

Hilbert Spaces of Solutions to Polynomial Dirac Equations, Fourier Transforms and Reproducing Kernel Functions for Cylindrical Domains

D. Constaes and R. S. Kraußhar

Abstract. In this paper, we consider L^2 spaces of functions that satisfy polynomial Dirac equations. Fourier transformation methods and methods from harmonic analysis are then applied to treat Hilbert spaces of Clifford algebra valued functions that are either square-integrable over a cylinder or square-integrable over its boundary, and which satisfy in its interior the generalized Cauchy-Riemann system. In particular, explicit representation formulas for the Bergman and Szegő reproducing kernel of several types of cylindrical domains are developed.

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1. Introduction and Basic Notions

Let $\{e_1, e_2, \dots, e_k\}$ be the standard basis of the Euclidean vector space \mathbb{R}^k and Cl_{0k} be the associated real Clifford algebra in which

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, \dots, k,$$

holds, δ_{ij} standing for the Kronecker symbol. Each element $a \in Cl_{0k}$ can be represented in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$, $A \subseteq \{1, \dots, k\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \dots < l_r \leq k$, $e_\emptyset = e_0 = 1$. The scalar part of

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a , $\text{Sc}(a)$, is defined as the a_0 term. The Clifford conjugate of a is defined by $\bar{a} = \sum_A a_A \bar{e}_A$, where $\bar{e}_A = \bar{e}_{l_r} \bar{e}_{l_{r-1}} \cdots \bar{e}_{l_1}$ and $\bar{e}_j = -e_j$ for $j = 1, \dots, k$, $\bar{e}_0 = e_0 = 1$. In contrast, we denote the complex conjugate of a complex number $\lambda \in \mathbf{C}$ by λ^\sharp .

We further write $\mathcal{A}_{k+1} := \text{span}_{\mathbb{R}}\{1, e_1, \dots, e_k\} = \mathbb{R} \oplus \mathbb{R}^k \subset Cl_{0k}$ for the space of paravectors $z = z_0 + z_1 e_1 + z_2 e_2 + \cdots + z_k e_k$, which in turn may be identified with \mathbb{R}^{k+1} . The standard scalar product between two Clifford numbers $a, b \in Cl_{0k}$ is defined by $\langle a, b \rangle := \text{Sc}(a\bar{b})$, inducing the standard norm of a Clifford number, viz $\|a\| = (\sum_A |a_A|^2)^{1/2}$.

Let $\Omega \subseteq \mathbb{R}^{k+1}$ be an open set. A real differentiable function $f : \Omega \rightarrow Cl_{0k}$ that satisfies inside of Ω the system $D_z f = 0$ (or $f D_z = 0$), where

$$D_z = \partial_{z_0} + \sum_{j=1}^k \partial_{z_j} e_j, \quad z_j \in \mathbb{R}, \quad j = 0, \dots, k,$$

stands for the Euclidean generalized Cauchy-Riemann operator in \mathcal{A}_{k+1} , are called left (right) monogenic with respect to the paravector variable z , respectively. In the two-dimensional case ($k = 1$) this operator coincides with the classical complex Cauchy-Riemann operator. In this sense, the set of monogenic functions can hence be regarded as a higher dimensional generalization of the set of complex-analytic functions.

In the sequel, we shall write D_z in the form $D_z = \partial_{z_0} + \mathbf{D}_{\mathbf{z}}$, where the vector part $\mathbf{D}_{\mathbf{z}} = \sum_{j=1}^k \partial_{z_j} e_j$ is the Euclidean Dirac operator in \mathbb{R}^k . Functions defined in an open subset of \mathbf{R}^k and satisfying there $\mathbf{D}_{\mathbf{z}} f = 0$ are also called left monogenic, now with respect to the vector variable \mathbf{z} . For details, see for example [12] and elsewhere. Both operators D_z and $\mathbf{D}_{\mathbf{z}}$ factorise the Euclidean Laplacian, viz $D_z \bar{D}_z = \Delta_z$ and $\mathbf{D}_{\mathbf{z}}^2 = -\Delta_{\mathbf{z}}$. Each real component of a monogenic function is therefore harmonic. This, in turn, allows us to apply methods from harmonic analysis to study monogenic functions.

Similarly to the classical complex case in two real dimensions, the treatment of Hilbert spaces in the higher dimensional setting is a central topic in the framework of this function theory – in particular, in view of many applications to physics and the applied sciences.

Important examples of Hilbert spaces of Clifford valued functions that satisfy in particular the Bergman condition

$$\|f(\mathbf{z})\| \leq C(\mathbf{z}) \|f\|_{L^2}$$

are the spaces of square integrable monogenic functions in a domain of \mathbb{R}^{k+1} or on its boundary; these are called Bergman spaces of monogenic functions or Hardy spaces, respectively. Each Bergman space is a Banach space which is endowed with the Clifford-valued inner product $(f, g) = \int_{\Omega} \overline{f(z)} g(z) dV$, where

f and g denote left monogenic functions that are square integrable on a given domain $\Omega \subseteq \mathbb{R}^{k+1}$, while each Hardy space is endowed with the inner product $(f, g) = \int_{\partial\Omega} \overline{f(z)}g(z)dS$, where dS is the k -dimensional positive scalar measure on $\partial\Omega$ induced by the ordinary Lebesgue measure.

Some of the first contributions to the investigation of these function spaces came from R. Delanghe and F. Brackx in 1976 and 1978, see [11] and [2]. In the following decades, the study of these types of function spaces has been extended significantly under a rich amount of different aspects. Just to give some very few examples, see for instance [6], [7], works from J. Ryan (for example [24]), from M. Shapiro and N. Vasilevski from 1993 onwards, as for instance [25, 26], or works by J. Cnops, such as, e.g., [5], papers of Bernstein and Lanzani, e.g., [1], among a lot of other contributions that point in this direction.

The reproducing kernel of the Bergman space, called the Bergman kernel, is uniquely defined and satisfies $f(z) = \int_{\Omega} B_{\Omega}(z, w)f(w)dV$ for any square integrable monogenic function f on the domain $\Omega \subseteq \mathbb{R}^{k+1}$.

Passing from volume to surface integrals, the corresponding reproducing kernel is called the Szegö kernel $S_{\Omega}(z, w)$; it satisfies $f(z) = \int_{\partial\Omega} S_{\Omega}(z, w)f(w)dS$ for any square integrable monogenic function f . A central aspect is the determination of explicit and closed formulas for these kernel functions, which is very difficult in general. In contrast to the Cauchy kernel, the Bergman and Szegö kernels both depend on the domain Ω .

For instance, in [2, 5], one finds explicit formulas for the monogenic kernel functions for the unit ball and the half-space. In 1996, D. Calderbank succeeded in deriving closed representation formulas for the kernel functions in annular domains [4].

Explicit formulas for the monogenic Bergman kernels for rectangular, strip and wedge-shaped domains have been worked out recently in [8, 10]. For the Szegö kernel associated to a strip domain that is bounded in one direction, a closed representation formula in terms of explicit monogenic generalized cosecant type functions has been developed in the paper [9], which in turn provides an extension to J. Peetre and P. Sjölin's results from [22], in particular in terms of explicitness. In [17], formulas have been developed for the cylindrical and toroidal monogenic and polymonogenic Bergman and Szegö kernel functions associated to the surface of projective half-cylinders of infinite extent that are constructed by factoring a half-space from \mathbb{R}^k by a translation group.

Analogues of these function spaces in the framework of more generally polynomial Dirac equations of the form $[\sum_{i=0}^m \alpha_i \mathbf{D}_z^i]f = 0$ with complex coefficients α_i are of current interest, too. In this case, the functions take values in the complex Clifford algebra $Cl_{0k} \otimes \mathbf{C}$, the Clifford-valued inner product must be adapted to

$$(f, g) = \int_{\Omega} \overline{f(z)}^{\sharp} g(z) dV$$

and the norm to

$$\|f\| = \sqrt{\text{Sc}(f, f)}.$$

Similar adaptations are performed in the context of Hardy spaces.

Important fundamentals for the general theory of function spaces in this setting have been developed for example by Xu Zhenyuan [28], by F. Brackx, F. Sommen, N. Van Acker [3] and by J. Ryan in [24]. See also [14], [19] and elsewhere.

In [3] an explicit representation formula for the Bergman kernel in the unit ball for the solutions of the special system $(\mathbf{D}_{\mathbf{z}} - \lambda)f = 0$, for arbitrary $\lambda \in \mathbf{C}$, has been developed. J. Ryan showed in [24] that the space of null solutions to $(\mathbf{D}_{\mathbf{z}} - \lambda)f = 0$ that are square-integrable over a domain that has a piecewise C^1 or Lipschitz boundary, has in general always a uniquely defined Bergman kernel function.

In this paper, we extend the study of these function spaces and the previous works in this field under two different aspects. In the first part of this paper, we complete the result from [3] by giving on the one hand an explicit formula for the Bergman kernel of the unit ball related to the more general system

$$(\mathbf{D}_{\mathbf{z}} - \lambda_1)(\mathbf{D}_{\mathbf{z}} - \lambda_2) \cdots (\mathbf{D}_{\mathbf{z}} - \lambda_p)f(\mathbf{z}) = 0, \quad (1)$$

where $\lambda_1, \dots, \lambda_p$ are arbitrary non-zero complex numbers, and, on the other hand, by providing explicit formulas for the Szegő kernel in the unit ball for the systems $(\mathbf{D}_{\mathbf{z}} - \lambda_1)f = 0$ and $(\mathbf{D} - \lambda_1)(\mathbf{D} - \lambda_2)f = 0$, where λ_1, λ_2 are again arbitrary complex numbers. This treatment includes also an explicit formula for the Szegő kernel for solutions to the Helmholtz equation $-\Delta f = \lambda^2 f$, with complex eigenvalues λ . Furthermore, we show that there are no Szegő kernel functions for systems of type (1) when $p > 2$.

In the second part of this paper, we study the Fourier transform of these kernels, which in turn allows the treatment of L^2 spaces of functions that satisfy the Euclidean Cauchy-Riemann equation in a $(k + 1)$ -dimensional cylinder of the form $z_1^2 + \dots + z_k^2 = 1$, $0 < z_0 < d$. By using harmonic analysis methods, we finally obtain explicit representation formulas for the reproducing kernel functions of the Bergman space of functions that satisfy the Euclidean Cauchy-Riemann equation in such cylinders.

In analogy to strip, block and wedge-shaped domains, the formulas for the Bergman kernel of a finite cylinder (with finite height) turn out to involve once again automorphic functions on discrete translation groups of the Vahlen group. This reflects the property that finite cylinders arise from infinite cylinders by applying periodizations in one and more directions.

