powers x and $x^{1+\kappa_j}$, $j = 1, \ldots, \nu$ in the brackets to zero. For the power x we obtain the relation $c = \lambda(A - N_0) - \lambda^2$ already used in the determination of λ by (4.6). For the power $x^{1+\kappa_1}$ the relation

$$\lambda \;=\; \frac{1}{2}\,[(1+\kappa_1)A - N_0] \tag{4.12}$$

between λ and κ_1 follows. This gives a positive value

$$\kappa_1 \;=\; \frac{1}{A}\sqrt{(A-N_0)^2 - 4c} \tag{4.13}$$

only for $\lambda = \lambda_1$, i.e.,

$$\lambda = \frac{1}{2}\left[A - N_0 + \sqrt{(A-N_0)^2 - 4c}\right]. \tag{4.14}$$

In case $\nu \geq 2$ for the power $x^{1+\kappa_j}$, $j = 2, \ldots, \nu$, it must be $\kappa_j = j\kappa_1$, $j = 2, \ldots, \nu$, which by $\kappa_\nu < \alpha$ yields the inequality $\kappa_1 < \frac{\alpha}{\nu}$ for κ_1. Under the further inequality $\kappa_1 \geq \frac{\alpha}{\nu+1}$ we get the recursive equations for μ_j, $j = 2, \ldots, \nu$,

$$\left(A - \frac{N_0 + 2\lambda}{1+\kappa_m}\right)\mu_m \;=\; \sum_{j=1}^{m-1} \mu_j \mu_{m-j} B\big(\kappa_j + 1, \kappa_{m-j} + 1\big)\,, \tag{4.15}$$

where $m = 2, \ldots, \nu$, the letter B denotes the Beta function, and μ_1 is arbitrary. In view of (4.12) we have

$$A - \frac{N_0 + 2\lambda}{1+\kappa_m} \;=\; \frac{(m-1)A\kappa_1}{1+m\kappa_1} \neq 0 \quad \text{for } \; m = 2, \ldots, \nu$$

so that (4.15) determines μ_j, $j = 2, \ldots, \nu$ uniquely for prescribed μ_1. Further, the value of $f_0(0)$ is given by the formulas

$$Af_0(0) \;=\; d - \lambda B + \lambda N_1 + \lambda \frac{N_2}{1+\alpha} + \lambda^2 M_1 + \lambda^2 \frac{M_2}{1+\alpha} \tag{4.16}$$

in case $\kappa_1 > \frac{\alpha}{\nu+1}$ and with the additional term

$$\sum_{j=1}^{\nu} \mu_j \mu_{\nu+1-j} B(\kappa_j + 1, \kappa_{\nu+1-j} + 1)$$

on the right-hand side of (4.16) in case $\kappa_1 = \frac{\alpha}{\nu+1}$.

We are now ready to formulate the *first existence theorem*.

Theorem 4. *Let the assumptions* (4.1) – (4.4) *be fulfilled and let the inequality*

$$(A-N_0)^2 - \frac{\alpha^2}{\nu^2}A^2 \, < \, 4c \, \le \, (A-N_0)^2 - \frac{\alpha^2}{(\nu+1)^2}A^2 \tag{4.17}$$

hold for $\nu \in \{1,2,\dots\}$. *Then equation* (1.1) *has a one-parametric family of solutions of the form* (4.5), *where* λ *is given by* (4.14), κ_1 *by* (4.13), $\mu_1 \in \mathbb{R}$ *is an arbitrary non-vanishing parameter, and for* $\nu \ge 2$ *there holds* $\kappa_j = j\kappa_1$, $j = 2,\dots,\nu$, *and the* μ_j, $j=2,\dots,\nu$, *are determined by* μ_1 *via relations* (4.15).

Proof. Due to (4.13) we have the relation

$$4c \, = \, (A-N_0)^2 - \kappa_1^2 A^2$$

between c and κ_1. Therefore, inequality (4.17) is equivalent to the above inequality $\frac{\alpha}{\nu+1} \le \kappa_1 < \frac{\alpha}{\nu}$. Further, (4.17) implies assumption (4.7).

