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Some Analysis of Tikhonov Regularization for the Inverse Problem of Option Pricing in the Price-Dependent Case

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Abstract. This paper deals with analytic studies for solving the inverse problem of identifying purely price-dependent volatilities from given option price data. Using the classical theory of parabolic differential equations we formulate and analyze the forward operator as a mapping between the Hilbert spaces $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$. We investigate continuity and Fréchet differentiability of this operator and prove the discontinuity of the inverse operator. We use Tikhonov regularization and present assertions to the stable solvability of this problem.

Keywords: *Inverse problem of option pricing, identification of local volatilities, Black-Scholes model, parabolic equations, fundamental solutions, ill-posed problem, regularization*

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1. Introduction

An European call option gives the holder the right to buy the underlying asset at the expiration date (or maturity) T for the strike price (or exercise price) K – independent of the actual price X of the asset at time T. We assume that the price X of the underlying asset follows a stochastic process of the form

$$\frac{dX}{X} = \mu(X)\,dt + \sigma(X)\,dW_t \tag{1}$$

with parameters drift μ and local volatility σ. Here W_t denotes a standard Wiener process. We suppose that the volatility σ is a deterministic function which depends on the asset price X. The model (1) represents a generalization of the model of geometric Brownian motion which forms the fundamentals for calculating option prices via the well-known Black-Scholes formula (see e.g. [4]).

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We denote by $c(X,t,K,T)$ the (fair) market price of a call option as a function of the variable asset price X, time $t \geq 0$, strike price K and expiration date $T \geq t$. Let the strike price K and expiration date T be fixed. Using the Black-Scholes analysis we can show that the option price function c satisfies the (generalized) *Black-Scholes equation*

$$\frac{\partial c}{\partial t} + \frac{1}{2} X^2 \sigma^2(X) \frac{\partial^2 c}{\partial X^2} + r\, X \frac{\partial c}{\partial X} - r\, c = 0, \qquad (X,t) \in (0,\infty) \times [0,T), \quad (2)$$

(see e.g. [4] or [18]). Furthermore, the final condition

$$c(X,T,K,T) = \max(X-K,0), \qquad X \in (0,\infty), \quad (3)$$

holds. The additional parameter r represents the interest rate of a risk-less investment and is assumed to be known. On the other hand we can fix the asset price X and the time t. Then the price function c fulfills the *Dupire equation*

$$\frac{\partial c}{\partial T} = \frac{1}{2} K^2 \sigma^2(K) \frac{\partial^2 c}{\partial K^2} - r\, K \frac{\partial c}{\partial K}, \qquad (K,T) \in (0,\infty) \times (t,\infty), \quad (4)$$

together with the initial condition

$$c(X,t,K,t) = \max(X-K,0), \qquad K \in (0,\infty). \quad (5)$$

Equation (4) was originally derived in [10] for the case $r=0$. For an alternative derivation we refer to [7].

We see that the volatility σ plays an important role in option pricing. The Cauchy problems (2)-(3) respectively (4)-(5) suggest to introduce a mapping

$$\sigma \mapsto c(X,t,K,T). \quad (6)$$

Calculating option prices by a given volatility σ is called the forward problem of option pricing. Otherwise the inverse problem seems to be of high interest. The volatility function σ is a market parameter which is not directly observable. On the other hand, options are traded on the stock market for a given asset price X and time t but different strike prices K and maturities T. Therefore we formulate the following question:

> Is it possible to identify the corresponding volatility function σ from given option price data?

This problem is known in the literature as the *inverse problem of option pricing* or *model calibrating problem.* It was first mentioned in [10] for the more general case that the volatility σ is a function as well of the asset price X as of the time t. Knowing the prices of European call options for all strike prices $K > 0$ and all maturities $T > t$ we can determine the corresponding volatility uniquely via the Dupire formula (see [2, 10]). For practical determination of local volatilities the Dupire formula does not play an important role. One reason is due to the nature of the given data: Normally option prices are given only on a discrete set of strike prices and maturities. The second point is more crucial. Calculating local volatilities via the Dupire formula demands the differentiation of the given data. As it is well-known this leads to instability phenonemons. The obtained results do not depend continuously on the given data.

In [2, 3] and [20] different approaches are suggested to solve the inverse option pricing problem in a stable way. In this context the development of numerical methods plays the principle part without studying the analytic background. First [9] gives an in-depth analytic study of this problem. The direct problem (6) is formulated as a mapping between the Hilbert space H^1 and Banach spaces L^p with $p \in (2, 3)$. In particular the case $p = 2$ is excluded. On the other hand mappings between Hilbert spaces are of particular interest in the regularization theory.

In [5, 6] and [7] another idea is suggested to study the inverse option pricing problem. To simplify the analysis the authors assume initially that the volatility is purely price-dependent. Later on these results are generalized to the case the volatility is a function which is piecewise constant in respect of time t. The inverse problem is analyzed in spaces of continuous functions. There is given a sufficient condition for uniqueness by using option prices for different strike prices but only one maturity. Some numerical methods for solving the inverse problem are introduced (see also [8] for a numerical implementation and a case study for these methods). Under strong conditions on the given data even stability was proved. In [21] a nonlinear Tikhonov regularization approach is used for stabilizing the inverse problem. Methods of optimal control are applied to derive necessary and sufficient optimality conditions for the corresponding optimization problem. But neither [7] nor [21] includes a deep analytic study of the direct mapping (6) in concrete function spaces. A first step in this direction is done in [12]. Specific Hilbert spaces are constructed to apply the well-known convergence analysis for nonlinear Tikhonov regularization (see e.g. [14]). The case that the volatility is purely time-dependent is analyzed in [15].

Our aim is to introduce a formulation of the mapping (6) as (nonlinear) operator between the Hilbert spaces H^1 and L^2 for purely price-dependent volatilities. A detailed analysis of the forward operator shall show that we can apply again the theory of nonlinear Tikhonov regularization to the inverse option pricing problem. Consequently all known stability and convergence results and assertions for convergence rates can be formulated.