2. Reproducing kernels for the unit ball for polynomial Dirac equations

2.1. Existence theorems. Let p be an arbitrary positive integer and $\alpha_0, \dots, \alpha_p$ arbitrary complex numbers, $\alpha_p \neq 0$; let us write $P(\mathbf{D})f := (\sum_{i=0}^p \alpha_i \mathbf{D}^i)f$, let $\Omega \subset \mathbb{R}^k$ be an arbitrary domain, and write

$$M_{P(\mathbf{D})}(\Omega) := \{f : \Omega \rightarrow Cl_{0k} : P(\mathbf{D})f = 0\}.$$

We start by showing the existence of the reproducing kernel function for the Hilbert space

$$B_{P(\mathbf{D})}^2(\Omega) := M_{P(\mathbf{D})}(\Omega) \cap L^2(\Omega).$$

The following proposition provides us with a generalization of the results from [24]:

Proposition 2.1. *Let $p \in \mathbb{N}$ be an arbitrary positive integer and let $\alpha_1, \dots, \alpha_p$ be arbitrary complex numbers, $\alpha_p \neq 0$. Let $\Omega \subset \mathbb{R}^k$ be a domain. Then the set of functions that are square-integrable over Ω and satisfy in the interior the equation $P(\mathbf{D})f := (\sum_{i=0}^p \alpha_i \mathbf{D}^i)f = 0$, satisfies the Bergman condition.*

Proof. Suppose $\Omega \subset \mathbb{R}^k$ is a domain. From [28] it follows that any f satisfying $P(\mathbf{D})f = 0$ in such an Ω belongs to $C^\infty(\Omega)$. Let $\mathbf{z} \in \Omega$. Let $0 < \rho < R$ be such that the closed k -dimensional ball centered around \mathbf{z} with radius R , denote it by $B_k(\mathbf{z}, R)$, is completely contained in Ω . Next, we take a C^∞ -function ϕ that satisfies $\phi \equiv 1$ in the open ball $B_k(\mathbf{z}, \rho)$ and $\phi \equiv 0$ in $\Omega \setminus B(\mathbf{z}, R)$.

According to [28], the operator $P(\mathbf{D})$ possesses a fundamental solution q in \mathbb{R}^k , which satisfies $qP(\mathbf{D}) = \delta$, where δ is the Dirac distribution. In view of the compact support of ϕf , we have that

$$\phi f = \delta * \phi f = qP(\mathbf{D}) * \phi f = q * (P(\mathbf{D})(\phi f)),$$

so that

$$(\phi f)(\mathbf{z}) = \int_{\mathbb{R}^k} q(\mathbf{z} - \mathbf{y})(P(\mathbf{D})(\phi f))(\mathbf{y}) dV_{\mathbf{y}},$$

which simplifies to

$$\int_{B_k(\mathbf{z}, R)} q(\mathbf{z} - \mathbf{y})(P(\mathbf{D})(\phi f))(\mathbf{y}) dV_{\mathbf{y}}$$

since $\phi \equiv 0$ in $\mathbb{R}^k \setminus B_k(\mathbf{z}, R)$. In view of $P(\mathbf{D})f = 0$ and $\phi \equiv 1$ in $B_k(\mathbf{z}, \rho)$, we may further conclude that

$$(\phi f)(\mathbf{z}) = \int_{B_k(\mathbf{z}, R) \setminus B_k(\mathbf{z}, \rho)} q(\mathbf{z} - \mathbf{y})(P(\mathbf{D})(\phi f))(\mathbf{y}) dV_{\mathbf{y}}. \tag{2}$$

Now introduce the following decomposition:

$$P(\mathbf{D})(\phi f) = (P(\mathbf{D})f)\phi + \mathcal{L},$$

where \mathcal{L} is a differential operator containing all terms in which ϕ is derived at least once. It has the form

$$\mathcal{L}[\phi, f] := \sum_{\alpha, \beta: |\alpha| \geq 1} C_{\alpha, \beta} \frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi \frac{\partial^{|\beta|}}{\partial \mathbf{y}^\beta} f,$$

where $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ are multi-indices and the $C_{\alpha, \beta}$ are Clifford constants. Equation (2) therefore has the form

$$f(\mathbf{z}) = \sum_{\alpha, \beta: |\alpha| \geq 1} \int_{B_k(\mathbf{z}, R) \setminus B_k(\mathbf{z}, \rho)} q(\mathbf{z} - \mathbf{y}) C_{\alpha, \beta} \left(\frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi \right) \left(\frac{\partial^{|\beta|}}{\partial \mathbf{y}^\beta} f \right) dV_{\mathbf{y}}.$$

We now apply successively Stokes' theorem on the integrals: after applying Stokes' theorem once with respect to y_j ($j = 1, \dots, k$), we obtain

$$\begin{aligned} & \int_{B_k(\mathbf{z}, R) \setminus B_k(\mathbf{z}, \rho)} q(\mathbf{z} - \mathbf{y}) C_{\alpha, \beta} \left(\frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi \right) \left(\frac{\partial^{|\beta|}}{\partial \mathbf{y}^\beta} f \right) dV_{\mathbf{y}} \\ &= \int_{\partial B_k(\mathbf{z}, R) \setminus \partial B_k(\mathbf{z}, \rho)} q(\mathbf{z} - \mathbf{y}) \langle e_j, n(\mathbf{y}) \rangle C_{\alpha, \beta} \left(\frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi \right) \frac{\partial^{|\beta|-1}}{\partial \mathbf{y}^{\beta-\tau(j)}} f(\mathbf{y}) dS_{\mathbf{y}} \\ & \quad - \int_{B_k(\mathbf{z}, R) \setminus B_k(\mathbf{z}, \rho)} \frac{\partial}{\partial y_j} \left(q(\mathbf{z} - \mathbf{y}) C_{\alpha, \beta} \frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi \right) \frac{\partial^{|\beta|-1}}{\partial \mathbf{y}^{\beta-\tau(j)}} f(\mathbf{y}) dV_{\mathbf{y}}, \end{aligned}$$

where $\tau(j)$ stands for the multi-index that has the entry 1 at the j -th position and the entry 0 at all other ones, and $n(\mathbf{y})$ stands for the outward oriented unit normal vector field at \mathbf{y} .

Since $\frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi|_{\partial B_k(\mathbf{z}, R)} \equiv 0$ and $\frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi|_{\partial B_k(\mathbf{z}, \rho)} \equiv 0$ for all partial derivatives, the boundary integral vanishes, so that we obtain after applying Stokes' theorem repeatedly, that

$$f(\mathbf{z}) = \sum_{\alpha, \beta} (-1)^{|\beta|} \int_{B_k(\mathbf{z}, R) \setminus B_k(\mathbf{z}, \rho)} \frac{\partial^{|\beta|}}{\partial \mathbf{y}^\beta} \left(q(\mathbf{z} - \mathbf{y}) C_{\alpha, \beta} \frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi \right) f(\mathbf{y}) dV_{\mathbf{y}}. \quad (3)$$

Putting $g(\mathbf{y}) := \sum_{\alpha, \beta} (-1)^{|\beta|} \frac{\partial^{|\beta|}}{\partial \mathbf{y}^\beta} \left(q(\mathbf{z} - \mathbf{y}) C_{\alpha, \beta} \frac{\partial^{|\alpha|}}{\partial \mathbf{y}^\alpha} \phi \right)$, (3) can be rewritten in the form

$$f(\mathbf{z}) = \int_{B_k(\mathbf{z}, R) \setminus B_k(\mathbf{z}, \rho)} g(\mathbf{y}) f(\mathbf{y}) dV_{\mathbf{y}}.$$

Therefore, we finally get, by applying the Cauchy-Schwarz inequality, that

$$\|f(\mathbf{z})\| = |\langle \bar{g}^\sharp, f \rangle_{L^2}| \leq 2^{\frac{k}{2}} \|g\|_{L^2} \|f\|_{L^2} \leq C_\Omega \|f\|_{L^2},$$

which is the Bergman condition. ■

A natural question arising in this context is whether a similar result can be established in the context of Hardy spaces. As we shall see next, an analogy can only be established for the cases $p = 1, 2$. The basis for all that follows is the following local representation theorem for eigenfunctions of the Dirac operator with a nonzero complex eigenvalue (see [28, Theorem 1.1]).

Lemma 2.2 (Local Representation Theorem). *Let f be a Cl_{0k} -valued function that satisfies in the k -dimensional open ball $B_k(0, R)$ the differential equation $(\mathbf{D} - \lambda)f(\mathbf{z}) = 0$ for a complex parameter $\lambda \in \mathbf{C} \setminus \{0\}$. Then there exists a sequence of spherical monogenics of total degree $n = 0, 1, 2, \dots$, say $P_n(\mathbf{z})$, such that in each open ball $B_k(0, r)$ with $0 < r < R$*

$$f(\mathbf{z}) = \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \left(J_{n+\frac{k}{2}-1}(\lambda\|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda\|\mathbf{z}\|) \right) P_n(\mathbf{z}).$$

Here, $J_{n+\frac{k}{2}}$ and $J_{n+\frac{k}{2}-1}$ denote the usual Bessel functions of the first kind with complex argument $\lambda\|\mathbf{z}\|$ and parameter $\nu = n + \frac{k}{2} - 1$ or $\nu = n + \frac{k}{2}$, respectively. For the proof of Lemma 2.2, see for instance [28].

For the sake of readability, we introduce the notation $S_n(\mathbf{z}, \mathbf{w})$ for the Szegő kernel for $\mathbf{D}_{\mathbf{z}}$ -monogenic homogeneous polynomials of total degree n in the k -dimensional unit ball $B_k(0, 1)$, which equals

$$\begin{aligned} S_n(\mathbf{z}, \mathbf{w}) &= \frac{1}{A_k} \left[\frac{1 + \mathbf{z}\mathbf{w}}{\|1 + \mathbf{z}\mathbf{w}\|^k} \right]_{z^n} \\ &= \frac{(-1)^n}{A_k} \sum_{m=0}^n \binom{\frac{k}{2} - 2 + m}{m} \binom{\frac{k}{2} - 1 + (n - m)}{n - m} (\mathbf{z}\mathbf{w})^m (\mathbf{w}\mathbf{z})^{n-m}, \end{aligned}$$

where $A_k = 2\pi^{\frac{k}{2}}/\Gamma(\frac{k}{2})$ denotes the ‘surface area’ of the unit ball in \mathbb{R}^k .