We split the linear operator G_0 in (4.10)

$$G_0[z](x) \, = \, \frac{\beta}{x^{1+\alpha}} \int_0^x \xi^\alpha z(\xi)\, d\xi + G_1[z](x)\,,$$

where $\beta = \frac{1}{A}[N_0 + 2\lambda] = 1 + \kappa_1$ observing (4.12) and

$$\begin{aligned} G_1[z](x) = {} & \frac{\beta}{x^{1+\alpha}} \frac{Ax - k(x)}{k(x)} \int_0^x \xi^\alpha z(\xi)\, d\xi \\ & + \frac{1}{x^\alpha k(x)} \int_0^x \Big\{ [n(x,\xi) - N_0]\xi^\alpha z(\xi) \\ & + \lambda[m(x,\xi)-1]\,[\xi^\alpha z(\xi) + (x-\xi)^\alpha z(x-\xi)] \Big\}\, d\xi \\ & + \frac{1}{x^\alpha k(x)} \int_0^x m(x,\xi) \\ & \times \sum_{j=1}^{\nu} \mu_j \,[\xi^{\kappa_j}(x-\xi)^\alpha z(x-\xi) + (x-\xi)^{\kappa_j}\xi^\alpha z(\xi)]\, d\xi \end{aligned} \tag{4.18}$$

and write equation (4.8) in the form

$$z(x) - \frac{\beta}{x^{1+\alpha}} \int_0^x \xi^\alpha z(\xi)\, d\xi = g(x)\,, \tag{4.19}$$

where $g(x) \, = \, f_0(x) + G_1[z](x) + L_0[z,z](x)$. Estimating (4.18) and (4.11) we get

$$|G_1[z](x)| \le \mathrm{Const}\, x^{\kappa_1}\|z\| \quad \text{and} \quad |L_0[z_1,z_2](x)| \le \mathrm{Const}\, x^\alpha \|z_1\|\|z_2\|\,.$$

So, for any $z \in C[0,T]$ we have $G_1[z] \in C[0,T]$ with $G_1[z](0) = 0$ and $L_0[z,z] \in C[0,T]$ with $L_0[z,z](0) = 0$, therefore $g \in C[0,T]$ with $g(0) = f_0(0)$.

The auxiliary equation (4.19) for known $g \in C[0,T]$ has the unique continuous solution

$$z(x) \;=\; g(x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} g(\xi)\, d\xi\,, \tag{4.20}$$

where $\alpha - \beta = \alpha - \kappa_1 - 1 > -1$. Hence we obtain instead of (4.8) the equivalent equation

$$z(x) \;=\; f(x) + G[z](x) + L[z,z](x) \tag{4.21}$$

where

$$f(x) = f_0(x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} f_0(\xi)\, d\xi$$

with $f(0) = \frac{1+\alpha}{1+\alpha-\beta} f_0(0) = \frac{\alpha+1}{\alpha-\kappa_1} f_0(0)$ and

$$G[z](x) = G_1[z](x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} G_1[z](\xi)\, d\xi$$
$$L[z_1,z_2](x) = L_0[z_1,z_2](x) + \beta x^{\beta-\alpha-1} \int_0^x \xi^{\alpha-\beta} L_0[z_1,z_2](\xi)\, d\xi\,.$$

We have the estimations

$$\|G[z]\|_\sigma \le \frac{\alpha+1}{\alpha-\kappa_1} \|G_1[z]\|_\sigma \quad \text{and} \quad \|L[z_1,z_2]\|_\sigma \le \frac{\alpha+1}{\alpha-\kappa_1} \|L_0[z_1,z_2]\|_\sigma\,.$$

So, for any $z \in C[0,T]$ we have $G[z] \in C[0,T]$ with $G[z](0) = 0$ and for any pair $z_1, z_2 \in C[0,T]$ also $L[z_1,z_2] \in C[0,T]$ with $L[z_1,z_2](0) = 0$. Hence $z(0) = f(0)$ for the solution z of (4.21).