The paper is organized as follows: In Section 2 we introduce appropriate variable transformations which allow us to apply the known (classical) solvability theory of parabolic equations. Additionally a close relationship between option prices and the fundamental solutions of the Black-Scholes equation and

Dupire equation is derived. This fact plays a crucial role in the further investigations. In Section 3 we deal with formulating the forward operator as a mapping from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$. Continuity of the forward operator is proved and the specific inverse problem is formulated. Section 4 is dedicated to the Fréchet differentiability of the forward operator. Finally we show that the inverse problem is ill-posed. We can prove that the well-known theory of nonlinear Tikhonov regularization (see e.g. [14]) is applicable to get stability and convergence results of regularized solution of the inverse problem.

2. Option prices and fundamental solutions

The (classical) solvability theory of parabolic differential equation (see e.g. [11] or [19]) is not directly applicable to the Cauchy problems (2)-(3) and (4)-(5). It's due to the coefficient of the second derivative, which in both equations is unbounded and tends to zero for $X \to 0$ respectively $K \to 0$. Therefore we introduce the following well-known transformations:

$$x := \ln X, \quad y := \ln K, \quad \tau := T - t, \quad u(x,\tau,y) := c(X,t,K,T)$$

and

$$a(x) := \frac{1}{2}\sigma^2(X).$$

In [21] a slightly modified transformation is used. Let $T > 0$ be fixed. Then we obtain from (2) and (4), respectively, the equations

$$\frac{\partial u}{\partial \tau} = a(x)\frac{\partial^2 u}{\partial x^2} + (r - a(x))\frac{\partial u}{\partial x} - r\,u, \quad (x,\tau) \in \mathbb{R} \times (0,T] \tag{7}$$

$$\frac{\partial u}{\partial \tau} = a(y)\frac{\partial^2 u}{\partial y^2} - (r + a(y))\frac{\partial u}{\partial y}, \quad (y,\tau) \in \mathbb{R} \times (0,T], \tag{8}$$

respectively. Additionally the initial condition

$$u(x,0,y) = \max(e^x - e^y, 0), \quad x, y \in \mathbb{R}, \tag{9}$$

holds for both equations.

Now conditions to the parameter a can be specified to obtain unique solvability of the Cauchy problems (7),(9) and (8),(9). In this context we define

$$\mathcal{D}_c^\lambda := \{a \in C^\lambda(\mathbb{R}) \ : \ \underline{c} \le a(y) \le \overline{c}, \ y \in \mathbb{R}\}$$

for two constants $0 < \underline{c} < \overline{c} < \infty$ and a Hölder index $0 < \lambda \le 1$. Thereby $C^\lambda(\mathbb{R})$ denotes as usual the Banach space of bounded and Hölder continuous functions (with Hölder index λ) with norm

$$\|a\|_{C^\lambda(\mathbb{R})} := \sup_{y\in\mathbb{R}} |a(y)| + \sup_{y_1 \ne y_2} \frac{|a(y_1) - a(y_2)|}{|y_1 - y_2|^\lambda} < \infty.$$

$\mathcal{D}_c^\lambda$ is a convex set as well in $C^\lambda(\mathbb{R})$ as in $C(\mathbb{R})$. In $C^\lambda(\mathbb{R})$ it is even closed.

Proposition 2.1. *Let $T > 0$, $r \in \mathbb{R}$ and $a \in \mathcal{D}_c^\lambda$ be fixed.*

(i) *For every $y \in \mathbb{R}$ the Cauchy problem*

$$\begin{cases} u_\tau(x,\tau) = a(x)\, u_{xx}(x,\tau) + (r - a(x))\, u_x(x,\tau) - r\, u(x,\tau), \\ u(x,0) = \max(e^x - e^y, 0), \end{cases}$$

for $(x,\tau) \in \mathbb{R} \times (0,T]$, $x \in \mathbb{R}$, possesses an unique (classical) continuous solution u on $\mathbb{R} \times [0,T]$ which satisfies a growth condition of the form

$$|u(x,\tau)| \le C_1 \exp\left(C_2\, |x|^2\right)$$

for two positive constants C_1 and C_2.

(ii) *For every $x \in \mathbb{R}$ the Cauchy problem*

$$\begin{cases} u_\tau(y,\tau) = a(y)\, u_{yy}(y,\tau) - (r + a(y))\, u_y(y,\tau), \\ u(y,0) = \max(e^x - e^y, 0), \end{cases}$$

for $(y,\tau) \in \mathbb{R} \times (0,T]$, $y \in \mathbb{R}$, possesses an unique (classical) bounded continuous solution u on $\mathbb{R} \times [0,T]$.

In both cases the existence of a classical solution follows directly from [11, Theorem 1.12]. The uniqueness we conclude from [11, Theorem 1.16].

In the next step we will derive a relationship between the fundamental solutions of the equations (7) and (8) and the transformed option price function $u(x,\tau,y)$ which is important for the further investigations. For a detailed reading on fundamental solutions we refer to [11, pp. 4ff.]. We prove the following lemma.

Lemma 2.2.