Now we prove

Proposition 2.3. *Let $\lambda \in \mathbf{C} \setminus \{0\}$ and let*

$$\begin{aligned} &\overline{M}_\lambda(B_k(0, 1)) \\ &:= \{f : B_k(0, 1) \rightarrow Cl_{0k} \mid f \in C(\overline{B}_k(0, 1)), (\mathbf{D} - \lambda)f = 0 \in B_k(0, 1)\}. \end{aligned}$$

Then the associated Hardy space

$$H_\lambda^2(B_k(0, 1)) := \text{cl} \{L^2(\partial B_k(0, 1)) \cap \overline{M}_\lambda(B_k(0, 1))\},$$

where cl stands for closure, satisfies the Bergman condition and therefore has a reproducing kernel function.

Proof. Let $f \in H_\lambda^2(B_k(0, 1))$. Since $f \in \overline{M}_\lambda(B_k(0, 1))$, there exist spherical monogenic functions of degree $n = 0, 1, 2, \dots$, say $(P_n(\mathbf{z}))_{n \in \mathbf{N}_0}$, such that

$$f(\mathbf{z}) = \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \frac{J_{n+\frac{k}{2}-1}(\lambda\|\mathbf{z}\|) + \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda\|\mathbf{z}\|)}{\sqrt{|J_{n+\frac{k}{2}-1}(\lambda)|^2 + |J_{n+\frac{k}{2}}(\lambda)|^2}} P_n(\mathbf{z}).$$

Next, we decompose the polynomials $P_n(\mathbf{z})$ in the following way:

$$P_n(\mathbf{z}) = a_n \tilde{P}_n(\mathbf{z}),$$

where $\|\tilde{P}_n(\mathbf{z})\|_{L^2(\partial B_k(0,1))} = 1$ and where the a_n are non-negative real numbers. In this notation, we have

$$\|f\|_{L^2(\partial B_k(0,1))} = \sum_{n=0}^{+\infty} a_n^2.$$

Next, let us estimate the Clifford norm of $f(\mathbf{z})$ at an arbitrary point $\mathbf{z} \in B_k(0, 1)$: clearly,

$$\|f(\mathbf{z})\| \leq \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \left\| \frac{J_{n+\frac{k}{2}-1}(\lambda\|\mathbf{z}\|) + \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda\|\mathbf{z}\|)}{\sqrt{|J_{n+\frac{k}{2}-1}(\lambda)|^2 + |J_{n+\frac{k}{2}}(\lambda)|^2}} \right\| a_n \|\tilde{P}_n(\mathbf{z})\|. \quad (4)$$

Now, let us first estimate $\|\tilde{P}_n(\mathbf{z})\|$: we know that

$$P_n(\mathbf{z}) = \int_{\partial B_k(0,1)} S_n(\mathbf{z}, \mathbf{w}) \tilde{P}_n(\mathbf{z}), dS_{\mathbf{w}},$$

from which we infer that

$$\|\tilde{P}_n(\mathbf{z})\| \leq \|S_n(\mathbf{z}, \mathbf{w})\|_{L^2(\partial B_k(0,1))} \underbrace{\|\tilde{P}_n(\mathbf{w})\|_{L^2(\partial B_k(0,1))}}_{=1}.$$

From

$$\int_{\partial B_k(0,1)} S_n(\mathbf{z}, \mathbf{w}) S_n(\mathbf{w}, \mathbf{z}) dS_{\mathbf{w}} = S_n(\mathbf{z}, \mathbf{z})$$

it follows that $\|S_n(\mathbf{z}, \mathbf{w})\|_{L^2(\partial B_k(0,1))} = \sqrt{S_n(\mathbf{z}, \mathbf{z})}$, and from

$$S(\mathbf{z}, \mathbf{z}) = \frac{1 - \|\mathbf{z}\|^2}{(1 - \|\mathbf{z}\|^2)^{k+1}} = \frac{1}{(1 - \|\mathbf{z}\|^2)^k} = \sum_{n=0}^{+\infty} \frac{(k+n)!}{k!n!} \|\mathbf{z}\|^{2n},$$

that

$$\|S_n(\mathbf{z}, \mathbf{w})\|_{L^2(\partial B_k(0,1))} = \sqrt{\frac{(k+n)!}{k!n!}} \|\mathbf{z}\|^n,$$

from which we now deduce directly that

$$\|\tilde{P}_n(\mathbf{z})\| \leq \sqrt{\frac{(k+n)!}{k!n!}} \|\mathbf{z}\|^n. \tag{5}$$

In order to proceed estimating the expression (4), we use the Carlini formula, which describes the asymptotic behavior of J_ν for large indices ν . According to [21], for large ν one has

$$J_\nu(z) = \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ze}{2\nu}\right)^\nu \left(1 + \mathcal{O}\left(\frac{1}{\nu}\right)\right), \quad \nu \rightarrow \infty, \tag{6}$$

so that

$$J_\nu(\lambda\|\mathbf{z}\|) = \frac{\lambda^\nu}{\sqrt{2\pi\nu}} \left(\frac{e}{2\nu}\right)^\nu \|\mathbf{z}\|^\nu \left(1 + \mathcal{O}\left(\frac{1}{\nu}\right)\right).$$

From this formula we obtain that the asymptotic behavior for large n of

$$G_{1,n}(\lambda; \mathbf{z}) := \|\mathbf{z}\|^{1-n-\frac{k}{2}} \frac{|J_{n+\frac{k}{2}-1}(\lambda\|\mathbf{z}\|)|}{\sqrt{|J_{n+\frac{k}{2}-1}(\lambda)|^2 + |J_{n+\frac{k}{2}}(\lambda)|^2}}$$

is

$$\frac{1}{\sqrt{1 + |\lambda|^2 \left[\left(1 - \frac{1}{n+\frac{k}{2}}\right)^{n+\frac{k}{2}} \right]^2 \left(\frac{e}{2(n+\frac{k}{2})}\right)^2}},$$

so that

$$G_{1,n}(\lambda; \mathbf{z}) = \|\mathbf{z}\|^{1-n-\frac{k}{2}} \frac{|J_{n+\frac{k}{2}-1}(\lambda\|\mathbf{z}\|)|}{\sqrt{|J_{n+\frac{k}{2}-1}(\lambda)|^2 + |J_{n+\frac{k}{2}}(\lambda)|^2}} \sim \frac{1}{\sqrt{1 + |\lambda|^2 \frac{1}{4n^2}}}. \tag{7}$$

In the same way, we obtain the asymptotic behavior for large n of the expression

$$G_{2,n}(\lambda; \mathbf{z}) = \|\mathbf{z}\|^{-n-\frac{k}{2}} \frac{|J_{n+\frac{k}{2}}(\lambda\|\mathbf{z}\|)|}{\sqrt{|J_{n+\frac{k}{2}-1}(\lambda)|^2 + |J_{n+\frac{k}{2}}(\lambda)|^2}} \sim \frac{1}{\sqrt{|\lambda|^2 + 1}}. \tag{8}$$

Inserting (5), (7) and (8) into (4) and applying the Cauchy-Schwarz inequality

then leads to the following estimate for $\|\mathbf{z}\| < 1$:

$$\begin{aligned} \|f(\mathbf{z})\| &\leq \sqrt{\sum_{n=0}^{+\infty} a_n^2} \sqrt{\sum_{n=0}^{+\infty} \|G_{1,n}(\lambda; \mathbf{z}) \tilde{P}_n(\mathbf{z})\|^2} \\ &\quad + \sqrt{\sum_{n=0}^{+\infty} a_n^2} \sqrt{\sum_{n=0}^{+\infty} \|G_{2,n}(\lambda; \mathbf{z}) \tilde{P}_n(\mathbf{z})\|^2} \\ &\leq 2^{\frac{k}{2}} \left(\sqrt{\sum_{n=0}^{+\infty} a_n^2} \sqrt{\sum_{n=0}^{+\infty} \|G_{1,n}(\lambda; \mathbf{z})\|^2 \frac{(k+n)!}{k!n!} \|\mathbf{z}\|^{2n}} \right. \\ &\quad \left. + \sqrt{\sum_{n=0}^{+\infty} a_n^2} \sqrt{\sum_{n=0}^{+\infty} \|G_{2,n}(\lambda; \mathbf{z})\|^2 \frac{(k+n)!}{k!n!} \|\mathbf{z}\|^{2n}} \right) \\ &\leq C \sqrt{\sum_{n=0}^{+\infty} a_n^2} = C \|f\|_{L^2(\partial B_k(0,1))}, \end{aligned}$$

which is the Bergman condition. ■

Next we prove

Proposition 2.4. *Let $\lambda_1, \lambda_2 \in \mathbf{C} \setminus \{0\}$ with $\lambda_1 \neq \lambda_2$, and let*

$$\begin{aligned} &\overline{M}_{\lambda_1, \lambda_2}(B_k(0, 1)) \\ &:= \left\{ f : B_k(0, 1) \rightarrow Cl_{0k} \mid \begin{array}{l} f \in C(\overline{B}_k(0, 1)), \\ (\mathbf{D} - \lambda_1)(\mathbf{D} - \lambda_2)f(z) = 0 \ \forall z \in B_k(0, 1) \end{array} \right\}. \end{aligned}$$

Then the associated Hardy space

$$H^2_{\lambda_1, \lambda_2}(B_k(0, 1)) := \text{cl} \{ L^2(\partial B_k(0, 1)) \cap \overline{M}_{\lambda_1, \lambda_2}(B_k(0, 1)) \}$$

satisfies the Bergman condition and has a reproducing kernel function.

Proof. This requires a more sophisticated argumentation than the proof of the previous proposition.