To apply Lemma 1 to equation (4.21) we have to prove the inequalities (1.3) – (1.5). We can show that

$$\|G[z]\|_\sigma \;\le\; \begin{cases} \text{Const}\, \frac{1}{\sigma} \|z\|_\sigma & \text{if} \;\; \kappa_1 \ge 1 \\ \text{Const}\, \frac{1}{\sigma^{\kappa_1}} \|z\|_\sigma & \text{if} \;\; 0 < \kappa_1 < 1 \end{cases}$$

and further $\|L[z_1,z_2]\|_\sigma \;\le\; \text{Const}\, \|z_1\|_\sigma \|z_2\|_\sigma$ and

$$\|L[z_1,z_2]\|_\sigma \;\le\; \begin{cases} \text{Const}\, \frac{1}{\sigma} \|z_1\| \|z_2\|_\sigma & \text{if} \;\; \alpha \ge 1 \\ \text{Const}\, \frac{1}{\sigma^{\alpha}} \|z_1\| \|z_2\|_\sigma & \text{if} \;\; 0 < \alpha < 1\,, \end{cases}$$

and also with z_1 and z_2 interchanged. These estimates verify (1.3) – (1.5) and by Lemma 1 the theorem is proved. ∎

Remark 4. By $z(0) = f(0) = \frac{\alpha+1}{\alpha-\kappa_1} f_0(0)$ and (4.16) all the solutions of the family of the form (4.5) have the common value

$$z(0) = \frac{1}{A}\frac{\alpha+1}{\alpha-\kappa_1}\left[d + \lambda\left(N_1 + \frac{N_2}{\alpha+1} - B\right) + \lambda^2\left(M_1 + \frac{M_2}{\alpha+1}\right)\right].$$

In the next theorem we prove the existence of solutions to equation (1.1) of the simpler form

$$y(x) = \lambda + x^\alpha z(x), \quad z \in C[0,T], \tag{4.22}$$

where $\lambda \in \mathbb{R}$. We again have the possible values $\lambda_{1,2}$ from (4.6) for λ assuming the assumption (4.7) for real solutions, too. In equation (4.8) for z the functions f_0 and G_0 are now defined by the formulas (4.9) and (4.10) without the terms with sums whereas the formula (4.11) for L_0 remains.

In contrast to the former case we now obtain solutions for both values $\lambda_{1,2}$ of λ. The value of $f_0(0)$ is given by (4.16). In the proof of existence of solutions we again split G_0 introducing G_1 by (4.18) without the last integral with sums. In the auxiliary equation (4.19) the parameter $\beta = \frac{1}{A}[N_0 + 2\lambda]$ now has the two possible values

$$\beta_{1,2} = 1 \pm \gamma_0, \quad \gamma_0 = \frac{1}{A}\sqrt{(A-N_0)^2 - 4c}\,. \tag{4.23}$$

In the following we distinguish the three cases $0 \le \gamma_0 < \alpha$, $\gamma_0 = \alpha$ and $\gamma_0 > \alpha$. In the case $0 \le \gamma_0 < \alpha$ we have $\beta_1 \in [1, 1+\alpha)$, $\beta_2 \in (1-\alpha, 1]$. For both $\beta = \beta_{1,2}$ the inversion formula (4.20) holds and we can proceed as above to obtain two solutions $z_{1,2}$ of equation (4.21) and hence solutions $y_{1,2}$ of form (4.22) to equation (1.1). Only if $\gamma_0 = 0$, the values β_1 and β_2 are equal (to 1) and the solutions y_1 and y_2 coincide.

In the case $\gamma_0 = \alpha$ we have $\beta_1 = 1+\alpha$, $\beta_2 = 1-\alpha$. For $\beta = \beta_2$, again the inversion formula (4.20) holds and we get a solution y of form (4.22). For $\beta = \beta_1$ instead of (4.20) the inversion formula

$$z_K(x) = K + g(x) + \beta_1 \int_0^x \frac{g(\xi)}{\xi}\, d\xi$$

is valid with an arbitrary $K \in \mathbb{R}$ if $g \in C[0,T]$ satisfies $g(0) = 0$ and $\int_0^T \frac{|g(x)|}{x} dx < \infty$. In view of (4.16) and the assumptions on the integrals of $C, \gamma, \delta, \epsilon$ this is fulfilled if the condition

$$d = \lambda_1\left(B - N_1 - \frac{N_2}{\alpha+1}\right) - \lambda_1^2\left(M_1 + \frac{M_2}{\alpha+1}\right) \tag{4.24}$$