(i) *For every $a \in \mathcal{D}_c^\lambda$ the parabolic equation (7) admits a fundamental solution $\Gamma(x,\tau,\xi,\eta)$ $(x,\xi \in \mathbb{R},\ \tau > \eta \ge 0)$ and*

$$\Gamma(x,T,y,t) = e^{-y}\left(u_{yy}(x,T-t,y) - u_y(x,T-t,y)\right).$$

(ii) *For every $a \in \mathcal{D}_c^\lambda$ the parabolic equation (8) admits a fundamental solution $\hat\Gamma(y,\tau,\xi,\eta)$ $(y,\xi \in \mathbb{R},\ \tau > \eta \ge 0)$ and*

$$\hat\Gamma(y,T,x,t) = e^{-x}\left(u_{xx}(x,T-t,y) - u_x(x,T-t,y)\right).$$

Proof. For the solution u of (7),(9) we derive from [11, Theorem 1.12] that

$$u(x,\tau,y) := \int_{-\infty}^{\infty} \max(e^\xi - e^y, 0)\, \Gamma(x,\tau,\xi,0)\, d\xi = \int_y^\infty (e^\xi - e^y)\, \Gamma(x,\tau,\xi,0)\, d\xi.$$

We differentiate twice by y to achieve

$$u_y(x,\tau,y) = -\int_y^\infty e^y\,\Gamma(x,\tau,\xi,0)\,d\xi$$
$$u_{yy}(x,\tau,y) = e^y\Gamma(x,\tau,y,0) - \int_y^\infty e^y\,\Gamma(x,\tau,\xi,0)\,d\xi.$$

We take the difference and obtain

$$u_{yy}(x,\tau,y) - u_y(x,\tau,y) = e^y\Gamma(x,\tau,y,0).$$

The first statement of the lemma we conclude now by using the relation

$$\Gamma(x,\tau,y,0) = \Gamma(x,\tau+t,y,0+t) = \Gamma(x,T,y,t),$$

which is valid for parabolic equations with time-independent coefficients. The proof of the second part occurs analogous. ∎

These results we use later to obtain L^2-estimates for solutions of appropriate Cauchy problems.

3. Formulation of the inverse problem

For evaluating option prices we consider the transformed Dupire equation

$$u_t = L(a)\,u \quad \text{on} \quad \mathbb{R}\times(0,T] \tag{10}$$

with the elliptic differential operator

$$L(a)\,u(x,t) := a(x)\,u_{xx}(x,t) - (r + a(x))\,u_x(x,t),$$

which depends on the parameter a. As consequence of Proposition 2.1 (ii) we can introduce the following notation.

Definition 3.1. Let $T > 0$, $r \in \mathbb{R}$ and the logarithmized asset price $x_0 \in \mathbb{R}$ be fixed. We set $Q_T := \mathbb{R}\times(0,T)$. For given parameter $a \in \mathcal{D}_c^\lambda$ we define $u(a) \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$ as *bounded classical solution* of the Cauchy problem

$$\begin{cases} u_t(x,t) = L(a)\,u(x,t), & (x,t)\in\mathbb{R}\times(0,T] \\ u(x,0) = \max(e^{x_0} - e^x, 0), & x \in \mathbb{R}. \end{cases} \tag{11}$$

Now we investigate how variations in the volatility function a influence the option price function $u(a)$. Let $a_1, a_2 \in \mathcal{D}_c^\lambda$, and let $u(a_1)$, $u(a_2)$ be the corresponding solutions of (11). We set $v := u(a_1) - u(a_2)$. Then

$$\begin{aligned}
v_t &= u_t(a_1) - u_t(a_2) \\
&= L(a_1)\, u(a_1) - L(a_2)\, u(a_2) \\
&= a_1\, u_{xx}(a_1) - (r + a_1) u_x(a_1) - [a_2\, u_{xx}(a_2) - (r + a_2) u_x(a_2)] \\
&= a_2\, [u_{xx}(a_1) - u_{xx}(a_2)] - (r + a_2)\, [u_x(a_1) - u_x(a_2)] \\
&\quad + (a_1 - a_2)\, [u_{xx}(a_1) - u_x(a_1)] \\
&= L(a_2)\, v + (a_1 - a_2)\, [u_{xx}(a_1) - u_x(a_1)] \\
&= L(a_1)\, v + (a_1 - a_2)\, [u_{xx}(a_2) - u_x(a_2)]\,.
\end{aligned}$$

Additionally we get the initial condition $v(x,0) = 0,\ x \in \mathbb{R}$. This motivates the following definition.

Definition 3.2. For given $a_0 \in \mathcal{D}_c^\lambda$ we set

$$\mathcal{D}(a_0) := \left\{ a \in C^\lambda(\mathbb{R}) \,:\, a + a_0 \in \mathcal{D}_c^\lambda \right\} \subset C^\lambda(\mathbb{R}). \tag{12}$$

Let $a \in \mathcal{D}(a_0)$. We define $v(a) \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$ as solution of the Cauchy problem

$$\begin{cases} v_t(a) = L(a_0)\, v(a) + a\, [u_{xx}(a_0 + a) - u_x(a_0 + a)] & \text{on } \mathbb{R} \times (0, T] \\ v(x, 0; a) = 0, & x \in \mathbb{R}. \end{cases} \tag{13}$$

Remark 3.3. As seen in the derivation of (13) we can define $v(a)$ analogously via the Cauchy problem

$$\begin{cases} v_t(a) = L(a_0 + a)\, v(a) + a\, [u_{xx}(a_0) - u_x(a_0)] & \text{on } \mathbb{R} \times (0, T] \\ v(x, 0; a) = 0, & x \in \mathbb{R}. \end{cases} \tag{14}$$

In the further investigations we will use both versions to derive continuity and differentiability properties of the option pricing problem.

As consequence of Lemma 2.2 we prove the following estimate.