Since f is a solution to $(\mathbf{D} - \lambda_1)(\mathbf{D} - \lambda_2)f = 0$, there are two families of spherical monogenic functions $(P_{1,n}(\mathbf{z}))_{n \in \mathbf{N}}$ and $(P_{2,n}(\mathbf{z}))_{n \in \mathbf{N}}$ such that, when $\|\mathbf{z}\| < 1$ (see [18, 19, 27]),

$$\begin{aligned} f(\mathbf{z}) &= \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \left(J_{n+\frac{k}{2}-1}(\lambda_1 \|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda_1 \|\mathbf{z}\|) \right) P_{1,n}(\mathbf{z}) \\ &\quad + \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \left(J_{n+\frac{k}{2}-1}(\lambda_2 \|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda_2 \|\mathbf{z}\|) \right) P_{2,n}(\mathbf{z}). \end{aligned} \tag{9}$$

On the boundary $\|\mathbf{z}\| = 1$, we have the following condition:

$$f(\mathbf{z}) = \sum_{n=0}^{+\infty} (P_n(\mathbf{z}) + Q_n(\mathbf{z})) \tag{10}$$

with inner spherical monogenics P_n of degree n and outer spherical monogenics Q_n of degree n . It is well known that, for $\|\mathbf{z}\| = 1$, we can write the outer spherical monogenic Q_n as

$$Q_n(\mathbf{z}) = \mathbf{z}P'_n(\mathbf{z}),$$

where P'_n is an inner spherical monogenic polynomial of degree n . Writing $P_n(\mathbf{z}) = a_n\tilde{P}_n(\mathbf{z})$ and $P'_n(\mathbf{z}) = b_n\tilde{P}'_n(\mathbf{z})$, where $\|\tilde{P}_n\|_{L^2(\partial B_k(0,1))} = \|\tilde{P}'_n\|_{L^2(\partial B_k(0,1))} = 1$ and where the a_n and b_n are non-negative real numbers, we can re-express (10) as

$$f(\mathbf{z}) = \sum_{n=0}^{+\infty} (a_n\tilde{P}_n(\mathbf{z}) + \mathbf{z}b_n\tilde{P}'_n(\mathbf{z})).$$

Note that in this notation, $\|f\|_{L^2(\partial B_k(0,1))} = \sum_{n=0}^{+\infty} (a_n^2 + b_n^2)$.

Since $f \in H^2_{\lambda_1, \lambda_2}(\partial B_k(0,1))$ it follows from the continuity of f up to the boundary that the following equation must be satisfied on $\|\mathbf{z}\| = 1$:

$$\begin{aligned} & \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \left(J_{n+\frac{k}{2}-1}(\lambda_1\|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda_1\|\mathbf{z}\|) \right) P_{1,n}(\mathbf{z}) \\ & + \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \left(J_{n+\frac{k}{2}-1}(\lambda_2\|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda_2\|\mathbf{z}\|) \right) P_{2,n}(\mathbf{z}) \\ & \stackrel{!}{=} \sum_{n=0}^{+\infty} a_n\tilde{P}_n(\mathbf{z}) + \mathbf{z}b_n\tilde{P}'_n(\mathbf{z}). \end{aligned}$$

A comparison of the inner and outer spherical monogenic parts produces the following 2×2 system of linear equations in $P_{1,n}$ and $P_{2,n}$:

$$\begin{aligned} J_{n+\frac{k}{2}-1}(\lambda_1)P_{1,n}(\mathbf{z}) + J_{n+\frac{k}{2}}(\lambda_2)P_{2,n}(\mathbf{z}) &= a_n\tilde{P}_n(\mathbf{z}) \\ -J_{n+\frac{k}{2}}(\lambda_1)P_{1,n}(\mathbf{z}) - J_{n+\frac{k}{2}-1}(\lambda_2)P_{2,n}(\mathbf{z}) &= b_n\tilde{P}'_n(\mathbf{z}). \end{aligned}$$

Having solved this system, one obtains

$$P_{1,n}(\mathbf{z}) = \frac{a_n\tilde{P}_n(\mathbf{z})J_{n+\frac{k}{2}}(\lambda_2) - b_n\tilde{P}'_n(\mathbf{z})J_{n+\frac{k}{2}-1}(\lambda_2)}{J_{n+\frac{k}{2}-1}(\lambda_1)J_{n+\frac{k}{2}}(\lambda_2) - J_{n+\frac{k}{2}-1}(\lambda_2)J_{n+\frac{k}{2}}(\lambda_1)} \tag{11}$$

$$P_{2,n}(\mathbf{z}) = \frac{a_n\tilde{P}_n(\mathbf{z})J_{n+\frac{k}{2}}(\lambda_1) - b_n\tilde{P}'_n(\mathbf{z})J_{n+\frac{k}{2}-1}(\lambda_1)}{J_{n+\frac{k}{2}-1}(\lambda_2)J_{n+\frac{k}{2}}(\lambda_1) - J_{n+\frac{k}{2}-1}(\lambda_1)J_{n+\frac{k}{2}}(\lambda_2)}. \tag{12}$$

Note that both equations (11) and (12) remain valid in the unit ball $\|\mathbf{z}\| < 1$, since the functions $P_{1,n}$ and $P_{2,n}$ are spherical monogenics.

This allows us to insert (11) and (12) into (9), which yields a decomposition of (9) into eight similar parts:

$$\begin{aligned}
 f(\mathbf{z}) = & \sum_{n=0}^{+\infty} a_n \frac{\|\mathbf{z}\|^{1-n-\frac{k}{2}} J_{n+\frac{k}{2}-1}(\lambda_1 \|\mathbf{z}\|) J_{n+\frac{k}{2}}(\lambda_2) \tilde{P}_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2) - J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1)} \\
 & - b_n \frac{\|\mathbf{z}\|^{1-n-\frac{k}{2}} J_{n+\frac{k}{2}-1}(\lambda_1 \|\mathbf{z}\|) J_{n+\frac{k}{2}-1}(\lambda_2) \tilde{P}'_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2) - J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1)} \\
 & - a_n \frac{\mathbf{z}}{\|\mathbf{z}\|^{n+\frac{k}{2}}} \frac{J_{n+\frac{k}{2}}(\lambda_1 \|\mathbf{z}\|) J_{n+\frac{k}{2}}(\lambda_2) \tilde{P}_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2) - J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1)} \\
 & + b_n \frac{\mathbf{z}}{\|\mathbf{z}\|^{n+\frac{k}{2}}} \frac{J_{n+\frac{k}{2}}(\lambda_1 \|\mathbf{z}\|) J_{n+\frac{k}{2}-1}(\lambda_2) \tilde{P}'_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2) - J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1)} \\
 & + a_n \frac{\|\mathbf{z}\|^{1-n-\frac{k}{2}} J_{n+\frac{k}{2}-1}(\lambda_2 \|\mathbf{z}\|) J_{n+\frac{k}{2}}(\lambda_1) \tilde{P}_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1) - J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2)} \\
 & - b_n \frac{\|\mathbf{z}\|^{1-n-\frac{k}{2}} J_{n+\frac{k}{2}-1}(\lambda_2 \|\mathbf{z}\|) J_{n+\frac{k}{2}-1}(\lambda_1) \tilde{P}'_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1) - J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2)} \\
 & - a_n \frac{\mathbf{z}}{\|\mathbf{z}\|^{n+\frac{k}{2}}} \frac{J_{n+\frac{k}{2}}(\lambda_2 \|\mathbf{z}\|) J_{n+\frac{k}{2}}(\lambda_1) \tilde{P}_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1) - J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2)} \\
 & + b_n \frac{\mathbf{z}}{\|\mathbf{z}\|^{n+\frac{k}{2}}} \frac{J_{n+\frac{k}{2}}(\lambda_2 \|\mathbf{z}\|) J_{n+\frac{k}{2}-1}(\lambda_1) \tilde{P}'_n(\mathbf{z})}{J_{n+\frac{k}{2}-1}(\lambda_2) J_{n+\frac{k}{2}}(\lambda_1) - J_{n+\frac{k}{2}-1}(\lambda_1) J_{n+\frac{k}{2}}(\lambda_2)}.
 \end{aligned} \tag{13}$$

This sum has therefore the form

$$f(\mathbf{z}) = \sum_{n=0}^{+\infty} (a_n G_{1,n}(\lambda_1, \lambda_2; \mathbf{z}) + \dots + b_n G_{8,n}(\lambda_1, \lambda_2; \mathbf{z})),$$

where the $G_{j,n}$ ($j = 1, \dots, 8$) denote the expressions that occur in (13) on the right-hand side of the a_n or b_n , respectively.

Now we need to show that the coefficients $G_{j,n}$ $j = 1, \dots, 8$, are all l^2 -summable. This can be done by again applying the asymptotic Carlini formula, eq. (6), from which we readily deduce that, writing

$$g_n(\mathbf{z}) = \frac{1}{\lambda_2 - \lambda_1} \sqrt{\frac{(k+n)!}{k!n!}} \|\mathbf{z}\|^n,$$

$$G_{1,n} \sim \lambda_2 g_n(\mathbf{z}), G_{2,n} \sim 2n g_n(\mathbf{z}), G_{3,n} \sim -\frac{\mathbf{z}\lambda_1\lambda_2}{2n} g_n(\mathbf{z}), G_{4,n} \sim \mathbf{z}\lambda_1 g_n(\mathbf{z})$$

$$G_{5,n} \sim -\lambda_1 g_n(\mathbf{z}), G_{6,n} \sim -2n g_n(\mathbf{z}), G_{7,n} \sim \frac{\mathbf{z}\lambda_1\lambda_2}{2n} g_n(\mathbf{z}), G_{8,n} \sim -\mathbf{z}\lambda_1 g_n(\mathbf{z}).$$

As one verifies directly, all these expressions are l^2 -summable for $\|\mathbf{z}\| < 1$. Applying the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} \|f(\mathbf{z})\| &\leq L \sqrt{\sum_{n=0}^{+\infty} a_n^2} \sqrt{\sum_{n=0}^{+\infty} \|G_{1,n}\|^2} + \dots + \sqrt{\sum_{n=0}^{+\infty} b_n^2} \sqrt{\sum_{n=0}^{+\infty} \|G_{8,n}\|^2} \\ &\leq L \sqrt{\sum_{n=0}^{+\infty} (a_n^2 + b_n^2)} \sum_{j=1}^8 \sqrt{\sum_{n=0}^{+\infty} \|G_{j,n}\|^2} \\ &\leq C \sqrt{\sum_{n=0}^{+\infty} (a_n^2 + b_n^2)} = C \|f\|_{L^2(\partial B_k(0,1))}. \quad \blacksquare \end{aligned}$$

A crucial difference with Bergman spaces is stated in the following

Proposition 2.5. *The space of functions that satisfy in the interior of the unit ball the equation $[\prod_{j=1}^p (\mathbf{D}_{\mathbf{z}} - \lambda_j)]f = 0$ and that have L^2 boundary values, does not have a reproducing kernel function whenever $p > 2$.*

Proof. To show that no Szegő kernel exists in this case, consider the sequence of the following p functions ($p > 2$):

$$f_{0,\lambda_j}(\mathbf{w}) = J_{\frac{k}{2}-1}(\lambda_j r) + \frac{\mathbf{w}}{\|\mathbf{w}\|} J_{\frac{k}{2}}(\lambda_j r), \quad j = 1, 2, \dots, p.$$