holds. Then as in the logarithmic case in Theorem 3 (or in Theorem 3 of [2]), for any $K \in \mathbb{R}$ we obtain a solution of the form (4.22), this means we have a one-parametric family of solutions y_K with parameter $K = z_K(0) \in \mathbb{R}$. If (4.24) does not hold, also as in Theorem 3 we can prove the existence of a family of solutions y_K of the form

$$y_K(x) \;=\; \lambda_1 + \mu x^\alpha \ln x + x^\alpha z_K(x)\,, \qquad z_K \in C[0,T] \tag{4.25}$$

with $\lambda_1 = \frac{1}{2}[(\alpha+1)A - N_0]$,

$$\mu \;=\; \frac{\alpha+1}{A}\left[d + \lambda_1\left(N_1 + \frac{N_2}{\alpha+1} - B\right) + \lambda_1^2\left(M_1 + \frac{M_2}{\alpha+1}\right)\right]$$

and arbitrary $K = z_K(0) \in \mathbb{R}$. Under the condition (4.24) we have $\mu = 0$ and the solutions (4.25) take the form (4.22).

In the remaining case $\gamma_0 > \alpha$ we have $\beta_1 > 1+\alpha$, $\beta_2 < 1-\alpha$. For $\beta = \beta_2$ again the inversion formula (4.20) holds leading to a solution y of form (4.22). For $\beta = \beta_1$ we take the inversion formula as follows:

$$\begin{aligned} z_K(x) = Kx^{\beta_1-\alpha-1} + g(x) - \beta_1 x^{\beta_1-\alpha-1} \int_x^T \xi^{\alpha-\beta_1} f_0(\xi)\,d\xi \\ + \beta_1 x^{\beta_1-\alpha-1} \int_0^x \xi^{\alpha-\beta_1} g_0(\xi)\,d\xi \end{aligned}$$

with an arbitrary $K \in \mathbb{R}$ and $g_0(x) = G_1[z](x) + L_0[z,z](x)$. Under the restriction $1+\alpha < \beta_1 < 1+2\alpha$ we can proceed as in the proof of Theorem 2 in [2] to obtain a family of solutions y_K of form (4.22) with parameter $K = \lim_{x\to 0} x^{\alpha+1-\beta_1} z_K(x) \in \mathbb{R}$.

So we have the following *second existence theorem.*

Theorem 5. *Let the assumptions* (4.1) – (4.4) *and the inequality* (4.7) *be satisfied. Then equation* (1.1) *has the following solutions, where γ_0 is given by* (4.23)*:*

1. *In case $\gamma_0 = 0$, i.e., $4c = (A - N_0)^2$: a unique solution y_0 of form* (4.22) *with $\lambda = \lambda_0 = \frac{1}{2}(A - N_0)$.*
2. *In case $0 < \gamma_0 < \alpha$, i.e., $0 < (A-N_0)^2 - 4c < \alpha^2 A^2$: two solutions $y_{1,2}$ of form* (4.22) *with $\lambda = \lambda_{1,2}$ given by* (4.6)*.*
3. *In case $\gamma_0 = \alpha$, i.e., $(A-N_0)^2 - 4c = \alpha^2 A^2$: for $\lambda = \lambda_2$ one solution y_2 of form* (4.22) *and for $\lambda = \lambda_1$ a one-parametric family of solutions y_K of form* (4.25) *with parameter $K = z_K(0) \in \mathbb{R}$.*
4. *In case $\gamma_0 > \alpha$, i.e., $(A-N_0)^2 - 4c > \alpha^2 A^2$: for $\lambda = \lambda_2$ one solution y_2 of form* (4.22) *and in case $\alpha < \gamma_0 < 2\alpha$, i.e., $\alpha^2 A^2 < (A-N_0)^2 - 4c < 4\alpha^2 A^2$, for $\lambda = \lambda_1$ a one-parametric family of solutions y_K of form* (4.22) *with parameter $K = \lim_{x\to 0} x^{\alpha-\gamma_0} z_K(x)$.*

Remark 5. In cases 1 and 2 and in cases 3 and 4 with $\lambda = \lambda_2$ of Theorem 5, the value $z(0)$ is given by $z_{1,2}(0) = \frac{1+\alpha}{\alpha \mp \gamma_0} f_0(0)$, where $f_0(0)$ follows from (4.16).