Lemma 3.4. *For given $a \in \mathcal{D}_c^\lambda$ and $u(a)$ as the classical solution of* (11) *the property $u_{xx}(a) - u_x(a) \in L^2(Q_T)$ holds.*

Proof. Using Lemma 2.2 we know

$$u_{xx}(x, t; a) - u_x(x, t; a) = e^x\, \Gamma(x_0, t, x, 0),$$

whereby $\Gamma(x_0, t, x, 0)$ is the fundamental solution of (7). From the estimate

$$0 < \Gamma(x_0, T, x, t) \leq \frac{c_1}{\sqrt{T-t}} \exp\left(-c_2 \frac{(x_0 - x)^2}{T-t}\right) \tag{15}$$

for two positive constants c_1 and c_2 (which depends only on $\mathcal{D}_c^\lambda$, see e.g., [11, Theorem 1.11]), we derive

$$\begin{aligned}
\|u_{xx}(a) - u_x(a)\|^2_{L^2(Q_T)} &\leq c_1^2 \iint_{Q_T} \frac{1}{t} \exp\left(-2\,c_2 \frac{(x_0-x)^2}{t} + 2\,x\right) dx\,dt \\
&\leq c_1^2 \sqrt{\frac{\pi}{2\,c_2}} \exp(2\,x_0) \int_0^T \frac{1}{\sqrt{t}} \exp\left(\frac{t}{2\,c_2}\right) dt \\
&\leq c_1^2 \sqrt{\frac{\pi}{2\,c_2}} \exp\left(2\,x_0 + \frac{T}{2\,c_2}\right) \int_0^T \frac{1}{\sqrt{t}}\, dt \\
&= c_1^2 \sqrt{\frac{2\,\pi\,T}{c_2}} \exp\left(2\,x_0 + \frac{T}{2\,c_2}\right) \\
&= C^2(x_0, T) < \infty.
\end{aligned}$$

∎

Now we can formulate a first important statement.

Theorem 3.5. *For given $a_0 \in \mathcal{D}_c^\lambda$, let $a \in \mathcal{D}(a_0)$ be arbitrarily and $v(a)$ be the solution of* (13)*. Then $v(a) \in L^2(0,T;H^2(\mathbb{R})) \cap C([0,T];H^1(\mathbb{R}))$. In particular, the estimate*

$$\|v(\cdot, T; a)\|_{L^2(\mathbb{R})} \leq C \|a\|_{L^\infty(\mathbb{R})} \tag{16}$$

holds for a constant $C > 0$ independent of a.

Proof. This is a result of the well-known theory of parabolic equations considering $v(a)$ as weak solution of (13). From [19, Theorem III.5.2] we conclude $v(a) \in L^2(0,T;H^1(\mathbb{R})) \cap C([0,T];H^1(\mathbb{R}))$, and from [19, Theorem III.2.1] we obtain the estimate

$$\begin{aligned}
\|v(\cdot, T; a)\|_{L^2(\mathbb{R})} &\leq \sup_{t \in [0,T]} \|v(\cdot, T; a)\|_{L^2(\mathbb{R})} \\
&\leq c\, \|a\,(u_{xx}(a_0 + a) - u_x(a_0 + a))\| \\
&\leq C\, \|a\|_{L^\infty(\mathbb{R})} \|u_{xx}(a_0 + a) - u_x(a_0 + a)\|_{L^2(\mathbb{R})} \\
&\leq c\, C(x_0, T)\, \|a\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

The higher regularity $v(a) \in L^2(0,T;H^2(\mathbb{R}))$ is a consequence of [19, Theorem III.6.1]. ∎

Using the fact that $H^1(\mathbb{R})$ is continuously embedded in $C(\mathbb{R})$ (and therefore in $L^\infty(\mathbb{R})$) we can define the following operator.

Definition 3.6. For given $a_0 \in \mathcal{D}_c^\lambda$, $x_0, r \in \mathbb{R}$ and $T > 0$ we define the nonlinear operator $F : \mathcal{D}(a_0) \subset H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ via

$$F(a) := v(\cdot, T; a) = u(\cdot, T; a_0 + a) - u(\cdot, T; a_0). \tag{17}$$

The choise of the spaces seems to be natural in the framework of Tikhonov regularization. Considering a 'smooth' parameter $a \in H^1(\mathbb{R})$ instead of $a \in L^2(\mathbb{R})$ usually leads to better regularization results. The following result is an immediate consequence of Theorem 3.5.

Corollary 3.7. *The operator F is Lipschitz continuous for every $a \in \mathcal{D}(a_0)$.*

Proof. For given $a \in \mathcal{D}(a_0)$ let $\tilde{a} \in \mathcal{D}(a_0)$ be arbitrarily. We set $h := \tilde{a} - a$ and consider $\tilde{v} := v(\tilde{a}) - v(a)$. Then

$$\begin{aligned} \tilde{v} &= v(a + h) - v(a) \\ &= v(a + h) - v(0) - [v(a) - v(0)] \\ &= u(a_0 + a + h) - u(a_0) - [u(a_0 + a) - u(a_0)] \\ &= u(a_0 + a + h) - u(a_0 + a). \end{aligned}$$

We can apply Theorem 3.5 with $a_0 + a$ instead of a_0 and h instead of a to obtain

$$\|F(\tilde{a}) - F(a)\|_{L^2(\mathbb{R})} = \|\tilde{v}\|_{L^2(\mathbb{R})} \leq C\,\|h\|_{L^\infty(\mathbb{R})} \leq \tilde{C}\,\|h\|_{H^1(\mathbb{R})} = \tilde{C}\,\|\tilde{a} - a\|_{H^1(\mathbb{R})},$$

whereby the constant $\tilde{C}$ does not depend on the element $\tilde{a}$. ∎

Now, Definition 3.6 of the operator F allows us to formulate the following inverse problem.

Definition 3.8 (Inverse Problem-(IP)). Let $x_0 \in \mathbb{R}$ be the actual logarithmized asset price at time $t = 0$. Furthermore let $u_d(x)$ be an option price function for a fixed maturity $T > 0$ and all logarithmized exercise prices $x \in \mathbb{R}$. For given interest rate r and given a priori guess $a_0 \in \mathcal{D}_c^\lambda$ of the unknown local volatility we try to find a function $a \in \mathcal{D}(a_0)$, which satisfies the equation

$$F(a) = u_d - u(\cdot, T; a_0). \tag{18}$$

4. Differentiability of the forward operator

We will examine the Fréchet differentiability of the operator F. To do this, let the Hölder coefficient λ in the definition of $\mathcal{D}_c^\lambda$ be in the interval $(0, \frac{1}{2})$. Then $H^1(\mathbb{R}) \subset \mathcal{D}_c^\lambda$, the interior of $\mathcal{D}(a_0)$ in $H^1(\mathbb{R})$ is not empty and

$$a \in \operatorname{int} \mathcal{D}(a_0) \iff \underline{c} < \gamma_1 \leq a + a_0 \leq \gamma_2 < \overline{c} \quad \text{on } \mathbb{R}$$

for two constants γ_1 and γ_2. Then we prove that F is Fréchet differentiable for all interior points a of $\mathcal{D}(a_0)$.