The set of functions $f_{0,\lambda_j}(\mathbf{w})$ is linearly independent on the unit ball, but its restriction to the unit sphere ($r = 1$) is linearly dependent whenever $p > 2$ (each $f_{0,\lambda_j}(\mathbf{w})$ being a linear combination there of the functions 1 and \mathbf{w}), so that there exists a non-trivial linear combination with scalar-valued complex coefficients such that $(\sum_{j=1}^p \alpha_j f_{0,\lambda_j}(\mathbf{w}))_{\|\mathbf{w}\|=1} = 0$. The function $g := \sum_{j=1}^p \alpha_j f_{0,\lambda_j}$ satisfies $[\prod_{j=1}^p (\mathbf{D}_{\mathbf{z}} - \lambda_j)]g = 0$ by construction. Suppose now that a Szegő kernel function for this operator on the unit sphere, say $S(\mathbf{z}, \mathbf{w})$, would exist. This function then needs to act on the unit sphere, and we have that

$$g(\mathbf{z}) = \int_{\partial B_k(0,1)} S(\mathbf{z}, \mathbf{w}) g(\mathbf{w}) dS_{\mathbf{w}}.$$

However, $g|_{\partial B_k(0,1)} \equiv 0$. Therefore $g \equiv 0$ in the whole unit ball. This is a contradiction, since $g \not\equiv 0$ due to the fact that the elements λ_j ($j = 1, \dots, p$) are pairwise different. \blacksquare

2.2. Explicit formulas. Next, we give explicit representation formulas for the Bergman kernel $B_{P(\mathbf{D})}^2(B_k(0, 1))$ for arbitrary complex polynomials P that split in pairwise different linear factors and for the Szegő kernel of the spaces $H_{\lambda_1}^2(B_k(0, 1))$ and $H_{\lambda_1, \dots, \lambda_2}^2(B_k(0, 1))$ where $\lambda_1 \neq \lambda_2$. What follows completes and extends the result from [3], in which an explicit representation formula for the Bergman kernel of the space $B_{\mathbf{D}-\lambda}^2(B_k(0, 1))$ was developed for real $\lambda \neq 0$.

In what follows, we use the abbreviation

$$f_{n,\lambda}(\mathbf{z}) = \|\mathbf{z}\|^{1-n-\frac{k}{2}} \left(J_{n+\frac{k}{2}-1}(\lambda\|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda\|\mathbf{z}\|) \right). \tag{14}$$

We prove

Proposition 2.6. *Let $\lambda_1, \dots, \lambda_p \in \mathbf{C} \setminus \{0\}$ be pairwise different numbers. Then the Bergman reproducing kernel function for the solutions to $(\mathbf{D}_{\mathbf{z}} - \lambda_1) \cdots (\mathbf{D}_{\mathbf{z}} - \lambda_p)f(\mathbf{z}) = 0$ in the unit ball is given by*

$$B_{\lambda_1, \dots, \lambda_p}(\mathbf{z}, \mathbf{w}) = \sum_{n=0}^{+\infty} \begin{pmatrix} f_{n,\lambda_1}(\mathbf{z}) \\ f_{n,\lambda_2}(\mathbf{z}) \\ \vdots \\ f_{n,\lambda_p}(\mathbf{z}) \end{pmatrix}^T \left[\mathcal{M}_{\lambda_1, \dots, \lambda_p} \right]^{-1} \begin{pmatrix} \bar{f}_{n,\lambda_1^\#}(\mathbf{w}) \\ \bar{f}_{n,\lambda_2^\#}(\mathbf{w}) \\ \vdots \\ \bar{f}_{n,\lambda_p^\#}(\mathbf{w}) \end{pmatrix}, \tag{15}$$

where \cdot^T indicates matrix transposition, $f_{n,\lambda}$ stands for the expression defined in (14) and $\mathcal{M}_{\lambda_1, \dots, \lambda_p}$ stands for the $p \times p$ matrix whose entries are given by

$$m_{ij} = \int_0^1 r^{2n+k-1} \text{Sc} \left\{ \bar{f}_{n,\lambda_i^\#}(\mathbf{z}) f_{n,\lambda_j}(\mathbf{z}) \right\}_{\|\mathbf{z}\|=r} dr,$$

i.e., explicitly,

$$m_{ij} = \begin{cases} \frac{1}{\lambda_i - \lambda_j^\#} \left(J_{n+\frac{k}{2}}(\lambda_i) J_{n+\frac{k}{2}-1}(\lambda_j^\#) - J_{n+\frac{k}{2}}(\lambda_j^\#) J_{n+\frac{k}{2}-1}(\lambda_i) \right), & (\lambda_i \neq \lambda_j^\#) \\ J_{n+\frac{k}{2}-1}^2(\lambda_i) J_{n+\frac{k}{2}}^2(\lambda_i) - \frac{(2n+k-1)}{\lambda} J_{n+\frac{k}{2}-1}(\lambda_i) J_{n+\frac{k}{2}}(\lambda_i), & (\lambda_i = \lambda_j^\#). \end{cases} \tag{16}$$

Proof. It suffices to show the reproduction of all $f_{n',\lambda_{q'}}(\mathbf{w}) S_{n'}(\mathbf{w}, \mathbf{v})$. For this, consider the expression

$$\int_{\mathbf{w} \in B_k(0,1)} S_n(\mathbf{z}, \mathbf{w}) \bar{f}_{n,\lambda_q^\#}(\mathbf{w}) f_{n',\lambda_{q'}}(\mathbf{w}) S_{n'}(\mathbf{w}, \mathbf{v}) dV_{\mathbf{w}} \tag{17}$$

in polar coordinates

$$\int_0^1 \int_{\mathbf{w} \in \partial B_k(0,1)} S_n(\mathbf{z}, r\mathbf{w}) \bar{f}_{n,\lambda_q^\#}(r\mathbf{w}) f_{n',\lambda_{q'}}(r\mathbf{w}) S_{n'}(r\mathbf{w}, \mathbf{v}) dS_{\mathbf{w}} r^{k-1} dr.$$

Now decompose the integrand into its scalar and pure vector part:

$$\bar{f}_{n,\lambda_q^\sharp}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w}) = \text{Sc}\{\bar{f}_{n,\lambda_q^\sharp}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w})\} + g(r)\mathbf{w}.$$

Since $\mathbf{w}S_{n'}(r\mathbf{w}, \mathbf{v})$ is an outer spherical monogenic on $\mathbf{w} \in \partial B_k(0, 1)$, it is orthogonal to $S_n(r\mathbf{w}, \mathbf{v})$, and the contribution of $g(r)\mathbf{w}$ to the integral therefore vanishes. We are finally left with the integral

$$\int_0^1 \int_{\mathbf{w} \in \partial B_k(0,1)} \text{Sc}\{\bar{f}_{n,\lambda_q^\sharp}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w})\} S_n(\mathbf{z}, r\mathbf{w})S_{n'}(r\mathbf{w}, \mathbf{v}) dS_{\mathbf{w}}r^{k-1} dr. \quad (18)$$

We observe that

$$\begin{aligned} &\text{Sc}\{\bar{f}_{n,\lambda_q^\sharp}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w})\} \\ &= r^{2-2n-k} \left(J_{n+\frac{k}{2}-1}(\lambda_q^\sharp)J_{n'+\frac{k}{2}-1}(\lambda_{q'}r) + J_{n+\frac{k}{2}}(\lambda_q^\sharp)J_{n'+\frac{k}{2}}(\lambda_{q'}r) \right), \end{aligned}$$

which is clearly independent of $\mathbf{w} \in \partial B_k(0, 1)$. Therefore, the previous integral can be rewritten as

$$\int_0^1 \text{Sc}\{\bar{f}_{n,\lambda_q^\sharp}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w})\} \left(\int_{\mathbf{w} \in \partial B_k(0,1)} S_n(\mathbf{z}, r\mathbf{w})S_{n'}(r\mathbf{w}, \mathbf{v}) dS_{\mathbf{w}} \right) r^{k-1} dr.$$

In view of the reproducing property of $S_n(\mathbf{z}, \mathbf{w})$ and the homogeneity, (18) may in turn be rewritten as

$$\delta_{n,n'}S_n(\mathbf{z}, \mathbf{v}) \int_0^1 \text{Sc}\{\bar{f}_{n,\lambda_q^\sharp}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w})\} r^{2n+k-1} dr,$$

so that we obtain

$$\begin{aligned} &\sum_{n=0}^{+\infty} \delta_{n,n'}S_n(\mathbf{z}, \mathbf{v}) \int_0^1 \text{Sc}\{\bar{f}_{n,\lambda_q^\sharp}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w})\} r^{2n+k-1} dr \\ &= S_{n'}(\mathbf{z}, \mathbf{v}) \int_0^1 \text{Sc}\{\bar{f}_{n',\lambda_{q'}}(r\mathbf{w})f_{n',\lambda_{q'}}(r\mathbf{w})\} r^{2n'+k-1} dr, \end{aligned}$$

which is just

$$S_{n'}(\mathbf{z}, \mathbf{v}) \int_0^1 \left(J_{n'+\frac{k}{2}-1}(\lambda_{q'}^\sharp)r J_{n'+\frac{k}{2}-1}(\lambda_{q'}r) + J_{n'+\frac{k}{2}}(\lambda_{q'}^\sharp)r J_{n'+\frac{k}{2}}(\lambda_{q'}r) \right) r dr.$$

The value of the integral is explicitly known, see for example [13]: it equals the expression m_{ij} defined in (16), i.e., the (q, q') -component of the matrix $\mathcal{M}_{\lambda_1, \dots, \lambda_p}$. Finally, we are left with proving that the matrix $\mathcal{M}_{\lambda_1, \dots, \lambda_p}$ is invertible. When this is done, the reproducing property readily follows. To proceed in this

direction, let us assume that $\mathcal{M}_{\lambda_1, \dots, \lambda_p}$ were not invertible. This would mean that for some non-trivial linear combination $\sum_{j=1}^p m_{ij} \alpha_j = 0$ is satisfied for all $i = 1, \dots, p$. Let us then consider the expression

$$\mathbf{g} := (f_{n, \lambda_1} \alpha_1 + \dots + f_{n, \lambda_p} \alpha_p) S_n(\mathbf{z}, \mathbf{w}).$$

Take an arbitrary $j \in \{1, \dots, p\}$ and evaluate

$$\langle \mathbf{g}, f_{n, \lambda_j} S_n(\mathbf{z}, \mathbf{w}) \rangle = \underbrace{\left[\sum_{j=1}^p m_{ij} \alpha_j \right]}_{=0} \|S_n(\mathbf{z}, \mathbf{w})\|_{L^2(B_k(0,1))} = 0.$$

