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General Orlicz–Pettis Theorem

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Abstract. In this paper, we establish two general Orlicz–Pettis theorems for G-valued duality pairs, where G is an Abelian topological group. These results give substantial improvements of many important results such as the Vitali–Hahn–Saks theorem, the Hahn–Schur theorem and the Stiles theorem.

Keywords: *Abstract duality pair, subseries convergence, Abelian topological groups.*
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1. Introduction

Let G be an abelian topological group. For a nonempty abstract set E and a nonempty $F \subseteq G^E$ we call (E, F) an abstract duality pair with respect to G or, simply, *G-valued duality pair.* Dual pairs in linear analysis are scalar valued duality pairs consisting of vector spaces, and the locally convex space theory is just the exhaustive discussion on dual pairs. In fact, each analysis field is discussing its own special duality pair such as the vector measure system $(\Sigma, ca(\Sigma, X))$, the abstract function system $(\Omega, C(\Omega, X))$ and the operator system $(X, L(X, Y))$.

In this paper we establish two powerful subseries convergence theorems for the general case of G-valued duality pairs. As well known, subseries convergence was a central problem in Functional Analysis. Since W. Orlicz gave out his first result in 1929, various Orlicz–Pettis type results were obtained in locally convex space theory [3, 9, 18, 19, 23]. Moreover, N. Kalton [8], G. Thomas [22], C. Swartz [20], Li Ronglu and C. Swartz [11] have discussed the subseries convergence problem in other special duality pairs such as in $(\Omega, C(\Omega, X))$ and $(X, L(X, Y))$. Especially, the measure theory contains a series of subseries convergence results, e.g., the classical Vitali–Hahn–Saks theorem, the Nikodym

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convergence theorem, the Phillips lemma and results on vector measures due to W. H. Graves and W. Ruess [4, 6]. We shall see that all of these results will be special cases of our general theorems and, especially, these special results will have a series of true improvements.

2. General subseries convergence theorems

If $\sum g_j$ is a series in G which is subseries convergent and if Δ is an infinite subset of $\mathbb{N}$, we write $\Sigma_{j\in\Delta} g_j = \sum_{k=1}^{\infty} g_{j_k}$ where $\Delta = \{j_1, j_2, \ldots\}$ with $j_1 < j_2 < \ldots$. For a G-valued duality pair (E, F), let $w(E, F)$ denotes the topology on E of pointwise convergence induced by F, i.e., $x_\alpha \to x$ in $(E, w(E, F))$ iff $f(x_\alpha) \to f(x)$ for every $f \in F$ ([17]). $w(E, F)$ is often abbreviated to wF. Let $x(f) = f(x)$ for $x \in E, f \in F$, we have $E \subseteq G^F$ and wE, the topology of pointwise convergence on E. A subset $B \subseteq F$ is said to be *conditionally wE-sequentially* compact if every sequence $\{f_j\}$ in B has a subsequence $\{f_{j_k}\}$ such that $\lim_k f_{j_k}(x)$ exists at each $x \in E$ ([5]).

A sequence $\{x_j\} \subseteq E$ is said to be *subseries wF-convergent* if for every nonempty $\Delta \subseteq \mathbb{N}$ there exists an $x_\Delta \in E$ such that $\sum_{j\in\Delta} f(x_j) = f(x_\Delta)$ for all $f \in F$. Let $\mathcal{F}$ be a family of subsets of F. Then we say a sequence $\{x_j\} \subseteq E$ is subseries convergent in the topology of uniform convergence on sets in $\mathcal{F}$ if for every nonempty $\Delta \subseteq \mathbb{N}$ there exists an $x_\Delta \in E$ such that the series $\sum_{j\in\Delta} f(x_j)$ converges to $f(x_\Delta)$ uniformly with respect to $f \in B$ for each $B \in \mathcal{F}$.

In order to shorten the proof of our general theorem we cite the following matrix theorem (see [1], Theorem 2.2.).

Theorem (Antosik-Mikusinski). *Let G be an Abelian topological group and $x_{ij} \in G$ for $i, j \in \mathbb{N}$. Suppose*

(I) $\lim_i x_{ij} = x_j$ *exists for each $j \in \mathbb{N}$, and*

(II) *for each increasing sequence $\{m_k\}$ in $\mathbb{N}$ there is a subsequence $\{n_k\}$ of $\{m_k\}$ such that $\{\sum_{k=1}^{\infty} x_{in_k}\}_{i=1}^{\infty}$ is Cauchy.*

Then $\lim_i x_{ij} = x_j$ uniformly for $j \in \mathbb{N}$ and, in particular, $x_{ii} \to 0$.

Lemma 1. *Let X be a countably compact Hausdorff space and (Y, d) a metric space. If $f : X \to Y$ is continuous, one to one and onto, then f is a homeomorphism between X and (Y, d).*

Proof. Let A be an open set in X and $\{f(x_n)\}$ a sequence in $f(X \setminus A)$. The sequence $\{x_n\}$ in $X \setminus A$ has a cluster point $x \in X \setminus A$ because A is open. Since f is continuous, for every $\delta > 0$ there is a neighborhood N_x of x such that $d(f(x), f(z)) < \delta$ for all $z \in N_x$. But x is a cluster point of $\{x_n\}$ so $d(f(x), f(x_n)) < \delta$ for infinite many x_n, i.e., $f(x)$ is a cluster point of $\{f(x_n)\}$. This shows that $(f(X\setminus A), d)$ is countably compact and, hence, compact and sequentially compact. Let $y_n \in f(X \setminus A)$ and $y_n \to y$. Then $\{y_n\}$ has a subsequence converging to $z \in f(X \setminus A)$ and, hence, $y = z \in f(X \setminus A)$. This shows that $f(X \setminus A)$ is closed in (Y, d) and $f(A) = Y \setminus f(X \setminus A)$ is open, i.e., f is an open mapping. ∎

Our main results are the following.