In a first step we show the existence of directional derivatives.

Theorem 4.1. *Let $a \in \mathcal{D}(a_0)$. Then for every $h \neq 0$ with $a + h \in \mathcal{D}(a_0)$ there exists the directional derivative $F'(a)\,h$. Let w be the solution of the Cauchy problem*

$$\begin{cases} \quad w_t = L(a_0 + a)\,w + h\,[u_{xx}(a_0 + a) - u_x(a_0 + a)] & \text{on } \mathbb{R} \times (0, T] \\ w(x, 0) = 0, & x \in \mathbb{R}, \end{cases} \tag{19}$$

then $[F'(a)\,h](x) = w(x, T),\ x \in \mathbb{R}$.

Proof. Let $h \neq 0$ with $a + h \in \mathcal{D}(a_0)$ be arbitrarily. Then $a + \varepsilon\,h \in \mathcal{D}(a_0)$ for every $0 \le \varepsilon \le 1$ since $\mathcal{D}(a_0)$ is a convex set in $H^1(\mathbb{R})$. We set

$$\begin{aligned} w^\varepsilon := \frac{1}{\varepsilon}\big(F(a + \varepsilon\,h) - F(a)\big) &= \frac{1}{\varepsilon}\big(v(a + \varepsilon\,h) - v(a)\big) \\ &= \frac{1}{\varepsilon}\big(u(a_0 + a + \varepsilon\,h) - u(a_0 + a)\big) \end{aligned}$$

and show $w^\varepsilon \to w$ for $\varepsilon \to 0$. Since

$$\begin{aligned} w_t^\varepsilon &= \frac{1}{\varepsilon}\big(L(a_0 + a + \varepsilon\,h)\,v(a + \varepsilon\,h) - L(a_0 + a)\,v(a)\big) \\ &\quad + \frac{1}{\varepsilon}\big[(a + \varepsilon\,h)(u_{xx}(a_0) - u_x(a_0)) - a\,(u_{xx}(a_0) - u_x(a_0))\big] \\ &= L(a_0 + a)\,w^\varepsilon + h\big(v_{xx}(a + \varepsilon\,h) - v_x(a + \varepsilon, h) + u_{xx}(a_0) - u_x(a_0)\big) \\ &= L(a_0 + a)\,w^\varepsilon + h\big(u_{xx}(a_0 + a + \varepsilon\,h) - u_x(a_0 + a + \varepsilon\,h)\big) \end{aligned}$$

because

$$v(a + \varepsilon\,h) + u(a_0) = u(a_0 + a + \varepsilon\,h) - u(a_0) + u(a_0) = u(a_0 + a + \varepsilon\,h),$$

we can represent w^ε as solution of the Cauchy problem

$$\begin{cases} \quad w_t^\varepsilon = L(a_0 + a)\,w^\varepsilon + h\,[u_{xx}(a_0 + a + \varepsilon\,h) - u_x(a_0 + a + \varepsilon\,h)] \\ w^\varepsilon(x, 0) = 0, \end{cases}$$

on $\mathbb{R} \times (0, T]$ and $x \in \mathbb{R}$. We consider the limit $\varepsilon \to 0$. Let w be the solution of the Cauchy problem (19) and $\tilde{w} := w^\varepsilon - w$. Then $\tilde{w}$ satisfies

$$\begin{cases} \quad \tilde{w}_t = L(a_0 + a)\,\tilde{w} + h\big[\big(u_{xx}(a_0 + a + \varepsilon\,h) - u_x(a_0 + a + \varepsilon\,h)\big) \\ \qquad\qquad - \big(u_{xx}(a_0 + a) - u_x(a_0 + a)\big)\big] \\ \tilde{w}(x, 0) = 0, \end{cases}$$

on $\mathbb{R} \times (0, T]$ and $x \in \mathbb{R}$. Obviously v with $v := u(a_0 + a + \varepsilon\,h) - u(a_0 + a)$ is a solution of the Cauchy problem (13) with $a_0 + a$ instead of a_0 and $\varepsilon\,h$ instead of a. Then $v \in C([0, T]; H^1(\mathbb{R}))$ and

$$\|v_{xx} - v_x\|_{L^2(Q_T)} \le C\,\varepsilon\,\|h\|_{L^\infty(Q_T)} \to 0 \quad \text{for } \varepsilon \to 0.$$

Consequently $\tilde{w} \to 0$ for $\varepsilon \to 0$ and therefore $w^\varepsilon \to w$. The proof is complete. ∎

Let now $a \in \mathcal{D}(a_0)$ be fixed. Theorem 4.1 motivates the definition of a linear operator $F'(a) : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ via

$$[F'(a)\,h](x) := w(x,T), \qquad x \in \mathbb{R} \tag{20}$$

whereby now $h \in H^1(\mathbb{R})$ is arbitrarily and w is the corresponding solution of (19). Since

$$\begin{aligned}\|F'(a)\,h\|_{L^2(\mathbb{R})} &= \|w(\cdot,T)\|_{L^2(\mathbb{R})} \\ &\le C\,\|h\|_{L^\infty(\mathbb{R})}\|u_{xx}(a_0+a) - u_x(a_0+a)\|_{L^2(Q_T)} \\ &\le \tilde{C}\,\|h\|_{H^1(\mathbb{R})}\,,\end{aligned}$$

the linear operator $F'(a)$ is bounded. Therefore F is Gâteaux differentiable for each $a \in \operatorname{int}\mathcal{D}(a_0)$, and $F'(a)$ denotes the Gâteaux derivative of F in a. Now we can prove the Fréchet differentiability.