We observe that

$$\mathbf{g} \perp f_{n, \lambda_j} S_n(\mathbf{z}, \mathbf{w}) \tag{19}$$

for all $j = 1, \dots, p$, where this orthogonality relation needs to be understood in the sense of the L^2 -inner product arising from the volume integral over the unit ball. Since (19) is satisfied for all $j = 1, \dots, p$, we furthermore infer that $\mathbf{g} \perp (f_{n, \lambda_1} \alpha_1 + \dots + f_{n, \lambda_p} \alpha_p)$, which means $\mathbf{g} \perp \mathbf{g}$, so $\mathbf{g} = 0$. Since $S_n(\mathbf{z}, \mathbf{w}) \neq 0$, we obtain that

$$f_{n, \lambda_1} \alpha_1 + \dots + f_{n, \lambda_p} \alpha_p = 0.$$

However, the set of functions $\{f_{n, \lambda_j}\}_{j=1, \dots, p}$ is linearly independent, since the elements λ_j are pairwise different. Thus, $\alpha_j = 0$ for all $j = 1, \dots, p$, and we have a contradiction, so that we conclude that the matrix $\mathcal{M}_{\lambda_1, \dots, \lambda_p}$ is always invertible. ■

Remarks: In the case $p = 1$ with $\lambda \in \mathbf{C} \setminus \{0\}$, the formula (15) simplifies to

$$\begin{aligned} B_\lambda(\mathbf{z}, \mathbf{w}) = & \sum_{n=0}^{+\infty} \left[\frac{\|\mathbf{z}\|^{1-n-\frac{k}{2}} \|\mathbf{w}\|^{1-n-\frac{k}{2}}}{\int_0^1 \left(|J_{n+\frac{k}{2}-1}(\lambda r)|^2 + |J_{n+\frac{k}{2}}(\lambda r)|^2 \right) r dr} \right. \\ & \times \left(J_{n+\frac{k}{2}-1}(\lambda \|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda \|\mathbf{z}\|) \right) S_n(\mathbf{z}, \mathbf{w}) \\ & \left. \times \left(J_{n+\frac{k}{2}-1}(\lambda^\# \|\mathbf{w}\|) + \frac{\mathbf{w}}{\|\mathbf{w}\|} J_{n+\frac{k}{2}}(\lambda^\# \|\mathbf{w}\|) \right) \right]. \end{aligned} \tag{20}$$

Here, the norm in the denominator of (20) does not arise from the inner product in the Clifford sense, but only from its scalar part. It is the complex Hermitian norm.

In the special case where $\lambda \in \mathbb{R} \setminus \{0\}$, this formula simplifies to the representation formula that was developed in [3] for the special case $p = 1$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Formula (15) provides therefore a canonical generalization.

In the case $p = 2$ with $\lambda_1 = -\lambda_2$, equation (15) offers an explicit formula for the Bergman kernel to the Helmholtz equation $-\Delta f = \lambda^2 f$ with complex eigenvalues λ .

Within the context of Hardy spaces, we obtain for the cases $p = 1, 2$ the following analogous result:

Proposition 2.7. *Let $\lambda \in \mathbf{C} \setminus \{0\}$. The Szegő kernel of the unit ball to $(\mathbf{D}_z - \lambda)f = 0$, $\lambda \neq 0$, is well defined and given explicitly by*

$$\begin{aligned} S_\lambda(\mathbf{z}, \mathbf{w}) &= \sum_{n=0}^{+\infty} \frac{\|\mathbf{z}\|^{1-n-\frac{k}{2}} \|\mathbf{w}\|^{1-n-\frac{k}{2}}}{|J_{n+\frac{k}{2}-1}(\lambda)|^2 + |J_{n+\frac{k}{2}}(\lambda)|^2} \\ &\times \left(J_{n+\frac{k}{2}-1}(\lambda \|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda \|\mathbf{z}\|) \right) \\ &\times S_n(\mathbf{z}, \mathbf{w}) \left(J_{n+\frac{k}{2}-1}(\lambda^\sharp \|\mathbf{w}\|) + \frac{\mathbf{w}}{\|\mathbf{w}\|} J_{n+\frac{k}{2}}(\lambda^\sharp \|\mathbf{w}\|) \right). \end{aligned} \quad (21)$$

Let $\lambda_1, \lambda_2 \in \mathbf{C} \setminus \{0\}$ with $\lambda_1 \neq \lambda_2$. The Szegő kernel of the unit ball for $(\mathbf{D}_z - \lambda_1)(\mathbf{D}_z - \lambda_2)f = 0$ is also well defined and given by

$$S_{\lambda_1, \lambda_2}(\mathbf{z}, \mathbf{w}) = \sum_{n=0}^{+\infty} \left(f_{n, \lambda_1}(\mathbf{z}) f_{n, \lambda_2}(\mathbf{z}) \right) S_n(\mathbf{z}, \mathbf{w}) [\mathcal{T}_{\lambda_1, \lambda_2}]^{-1} \begin{pmatrix} \bar{f}_{n, \lambda_1^\sharp}(\mathbf{w}) \\ f_{n, \lambda_2^\sharp}(\mathbf{w}) \end{pmatrix}, \quad (22)$$

where $f_{n, \lambda}$ stands for the expression defined in (14) and $\mathcal{T}_{\lambda_1, \lambda_2}$ stands for the 2×2 matrix whose entries are given by

$$\begin{aligned} t_{ij} &= \text{Sc} \left\{ \bar{f}_{n, \lambda_i^\sharp}(\mathbf{w}) f_{n, \lambda_j}(\mathbf{w}) \right\} \Big|_{\|\mathbf{w}\|=1} \\ &= J_{n+\frac{k}{2}-1}(\lambda_i^\sharp) J_{n+\frac{k}{2}-1}(\lambda_j) + J_{n+\frac{k}{2}}(\lambda_i^\sharp) J_{n+\frac{k}{2}}(\lambda_j). \end{aligned} \quad (23)$$

Remark. When $\lambda_1 = -\lambda_2$, equation (22) provides an explicit formula for the Szegő kernel to the Helmholtz equation $-\Delta f = \lambda^2 f$.

Proof. Since (21) can be deduced as a special case from the more general equation (22), one only needs to prove (22). This can be done analogously to the proof of Proposition 2.6. Instead of considering the volume integral (17), we only have to perform the integration of

$$S_n(\mathbf{z}, \mathbf{w}) \bar{f}_{n, \lambda_i^\sharp}(\mathbf{w}) f_{n', \lambda_j} S_{n'}(\mathbf{w}, \mathbf{v})$$

over the boundary of the unit ball. Proceeding as in the proof of Proposition 2.6, we obtain that

$$\begin{aligned} \int_{\mathbf{w} \in \partial B(0,1)} S_n(\mathbf{z}, \mathbf{w}) \bar{f}_{n, \lambda_i^\sharp}(\mathbf{w}) f_{n', \lambda_j} S_{n'}(\mathbf{w}, \mathbf{v}) dS_{\mathbf{w}} \\ = \delta_{n, n'} S_n(\mathbf{z}, \mathbf{v}) \text{Sc} \left\{ \bar{f}_{n, \lambda_i^\sharp}(\mathbf{w}) f_{n', \lambda_j}(\mathbf{w}) \right\}, \end{aligned} \quad (24)$$

so that a summation over $n = 0, 1, 2, \dots$ leads to

$$\begin{aligned} \sum_{n=0}^{+\infty} \delta_{n,n'} S_n(\mathbf{z}, \mathbf{v}) \text{Sc}\{\bar{f}_{n,\lambda_i^\#}(\mathbf{w}) f_{n',\lambda_j}(\mathbf{w})\} \\ = S_{n'}(\mathbf{z}, \mathbf{v}) \text{Sc}\{\bar{f}_{n',\lambda_i^\#}(\mathbf{w}) f_{n',\lambda_j}(\mathbf{w})\} \\ = S_{n'}(\mathbf{z}, \mathbf{v}) t_{ij}. \end{aligned}$$

With a similar argumentation as in the proof of Proposition 2.6, we infer that the matrix $\mathcal{T}_{\lambda_1,\lambda_2}$ has rank 2, and is therefore invertible, being a 2×2 matrix. ■

Remark. Let us analyze why this construction does not extend to systems of the form $[\prod_{j=1}^p (\mathbf{D}_z - \lambda_j)]f = 0$ with $p > 2$. If $p > 2$, then the matrix $\mathcal{T}_{\lambda_1,\dots,\lambda_p}$ built with the coefficients t_{ij} is always singular, as follows from the fact that the rank of the matrix $\mathcal{T}_{\lambda_1,\dots,\lambda_p}$ can never exceed 2 because it can always be decomposed as

$$\underbrace{\begin{pmatrix} b_1^\# \\ \vdots \\ b_p^\# \end{pmatrix}}_{=:B} (b_1 \cdots b_p) + \underbrace{\begin{pmatrix} a_1^\# \\ \vdots \\ a_p^\# \end{pmatrix}}_{=:A} (a_1 \cdots a_p),$$

where we put for simplicity $a_i = J_{n+\frac{k}{2}}(\lambda_i)$ and $b_i = J_{n+\frac{k}{2}-1}(\lambda_i)$. Both matrices A and B have rank 1 at most, so their sum $A + B = \mathcal{T}_{\lambda_1,\dots,\lambda_p}$ can have rank 2 at most. Therefore, $\mathcal{T}_{\lambda_1,\dots,\lambda_p}$ is always singular for $p > 2$.

3. Reproducing kernels for cylinders

3.1. Fourier transform of reproducing kernels for $(\mathbf{D} - \lambda)$. From the formulas of the previous section we can now derive, using Fourier transformation methods, explicit representation formulas for the Bergman and the Szegő kernel of monogenic square-integrable functions on the unit infinite cylinder

$$\mathcal{C} = \left\{ z \in \mathcal{A}_{k+1} \mid \sum_{j=1}^k z_j^2 = 1 \right\}.$$

The crucial points are that the Fourier transform of the equation $D_z f = 0$ with respect to the scalar variable z_0 alone reads

$$(\mathbf{D}_z + i\zeta_0)g = 0,$$

where $g = (\mathcal{F}f)(\zeta_0, \mathbf{z})$ stands for the Fourier image of f , and that the Fourier transform is unitary.