Summarizing the results of Theorem 4 and 5 we get the following picture of solvability of equation (1.1), where we take the solution in case 2 of Theorem 5 for $\lambda = \lambda_1$ as the member of the family of solutions in Theorem 4 with parameter $\mu_1 = 0$.

Corollary 2. *Under the assumptions* (4.1) – (4.4) *and the inequality* (4.7) *the following solutions to equation* (1.1) *exist.*

1. *In case* $4c = (A - N_0)^2$*: a solution of form* (4.22).
2. *In case* $(A - N_0)^2 - \alpha^2 A^2 < 4c < (A - N_0)^2$*: a one-parametric family of solutions of form* (4.5) *with parameter* $\mu = \mu_1 \in \mathbb{R}$ *for* $\lambda = \lambda_1$ *choosing a corresponding* $\nu \in \{1, 2, \ldots\}$ *in* (4.17) *and an additional solution of form* (4.22) *for* $\lambda = \lambda_2$.
3. *In case* $4c = (A - N_0)^2 - \alpha^2 A^2$*: a one-parametric family of solutions* y_K *of form* (4.25) *with parameter* $K \in \mathbb{R}$ *for* $\lambda = \lambda_1$ *and an additional solution of form* (4.22) *for* $\lambda = \lambda_2$.
4. *In case* $(A - N_0)^2 - 4\alpha^2 A^2 < 4c < (A - N_0)^2 - \alpha^2 A^2$*: a one-parametric family of solutions* y_K *of form* (4.22) *with parameter* $K \in \mathbb{R}$ *for* $\lambda = \lambda_1$ *and in case* $4c < (A - N_0)^2 - \alpha^2 A^2$ *a solution of form* (4.22) *for* $\lambda = \lambda_2$.

Remark 6. In case 1 of Corollary 2 there may exist other continuous solutions of equation (1.1) which are not of form (4.5), (4.22) or (4.25). So the equation

$$xy(x) = \int_0^x y(\xi)y(x-\xi)\,d\xi + \int_0^x y(\xi)\,d\xi \tag{4.26}$$

has besides $y_0(x) \equiv 0$ the family of solutions $y(x) = \nu\left(\frac{x}{\gamma}\right)$, $\gamma > 0$, with Volterra's function

$$\nu(x) = \int_0^\infty \frac{x^t}{\Gamma(t+1)} dt \sim \frac{1}{-\ln x} \quad \text{as} \quad x \to 0\,.$$

This follows applying the method of Laplace transform to the equation. We remark that any equation of the form

$$Axw(x) = \int_0^x w(\xi)w(x-\xi)\,d\xi + N_0 \int_0^x w(\xi)\,d\xi + c \tag{4.27}$$

with $4c = (A - N_0)^2$ can be reduced to equation (4.26) substituting $y(x) = \frac{1}{A}[w(x) - \frac{1}{2}(A - N_0)]$. The general equation (1.1) in case 1 of Corollary 2 can be treated as a perturbation of equation (4.27).

Remark 7. If $4c > (A - N_0)^2$ we have the conjugate complex values

$$\lambda_{1,2} = \frac{1}{2}[A - N_0 \pm iA\omega_0], \quad \beta_{1,2} = 1 \pm i\omega_0$$

where $\omega_0 = \frac{1}{A}\sqrt{4c - (A - N_0)^2}$. From $\operatorname{Re}\beta_{1,2} = 1$ it follows that there exist two complex solutions of form (4.22) now as in case 2 of Theorem 5.

Remark 8. The assumptions (4.1) – (4.4) on the data of equation (1.1) allow to handle as a special case the equation of Bernstein and Doetsch [4]

$$xy(x) = \gamma \int_0^x y(\xi)y(x-\xi)\,d\xi + (1-\gamma)\int_0^x y(\xi)\,d\xi \qquad (0 < \gamma < 1)$$

with the solutions $y(x) = E_\gamma(Cx^\gamma)$, where $C \in \mathbb{R}$ is an arbitrary parameter and E_γ denotes the Mittag-Leffler function. But the integral equation for the elliptic theta zero function [3 - 5]

$$2xy(x) = \int_0^x y(\xi)y(x-\xi)\,d\xi + \int_0^x y(\xi)\,d\xi - 1$$

cannot be dealt with by the present method because of the free term $p(x) \equiv -1$ and requires further investigation.

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