Lemma 4.2. *For arbitrarily $a, a+h \in \mathcal{D}(a_0)$ the estimate*

$$\|F'(a+h) - F'(a)\| \le C\,\|h\|_{H^1(\mathbb{R})}$$

holds for a constant $C > 0$, which does not depend on h.

Proof. Let $\tilde{h} \in H^1(\mathbb{R})$ be arbitrarily. We set $w(a) := F'(a)\,\tilde{h}$, $w(a+h) := F'(a+h)\,\tilde{h}$ and $\hat{w} := w(a+h) - w(a)$. Then $\hat{w}$ satisfies

$$\begin{aligned}\hat{w}_t &= L(a_0+a+h)\,w(a+h) + \tilde{h}\big[u_{xx}(a_0+a+h) - u_x(a_0+a+h)\big] \\ &\quad - L(a_0+a)\,w(a) - \tilde{h}\big[u_{xx}(a_0+a) - u_x(a_0+a)\big] \\ &= L(a_0+a)\,\hat{w} + h\big[(w_{xx}(a+h) - w_x(a+h)\big] \\ &\quad + \tilde{h}\big[\,[u_{xx}(a_0+a+h) - u_x(a_0+a+h)] - [u_{xx}(a_0+a) - u_x(a_0+a)]\,\big]\end{aligned}$$

on $\mathbb{R} \times (0,T]$, together with the initial condition $\hat{w}(x,0) = 0$, $x \in \mathbb{R}$. From Theorem 3.5 with $a_0 + a$ instead of a_0 and h instead of a we derive

$$\begin{aligned}&\Big\|\big[u_{xx}(a_0+a+h) - u_x(a_0+a+h)\big] \\ &\quad - \big[u_{xx}(a_0+a) - u_x(a_0+a)\big]\Big\|_{L^2(Q_T)} \le C\,\|h\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Furthermore, from Theorem 4.1 follows $w(a+h) \in L^2(0,T;H^2(\mathbb{R}))$ and therefore

$$\Big\|h\big[w_{xx}(a+h) - w_x(a+h)\big]\Big\|_{L^2(Q_T)} \le C\,\|h\|_{L^\infty(\mathbb{R})}\,\|\tilde{h}\|_{L^\infty(\mathbb{R})}.$$

Analogously to the proof of Theorem 3.5 we obtain

$$\|\hat{w}(\cdot,T)\|_{L^2(\mathbb{R})} = \|F'(a+h)\,\tilde{h} - F'(a)\,\tilde{h}\|_{L^2(\mathbb{R})} \le C\,\|h\|_{H^1(\mathbb{R})}\,\|\tilde{h}\|_{H^1(\mathbb{R})}.$$

This proves the lemma. ∎

The Fréchet differentiability of F follows now immediately from this lemma (see e.g. [17, Proposition 1]).

Theorem 4.3. *For every $a \in \mathrm{int}\,\mathcal{D}(a_0)$ the operator F is Fréchet differentiable. $F'(a)$ is defined via (20), and for every h with $a+h \in \mathcal{D}(a_0)$ the estimate*

$$\|F(a+h) - F(a) - F'(a)\,h\|_{L^2(\mathbb{R})} \leq \frac{L}{2}\|h\|^2_{H^1(\mathbb{R})}$$

holds for a constant $L > 0$, which does not depend on h.

5. Tikhonov regularization of the inverse problem

First of all we show that the problem (IP) is ill-posed. Following [17, Definition 2] we prove the local ill-posedness of equation (18) for every function $a \in \mathcal{D}(a_0)$. Consequently solutions a of (18) does not depend continuously on the given data u_d.

Theorem 5.1. *For every $a \in \mathcal{D}(a_0)$ and every ball $B_r(a) := \{\hat{a} \in H^1(\mathbb{R}) : \|\hat{a} - a\|_{H^1(\mathbb{R})} < r\}$ ($r > 0$) there exists a sequence $\{a_n\} \subset \mathcal{D}(a_0) \cap B_r(a)$ with $a_n \not\to a$ but $F(a_n) \to F(a)$, in particular equation (18) is locally ill-posed.*

Proof. Let $a \in \mathcal{D}(a_0)$ be arbitrarily. Then

$$\int_{-\infty}^{\infty} [a_0(x) + a(x) - \underline{c}]^2\,dx = \infty \quad \text{or} \quad \int_{-\infty}^{\infty} [\overline{c} - (a_0(x) + a(x))]^2\,dx = \infty.$$

We assume the first case. Furthermore we define a sequence of functions $\{a_n\}$, $n = 1, 2, \ldots$, with the following properties

$$a_n(x) := \begin{cases} \underline{c} - a_0(x), & |x| < n \\ a(x), & |x| > n+1. \end{cases}$$

On the intervals $[-n-1, -n]$ and $[n, n+1]$ we choose $a_n(x)$ in the way that they are elements of $\mathcal{D}(a_0) \subset H^1(\mathbb{R})$ and

$$\|a_n - a\|_{L^\infty(\mathbb{R})} = \sup_{x \in [-n,n]} |a_n(x) - a(x)|.$$

For given $0 < r \leq 1$ we introduce

$$\tilde{a}_n := a + \frac{r}{2}\,\frac{a_n - a}{\max\left\{1, \|a_n - a\|_{H^1(\mathbb{R})}\right\}}.$$