Substituting $\lambda = -i\zeta_0$, $\zeta_0 \in \mathbb{R} \setminus \{0\}$, we obtain immediately from Proposition 2.7 that the Szegő kernel for the solutions to $(\mathbf{D}_z + i\zeta_0)g = 0$ reads

$$\begin{aligned}
 S_{-i\zeta_0}(z, w) &= \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \frac{i \left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{z}\|) + \frac{i\mathbf{z}}{\|\mathbf{z}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{z}\|) \right) S_n(\mathbf{z}, \mathbf{w})}{I_{n+\frac{k}{2}-1}^2(\zeta_0) + I_{n+\frac{k}{2}}^2(\zeta_0)} \\
 &\quad \times \left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{w}\|) + \frac{i\mathbf{w}}{\|\mathbf{w}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{w}\|) \right) \|\mathbf{w}\|^{1-n-\frac{k}{2}}.
 \end{aligned} \tag{25}$$

Here, $I_{n+\frac{k}{2}}$ and $I_{n+\frac{k}{2}-1}$ stand for the usual modified Bessel functions of index $\nu = n + \frac{k}{2}$ or $\nu = n + \frac{k}{2} - 1$, respectively, which are given by $I_\nu(\zeta_0 \|\mathbf{z}\|) = e^{-\pi i \nu / 2} J_\nu(i\zeta_0 \|\mathbf{z}\|)$.

Applying the inverse Fourier transformation to (25) gives the Szegő kernel for D_z -monogenic functions in the infinite cylinder \mathcal{C} :

$$\begin{aligned}
 S_{\mathcal{C}}(z, w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \|\mathbf{w}\|^{1-n-\frac{k}{2}} \\
 &\quad \times \left[\int_{-\infty}^{+\infty} \frac{\left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{z}\|) + \frac{i\mathbf{z}}{\|\mathbf{z}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{z}\|) \right) S_n(\mathbf{z}, \mathbf{w})}{I_{n+\frac{k}{2}-1}^2(\zeta_0) + I_{n+\frac{k}{2}}^2(\zeta_0)} \right. \\
 &\quad \left. \times \left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{w}\|) + \frac{i\mathbf{w}}{\|\mathbf{w}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{w}\|) \right) e^{i\zeta_0(z_0-w_0)} d\zeta_0 \right].
 \end{aligned} \tag{26}$$

Similarly, the inverse Fourier transform applied to (20) provides us with a formula for the Bergman kernel for D_z -monogenic functions in the infinite cylinder \mathcal{C} :

$$\begin{aligned}
 B_{\mathcal{C}}(z, w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \|\mathbf{w}\|^{1-n-\frac{k}{2}} \\
 &\quad \times \left[\int_{-\infty}^{+\infty} \frac{\left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{z}\|) + \frac{i\mathbf{z}}{\|\mathbf{z}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{z}\|) \right) S_n(\mathbf{z}, \mathbf{w})}{\int_0^1 \left(I_{n+\frac{k}{2}-1}^2(\zeta_0 r) + I_{n+\frac{k}{2}}^2(\zeta_0 r) \right) r dr} \right. \\
 &\quad \left. \times \left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{w}\|) + \frac{i\mathbf{w}}{\|\mathbf{w}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{w}\|) \right) e^{i\zeta_0(z_0-w_0)} d\zeta_0 \right].
 \end{aligned} \tag{27}$$

The integral in the denominator of equation (27) can be evaluated explicitly:

$$\int_0^1 \left(I_{n+\frac{k}{2}-1}^2(\zeta_0 r) + I_{n+\frac{k}{2}}^2(\zeta_0 r) \right) r dr = \frac{1}{\zeta_0} I_{n+\frac{k}{2}-1}(\zeta_0) I_{n+\frac{k}{2}}(\zeta_0),$$

so that equation (27) simplifies to

$$\begin{aligned}
 B_C(z, w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-n-\frac{k}{2}} \|\mathbf{w}\|^{1-n-\frac{k}{2}} \\
 &\times \left[\int_{-\infty}^{+\infty} \frac{\left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{z}\|) + \frac{i\mathbf{z}}{\|\mathbf{z}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{z}\|) \right) S_n(\mathbf{z}, \mathbf{w})}{\frac{1}{\zeta_0} I_{n+\frac{k}{2}-1}(\zeta_0) I_{n+\frac{k}{2}}(\zeta_0)} \right. \\
 &\times \left. \left(I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{w}\|) + \frac{i\mathbf{w}}{\|\mathbf{w}\|} I_{n+\frac{k}{2}}(\zeta_0 \|\mathbf{w}\|) \right) e^{i\zeta_0(z_0-w_0)} d\zeta_0 \right]. \tag{28}
 \end{aligned}$$

From the equality

$$\begin{aligned}
 &\|\mathbf{z}\|^{1-n-\frac{k}{2}} \left(J_{n+\frac{k}{2}-1}(\lambda \|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda \|\mathbf{z}\|) \right) P_n(\mathbf{z}) \\
 &= \frac{1}{\lambda} (\mathbf{D}_z + \lambda) \left(\|\mathbf{z}\|^{1-n-\frac{k}{2}} J_{n+\frac{k}{2}-1}(\lambda \|\mathbf{z}\|) P_n(\mathbf{z}) \right), \quad \lambda \neq 0,
 \end{aligned}$$

we obtain that

$$\begin{aligned}
 &\|\mathbf{z}\|^{1-n-\frac{k}{2}} e^{-\lambda z_0} \left(J_{n+\frac{k}{2}-1}(\lambda \|\mathbf{z}\|) - \frac{\mathbf{z}}{\|\mathbf{z}\|} J_{n+\frac{k}{2}}(\lambda \|\mathbf{z}\|) \right) P_n(\mathbf{z}) \\
 &= -\frac{1}{\lambda} \overline{D}_z \left(\|\mathbf{z}\|^{1-n-\frac{k}{2}} e^{-\lambda z_0} J_{n+\frac{k}{2}-1}(\lambda \|\mathbf{z}\|) P_n(\mathbf{z}) \right), \quad \lambda \neq 0. \tag{29}
 \end{aligned}$$

Applying now (29) to both the variables z and w in (26) and in (28), respectively, we obtain

$$\begin{aligned}
 S_C(z, w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} \overline{D}_z \left\{ \|\mathbf{z}\|^{1-n-\frac{k}{2}} S_n(\mathbf{z}, \mathbf{w}) \|\mathbf{w}\|^{1-n-\frac{k}{2}} \right. \\
 &\times \left. \frac{I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{z}\|) I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{w}\|)}{\zeta_0^2 \left(I_{n+\frac{k}{2}-1}^2(\zeta_0) + I_{n+\frac{k}{2}}^2(\zeta_0) \right)} e^{i\zeta_0(z_0-w_0)} \right\} D_w d\zeta_0 \\
 B_C(z, w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} \overline{D}_z \left\{ \|\mathbf{z}\|^{1-n-\frac{k}{2}} S_n(\mathbf{z}, \mathbf{w}) \|\mathbf{w}\|^{1-n-\frac{k}{2}} \right. \\
 &\times \left. \frac{I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{z}\|) I_{n+\frac{k}{2}-1}(\zeta_0 \|\mathbf{w}\|)}{\zeta_0 (I_{n+\frac{k}{2}-1}(\zeta_0) I_{n+\frac{k}{2}}(\zeta_0))} e^{i\zeta_0(z_0-w_0)} \right\} D_w d\zeta_0.
 \end{aligned}$$

Here D_w stands for the same operator as D_z but with the z_j replaced by the w_j ; note also that it acts from the right in these formulas.

3.2. Harmonic Green’s functions and Bergman kernels. The Bergman kernel B_Ω for square-integrable monogenic functions defined over a sufficiently smooth domain $\Omega \subset \mathbb{R}^{k+1}$ can also be obtained from the harmonic Green’s function $G(z, w)$ of Ω , which is defined uniquely by the requirements that for all $w \in \Omega$,

$$\Delta_z G(z, w) = 0, \quad z \in \Omega \tag{30}$$

$$G(z, w) = \frac{1}{(1 - k)A_{k+1}} \frac{1}{\|z - w\|^{k-1}}, \quad z \in \partial\Omega, \tag{31}$$

(see [26]). Then $B_\Omega = \overline{D}_z G(z, w) D_w$, as can be verified directly by applying Stokes’ theorem and the Cauchy representation formula: if f is monogenic and square-integrable in Ω ,

$$\begin{aligned} \int_\Omega B_\Omega(z, w) f(w) dV_w &= \int_\Omega (\overline{D}_z G(z, w) D_w) f(w) dV_w \\ &= \int_{\partial\Omega} \underbrace{\overline{D}_z G(z, w)}_{=q_0(z-w) \text{ when } w \in \partial\Omega} d\sigma_w f(w) dV_w \\ &\quad \underbrace{\hspace{10em}}_{=f(z)} \\ &\quad - \int_\Omega \overline{D}_z G(z, w) \underbrace{(D_w f(w))}_{=0} dV_w, \end{aligned}$$

where $q_0(z - w) = \frac{1}{A_{k+1}} \frac{\overline{z-w}}{\|z-w\|^{k+1}}$ denotes the fundamental solution to the generalized Cauchy-Riemann operator D_z .

When using this method in the case of the cylinder \mathcal{C} , it is again advantageous to apply the Fourier transform in z_0 to the defining properties of the Green’s function:

$$(\Delta_z - \zeta_0^2)(\mathcal{F}G)(\zeta_0, \mathbf{z}, w) = 0, \quad \|\mathbf{z}\| < 1$$

$$\begin{aligned} \mathcal{F}G(\zeta_0, \mathbf{z}, w) &= \frac{e^{-i\zeta_0 w_0}}{2^{\frac{k}{2}-1} \Gamma(\frac{k}{2})} \left(\frac{\zeta_0}{\|\mathbf{z} - \mathbf{w}\|} \right)^{\frac{k}{2}-1} \\ &\quad \times K_{\frac{k}{2}-1}(\zeta_0 \|\mathbf{z} - \mathbf{w}\|), \quad \|\mathbf{z}\| = 1, \end{aligned}$$

where we have relied on the integral representation of the Bessel function K_ν .

To analyse these transformed defining properties, we apply the Gegenbauer addition theorem for K_ν , from which it follows that for $\|\mathbf{z}\| = 1$

$$\begin{aligned} (\mathcal{F}G)(\zeta_0, \mathbf{z}, w) &= \frac{1}{\frac{k}{2} - 1} e^{-i\zeta_0 w_0} \\ &\quad \times \sum_{n=0}^{+\infty} \frac{I_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{w}\|) K_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{z}\|)}{\|\mathbf{w}\|^{\frac{k}{2}-1} \|\mathbf{z}\|^{\frac{k}{2}-1}} \left(\frac{k}{2} + n - 1 \right) C_n^{\frac{k}{2}-1}(\cos \theta), \end{aligned}$$

where C_n^p denotes a Gegenbauer polynomial (see [13]), and θ the angle between \mathbf{w} and \mathbf{z} .