Then $\{\tilde{a}_n\} \subset \mathcal{D}(a_0) \cap B_r(a)$, $\|\tilde{a}_n - a\|_{H^1(\mathbb{R})} = \frac{r}{2}$ for n large enough and $\|\tilde{a}_n - a\|_{L^\infty(\mathbb{R})} \to 0$ for $n \to \infty$ since $\|a_n - a\|_{L^2(\mathbb{R})} \to \infty$ for $n \to \infty$ (and therefore $\|a_n - a\|_{H^1(\mathbb{R})} \to \infty$). From Theorem 3.4 we obtain now

$$\|F(\tilde{a}_n) - F(a)\|_{L^2(\mathbb{R})} \leq C\,\|\tilde{a}_n - a\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{for} \quad n \to \infty$$

for a constant $C > 0$. Thus we have proved the instability of equation (18). ∎

Theorem 5.2. *The operator F is weakly closed. Moreover for $a_n \in \mathcal{D}(a_0)$, $a_n \rightharpoonup a$ it follows that $a \in \mathcal{D}(a_0)$ and $F(a_n) \to F(a)$.*

Proof. From the convexity and closedness of $\mathcal{D}(a_0)$ follows the weak closedness of $\mathcal{D}(a_0)$. Let $\{a_n\} \subset \mathcal{D}(a_0)$ be a sequence which converges weakly to an element $a \in H^1(\mathbb{R})$. Consequently $a \in \mathcal{D}(a_0)$ holds. We consider $F(a_n) - F(a)$. Again we have

$$F(a_n) - F(a) = v_n(\cdot, T),$$

where v_n is the solution of (13) with $a_0 + a$ instead of a_0 and $a_n - a$ instead of a. We show

$$\| \underbrace{(a_n - a)\,(u_{xx}(a_0 + a) - u_x(a_0 + a))}_{=: f_n} \|_{L^2(Q_T)} \to 0 \quad \text{for} \;\; n \to \infty.$$

Then – from [19, Theorem III.2.1] – we can conclude $v_n \to 0$ and therefore $F(a_n) \to F(a)$ for $n \to \infty$. We consider

$$\begin{aligned}
&\|f_n\|^2_{L^2(Q_T)} \\
&\quad = \int_0^T \int_{-\infty}^{\infty} f_n^2(x,t)\; dx\, dt \\
&\quad = \underbrace{\int_0^T \int_{-\infty}^{y_1} f_n^2(x,t)\; dx\, dt}_{=: I_1(y_1)} + \underbrace{\int_0^T \int_{y_1}^{y_2} f_n^2(x,t)\; dx\, dt}_{=: I_2(y_1,y_2)} + \underbrace{\int_0^T \int_{y_2}^{\infty} f_n^2(x,t)\; dx\, dt}_{=: I_3(y_2)} .
\end{aligned}$$

We prove: for every $\varepsilon > 0$ there exist $y_1, y_2 \in \mathbb{R}$ and $n \in \mathbb{N}$ such that

$$I_1(y_1) < \frac{\varepsilon}{3}, \quad I_2(y_1, y2) < \frac{\varepsilon}{3}, \quad I_3(y_2) < \frac{\varepsilon}{3}.$$

Since $u_{xx}(a_0 + a) - u_x(a_0 + a) \in L^2(Q_T)$ there exist $y_1, y_2 \in \mathbb{R}$ such that

$$\begin{aligned}
\|u_{xx}(a_0 + a) - u_x(a_0 + a)\|_{L^2((-\infty,y_1)\times(0,T))} &\le \frac{\varepsilon}{3(\overline{c} - \underline{c})} \\
\|u_{xx}(a_0 + a) - u_x(a_0 + a)\|_{L^2((y_2,\infty)\times(0,T))} &\le \frac{\varepsilon}{3(\overline{c} - \underline{c})} \,,
\end{aligned}$$

and therefore

$$I_1(y_1) \le \frac{\varepsilon}{3}\,, \quad I_3(y_2) \le \frac{\varepsilon}{3}$$

for all $n \in \mathbb{R}$. Let now y_1, y_2 be fixed. Then we have

$$\begin{aligned}
I_2(y_1, y_2) &= \int_0^T \int_{y_1}^{y_2} (a_n - a)^2 \big(u_{xx}(a_0 + a) - u_x(a_0 + a)\big)^2\, dx\, dt \\
&\le \|a_n - a\|^2_{L^\infty(y_1,y_2)} \|u_{xx}(a_0 + a) - u_x(a_0 + a)\|^2_{L^2(0,T;L^2(y_1,y_2))} \\
&\le \|a_n - a\|^2_{L^\infty(y_1,y_2)} \|u_{xx}(a_0 + a) - u_x(a_0 + a)\|^2_{L^2(Q_T)}.
\end{aligned}$$

Since $a_n - a$ converges weakly to 0 in $H^1(\mathbb{R})$ we can conclude the strong convergence $a_n - a|_{(y_1,y_2)} \to 0$ in $L^\infty(y_1,y_2)$ by the Rellich-Kondrachov Theorem (see [1, Theorem 6.2]), and consequently $I_2(y_1,y_2) \to 0$ for $n \to \infty$. ∎

Theorem 5.2 allows us to apply the convergence and stability results of the well-known theory of nonlinear Tikhonov regularization (see [14]). To do this, we consider instead of equation (18) the minimization problem

$$\|F(a) - u^\delta - u(\cdot,T;a_0)\|^2_{L^2(\mathbb{R})} + \alpha\,\|a - a_0\|^2_{H^1(\mathbb{R})} \longrightarrow \min_{a\in\mathcal{D}(a_0)} \tag{21}$$

for given noisy data u^δ (instead of the exact data u_d) with $\|u^\delta - u_d\| \le \delta$, $\delta > 0$. Then (21) admits for every $\alpha > 0$ a (not necessarily unique) solution $a^\delta_\alpha \in \mathcal{D}(a_0)$ which depends continuously of the data u^δ (see [14, Theorem 2.1]). We formulate the following lemma.

Lemma 5.3. *Let $\alpha > 0$, $\{u_k\}$ and $\{a_k\}$ are sequences with $u_k \to u^\delta$ and a_k is a solution of (21) with u_k instead of u^δ. Then there exists a convergent subsequence of $\{a_k\}$, and the limit of each convergent subsequence is a solution of (21).*

Under the additional assumption that equation (18) admits a solution for exact data u_d we can prove the convergence a^δ_α to a solution of (18) for $u^\delta \to u_d$ and an appropriate parameter choice $\alpha = \alpha(\delta)$ (see [14, Theorem 2.3]).

Now we turn to convergence rates. As a consequence of Theorem 4.3 we can apply [14, Theorem 2.4] to present the following statement.

Proposition 5.4. *Assume there exists a solution $a^* \in \mathcal{D}$ of (18) for exact data u_d and an element $\omega \in L^2(\mathbb{R})$ such that*

- **(i)** *$a^* - a_0 = F'(a^*)^*\,\omega$, whereby $F'(a^*)^*$ denotes the adjoint operator of $F'(a^*)$ and*
- **(ii)** *$L\|\omega\|_{L^2(\mathbb{R})} < 1$, where L is the constant in Theorem 4.3.*

Then, for an a-priory parameter choice $\alpha \sim \delta$, we can verify a convergence rate

$$\|a^\delta_\alpha - a^*\|_{H^1(\mathbb{R})} = O\big(\sqrt{\delta}\big).$$

We will examine the conditions (i) and (ii). Let $a \in \mathcal{D}(a_0)$ be a fixed element. To give an interpretation of the source condition (i) we decompose the operator $F'(a)$ into

$$F'(a) = \tilde{F}'(a) \circ \mathcal{I},$$

where $\mathcal{I} : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ denotes the embedding operator from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$. Furthermore $\tilde{F}'(a)$ is defined analogously to $F'(a)$ by (20) with

$\omega \in L^2(\mathbb{R})$ instead of $h \in H^1(\mathbb{R})$. We introduce the adjoint equation of (10) which is given by

$$u_t + L^*(a)\, u = u_t + (a\, u)_{xx} + ((r + a)\, u)_x = 0 \quad \text{on } \mathbb{R} \times [0, T).$$

For given $\omega \in L^2(\mathbb{R})$ let $z(a, \omega)$ denotes the solution of the Cauchy problem

$$\begin{cases} z_t(a,\omega) + L^*(a)\, z(a,\omega) = 0 & \text{on } \mathbb{R} \times [0,T) \\ \qquad\qquad z(x,T;a,\omega) = \omega(x), & x \in \mathbb{R}. \end{cases} \tag{22}$$

Furthermore, let $g(a, \omega)$ be defined via

$$g(x;a,\omega) := \int_0^T z(x,t;a,\omega)\,(u_{xx}(x,t;a_0+a) - u_x(x,t;a_0+a))\; dt, \quad x \in \mathbb{R}.$$

Then we can show (see [16, Lemma 7.3]) that $g(a,\omega) \in L^2(\mathbb{R})$ and the adjoint operator $\tilde{F}'(a)^* : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is given by

$$F'(a)^* \omega = g(a,\omega) \quad \forall\, \omega \in \mathbb{R},$$

(see [16, Theorem 6.1] and the proof therein). It is well-known that the adjoint $\mathcal{I}^* : L^2(\mathbb{R}) \longrightarrow H^1(\mathbb{R})$ is given by $\mathcal{I}^* g = v$, $g \in L^2(\mathbb{R})$, where $v \in H^1(\mathbb{R})$ solves

$$v - v'' = g \qquad \text{a.e. on } \mathbb{R}, \tag{23}$$

(see e.g. [14, Example 3.2]). Using the fact that $F'(a)^* = (\tilde{F}'(a) \circ \mathcal{I})^* = \mathcal{I}^* \circ \tilde{F}'(a)^*$ we now derive that

$$F'(a)\, \omega = v(a,\omega)\,,$$

where $v(a,\omega)$ is given by (23) with $g(a,\omega)$ instead of g. Applying this result we can formulate the following consequence out of Proposition 5.4.

Corollary 5.5. *Assume there exists a solution $a^* \in \mathcal{D}$ of* (18) *for exact data u_d. Then the conditions* (i) *and* (ii) *of Proposition* 5.4 *say that $(a^* - a_0)'' \in L^2(\mathbb{R})$, and there exists a constant $C > 0$ such that*

$$|a^*(x) - a_0(x)| \le \frac{C}{L} \exp(-|x|) \quad \forall\, x \in \mathbb{R}.$$

The regularity condition $(a^* - a_0)'' \in L^2(\mathbb{R})$ follows immediately out of the differential equation (23). The exponential decay was proven in [16] using the exponential decay of fundamental solutions of parabolic equations.

6. Concluding remarks

With a deeply analytic study we have shown that instability effects arising by the numerical determination of price-dependent volatilities are a consequence of the ill-posedness of equation (18). Furthermore, we have proved that the Tikhonov regularization approach (21) provides a stable way for solving the inverse option pricing problem (18) for purely price-dependent volatilities. Therefore we can close the gap between the numerical results of Tikhonov regularization as presented, e.g., in [20] and the convergence analysis behind this approach. Finally, we have formulated conditions to the a-priori guess a_0 in (21) to obtain convergence rates when the noise levels δ decays to zero.

On the other hand, there is still the open problem of uniqueness of the inverse problem (18). The uniqueness result of [6] is based on the additional assumption that the volatility σ is known on a interval $(a, b) \subset \mathbb{R}$. Recently in [13] a different way is applied to obtain convergence rates. The problem (21) is reformulated in that way that data on a stripe $\mathbb{R} \times [T - \Delta T, T]$ are used to reconstruct price-dependent volatilities. Note that in this overdetermined case the Dupire formula (see, e.g., [7]) gives an explicite expression for estimating the volatility σ uniquely.

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