The Poisson kernel in the unit ball of \mathbb{R}^k is given by

$$P(\mathbf{z}, \mathbf{w}) = \frac{1 - \|\mathbf{z}\|^2 \|\mathbf{w}\|^2}{\|1 + \mathbf{z}\mathbf{w}\|^k} = \sum_{n=0}^{+\infty} \frac{\frac{k}{2} + n - 1}{\frac{k}{2} - 1} \|\mathbf{z}\|^n \|\mathbf{w}\|^n C_n^{\frac{k}{2}-1}(\cos \theta),$$

so that its homogeneous part of total degree n in \mathbf{z} satisfies

$$P_n(\mathbf{z}, \mathbf{w}) = \frac{\frac{k}{2} + n - 1}{\frac{k}{2} - 1} \|\mathbf{z}\|^n \|\mathbf{w}\|^n C_n^{\frac{k}{2}-1}(\cos \theta),$$

and that we can write for $\|\mathbf{z}\| = 1$

$$(\mathcal{F}G)(\zeta_0, \mathbf{z}, w) = e^{-i\zeta_0 w_0} \sum_{n=0}^{+\infty} \frac{I_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{w}\|) K_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{z}\|)}{\|\mathbf{w}\|^{\frac{k}{2}+n-1} \|\mathbf{z}\|^{\frac{k}{2}+n-1}} P_n(\mathbf{z}, \mathbf{w}),$$

applying the inverse Fourier transform then yields for $\|\mathbf{z}\| = 1$

$$G(z, w) = \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} e^{i\zeta_0(z_0-w_0)} \frac{I_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{w}\|) K_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{z}\|)}{\|\mathbf{w}\|^{\frac{k}{2}+n-1} \|\mathbf{z}\|^{\frac{k}{2}+n-1}} P_n(\mathbf{z}, \mathbf{w}) d\zeta_0.$$

In order to extend this expression harmonically to $\|\mathbf{z}\| < 1$, it suffices to replace $K_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{z}\|)$ by $I_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{z}\|)$, while multiplying by $K_{\frac{k}{2}+n-1}(\zeta_0)/I_{\frac{k}{2}+n-1}(\zeta_0)$ in order to preserve the values at the boundary $\|\mathbf{z}\| = 1$:

$$\begin{aligned} G(z, w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} e^{i\zeta_0(z_0-w_0)} \frac{I_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{w}\|) I_{\frac{k}{2}+n-1}(\zeta_0 \|\mathbf{z}\|)}{\|\mathbf{w}\|^{\frac{k}{2}+n-1} \|\mathbf{z}\|^{\frac{k}{2}+n-1}} \\ &\quad \times \frac{K_{\frac{k}{2}+n-1}(\zeta_0)}{I_{\frac{k}{2}+n-1}(\zeta_0)} P_n(\mathbf{z}, \mathbf{w}) d\zeta_0. \end{aligned} \tag{32}$$

The Poisson and Szegö kernels in the unit ball of \mathbb{R}^k satisfy the relation

$$P(\mathbf{z}, \mathbf{w}) = S(\mathbf{z}, \mathbf{w}) - \mathbf{z}S(\mathbf{z}, \mathbf{w})\mathbf{w},$$

which means that, in terms of the homogeneous components,

$$P_n(\mathbf{z}, \mathbf{w}) = S_n(\mathbf{z}, \mathbf{w}) - \mathbf{z}S_{n-1}(\mathbf{z}, \mathbf{w})\mathbf{w};$$

substitution then leads to

$$\begin{aligned}
 G(z, w) &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} e^{i\zeta_0(z_0-w_0)} \frac{I_{\frac{k}{2}+n-1}(\zeta_0\|\mathbf{w}\|)I_{\frac{k}{2}+n-1}(\zeta_0\|\mathbf{z}\|)}{\|\mathbf{w}\|^{\frac{k}{2}+n-1}\|\mathbf{z}\|^{\frac{k}{2}+n-1}} \\
 &\quad \times \frac{K_{\frac{k}{2}+n-1}(\zeta_0)}{I_{\frac{k}{2}+n-1}(\zeta_0)} S_n(\mathbf{z}, \mathbf{w}) d\zeta_0 \\
 &\quad - \frac{1}{2\pi} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} e^{i\zeta_0(z_0-w_0)} \frac{I_{\frac{k}{2}+n}(\zeta_0\|\mathbf{w}\|)I_{\frac{k}{2}+n}(\zeta_0\|\mathbf{z}\|)}{\|\mathbf{w}\|^{\frac{k}{2}+n}\|\mathbf{z}\|^{\frac{k}{2}+n}} \\
 &\quad \times \frac{K_{\frac{k}{2}+n}(\zeta_0)}{I_{\frac{k}{2}+n}(\zeta_0)} \mathbf{z} S_n(\mathbf{z}, \mathbf{w}) \mathbf{w} d\zeta_0.
 \end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}
 \bar{D}_z G(z, w) D_w &= \frac{1}{2\pi} \sum_{n=0}^{+\infty} \|\mathbf{z}\|^{1-\frac{k}{2}-n} \|\mathbf{w}\|^{1-\frac{k}{2}-n} \int_{-\infty}^{+\infty} e^{i\zeta_0(z_0-w_0)} \\
 &\quad \times (I_{\frac{k}{2}+n-1}(\zeta_0\|\mathbf{z}\|) + i(\mathbf{z}/\|\mathbf{z}\|)I_{\frac{k}{2}+n}(\zeta_0\|\mathbf{z}\|)) \\
 &\quad \times S_n(\mathbf{z}, \mathbf{w}) (I_{\frac{k}{2}+n-1}(\zeta_0\|\mathbf{w}\|) + i(\mathbf{w}/\|\mathbf{w}\|)I_{\frac{k}{2}+n}(\zeta_0\|\mathbf{w}\|)) \\
 &\quad \times \zeta_0^2 \left(\frac{K_{\frac{k}{2}+n-1}(\zeta_0)}{I_{\frac{k}{2}+n-1}(\zeta_0)} + \frac{K_{\frac{k}{2}+n}(\zeta_0)}{I_{\frac{k}{2}+n}(\zeta_0)} \right) d\zeta_0.
 \end{aligned}$$

The Wronskian identity $K_\nu(x)I_{\nu+1}(x) + I_\nu(x)K_{\nu+1}(x) = \frac{1}{x}$ then lets us simplify

$$\zeta_0^2 \left(\frac{K_{\frac{k}{2}+n-1}(\zeta_0)}{I_{\frac{k}{2}+n-1}(\zeta_0)} + \frac{K_{\frac{k}{2}+n}(\zeta_0)}{I_{\frac{k}{2}+n}(\zeta_0)} \right) = \frac{\zeta_0}{I_{\frac{k}{2}+n-1}(\zeta_0)I_{\frac{k}{2}+n}(\zeta_0)},$$

so that equation (28) is confirmed.

3.3. Half-infinite cylinders. In the case of a half-infinite cylinder

$$\mathcal{C}^+ = \left\{ z \in \mathcal{A}_{k+1} \mid \sum_{j=1}^k z_j^2 = 1, \quad z_0 > 0 \right\}$$

the corresponding Green’s function $G^+(z, w)$ can be obtained using a symmetry trick such as

$$G^+(z, w) = G(z, w) - G(-\bar{z}, w) + \frac{1}{(1-k)A_{k+1}} \frac{1}{\|\bar{z} + w\|^{k-1}}, \quad z, w \in \mathcal{C}^+,$$

because this expression is readily seen to satisfy the defining properties (30) and (31) on \mathcal{C}^+ and $\partial\mathcal{C}^+$, respectively. The corresponding function $\bar{D}_z G^+(z, w) D_w$ is then the Bergman kernel for \mathcal{C}^+ , and substitution of (32) provides an explicit formula for it.

3.4. Bounded finite cylinders. Now we have the tools in hand to treat cylinders that have a finite height $d > 0$. We consider without loss of generality cylinders of the form

$$\mathcal{C}_d : z_1^2 + \dots + z_k^2 = 1, \quad 0 < z_0 < d.$$

The symmetries involved here concern the hyperplanes $z_0 = 0$ and $z_0 = d$, with corresponding reflections $z \rightarrow -\bar{z}$ and $z \rightarrow 2d - \bar{z}$. The resulting expression for the Bergman kernel is

$$B_{\mathcal{C}_d}(z, w) = \sum_{m=-\infty}^{\infty} \left(\bar{D}_z G(z + 2md, w) D_w - \bar{D}_z G(-\bar{z} + 2md, w) D_w + \bar{D}_z \frac{1}{(1-k)A_{k+1}} \frac{1}{\|\bar{z} - 2md + w\|^{k-1}} D_w \right),$$

where the third term is precisely the Bergman kernel for the strip domain $0 < z_0 < d$ (expressible as a finite combination of Eisenstein series for the translation group with one generator $z \rightarrow z + 2d$, see [16, 8, 9]), and the first two terms are the cylindrical corrections to it. Substitution of (32) gives rise to the opportunity of applying the Poisson summation formula, thus eliminating the Fourier integrals. We finally arrive at the following main result:

Theorem 3.1. *The reproducing Bergman kernel of the space of square-integrable monogenic functions within a bounded cylinder of the form \mathcal{C}_d is given by*

$$B_{\mathcal{C}_d}(z, w) = \bar{D}_z \left(\frac{1}{2d} \sum_{n=0}^{+\infty} \sum_{m=-\infty}^{+\infty} \left(e^{i(\frac{\pi m}{d})(z_0-w_0)} - e^{i(\frac{\pi m}{d})(-z_0-w_0)} \right) \times \frac{I_{\frac{k}{2}+n-1}(\frac{\pi m}{d}\|\mathbf{w}\|) I_{\frac{k}{2}+n-1}(\frac{\pi m}{d}\|\mathbf{z}\|)}{\|\mathbf{w}\|^{\frac{k}{2}+n-1} \|\mathbf{z}\|^{\frac{k}{2}+n-1}} \frac{K_{\frac{k}{2}+n-1}(\frac{\pi m}{d})}{I_{\frac{k}{2}+n-1}(\frac{\pi m}{d})} P_n(\mathbf{z}, \mathbf{w}) + \frac{1}{(1-k)A_{k+1}} \sum_{m=-\infty}^{\infty} \frac{1}{\|\bar{z} - 2md + w\|^{k-1}} \right) D_w.$$

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