Theorem 1. *Let G be an Abelian topological group, E a nonempty abstract set and F a family of mappings from E into G. If a sequence $\{x_j\}$ in E is subseries wF-convergent, then $\{x_j\}$ is subseries convergent in the topology of uniform convergence on wE-compact subsets of F, wE-countably compact subsets of F and conditionally wE-sequentially compact subsets of F.*

Proof. `Step 1.` Let $\{q_1, q_2, \ldots\}$ be a strictly increasing sequence in $\mathbb{N}$ and B a conditionally wE-sequentially compact subset of F. If the convergence of $\sum_{k=1}^{\infty} f(x_{q_k})$ is not uniform with respect to $f \in B$, then there exist sequences $n_1 < m_1 < n_2 < m_2 < \ldots$ in $\mathbb{N}$, $\{f_p\} \subseteq B$ and a neighborhood U of 0 in G such that

$$\sum_{k=n_p}^{m_p} f_p(x_{q_k}) \notin U, \qquad p = 1, 2, 3, \ldots . \tag{$*$}$$

Set $\Delta_p = \{q_k : n_p \leq k \leq m_p\}$. For every nonempty $\Delta \subseteq \mathbb{N}$, let x_Δ denote an element in E for which $\sum_{q \in \Delta} f(x_q) = f(x_\Delta)$ holds for each $f \in F$, and let $S = \{x_\Delta : \Delta \subseteq \mathbb{N}, \Delta \neq \phi\}$, be the set of partial sums of $\{x_q\}$. Clearly, we can require that $\{x_q\} \subseteq S$.

There exist a subsequence $\{f_{p_i}\} \subseteq \{f_p\}$ and $f \in G^E$ such that $\lim_i f_{p_i}(x) = f(x)$ for each $x \in E$. We consider the matrix $\left[f_{p_i}(x_{\Delta_{p_j}})\right]_{i,j}$. For each j, $\lim_i f_{p_i}(x_{\Delta_{p_j}}) = f(x_{\Delta_{p_j}})$. If $\{j_k\}$ is a strictly increasing sequence in $\mathbb{N}$, then for each i we have

$$\sum_{k=1}^{\infty} f_{p_i}(x_{\Delta_{p_{j_k}}}) = \sum_{k=1}^{\infty} \sum_{q \in \Delta_{p_{j_k}}} f_{p_i}(x_q) = \sum_{q \in \bigcup_{k=1}^{\infty} \Delta_{p_{j_k}}} f_{p_i}(x_q) = f_{p_i}\big(x_{\bigcup_{k=1}^{\infty} \Delta_{p_{j_k}}}\big).$$

Therefore, $\lim_i \sum_{k=1}^{\infty} f_{p_i}(x_{\Delta_{p_{j_k}}}) = \lim_i f_{p_i}(x_{\bigcup_{k=1}^{\infty} \Delta_{p_{j_k}}}) = f(x_{\bigcup_{k=1}^{\infty} \Delta_{p_{j_k}}})$. Now, using the Antosik-Mikusinski matrix theorem, $\lim_i f_{p_i}(x_{\Delta_{p_i}}) = 0$. This contradicts $(*)$ and, hence, $\sum_{k=1}^{\infty} f(x_{q_k})$ converges uniformly for $f \in B$.

`Step 2.` Suppose the group topology on G is generated by a group norm $|\cdot|$. Note that $|\cdot|$ is a group seminorm but $|g| = 0$ implies $g = 0$. Let B be a wE-countably compact subset of F and S the set of partial sums of $\{x_j\}$ as was stated in Step 1. Define an equivalence relation on B by $f \sim g$ iff $f(x_j) = g(x_j)$

for all j. Let $\hat{B}$ be the collection of equivalence classes and $\hat{f}$ the class to which f belongs. Now define a metric $d(\cdot,\cdot)$ on $\hat{B}$ by

$$d(\hat{f},\hat{g})=\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{|f(x_j)-g(x_j)|}{1+|f(x_j)-g(x_j)|}.$$

Then $\hat{f}_\alpha \xrightarrow{d} \hat{f}$ iff $\lim_\alpha f_\alpha(x_j)=f(x_j)$ for each j.

If $f,g\in B$ and $f\sim g$, then $f(x)=g(x)$ for all $x\in S$ because $\{x_j\}$ is subseries wF-convergent. So we may define $\hat{f}$ on S by $\hat{f}(x)=f(x)$ for each $x\in S$, and then $(S,\hat{B})$ becomes a $G-$valued duality pair. The convergence $f_\alpha \xrightarrow{wE} f$ implies $f_\alpha \xrightarrow{wS} f$ and, hence, $\hat{f}_\alpha \xrightarrow{wS} \hat{f}$, so the correspondence $f\longmapsto \hat{f}$ is a continuous mapping from (B,wE) onto $(\hat{B},wS)$ and $\hat{B}$ is wS-countably compact because B is wE-countably compact. Moreover, if $\hat{f}_\alpha \xrightarrow{wS} \hat{f}$, then $\lim_\alpha f_\alpha(x_j)=\lim_\alpha \hat{f}_\alpha(x_j)=\hat{f}(x_j)=f(x_j)$ for each j because $\{x_j\}\subseteq S$, i.e., $\hat{f}_\alpha \xrightarrow{d} \hat{f}$. This shows that the correspondence $\hat{f}\longmapsto\hat{f}$ is a continuous mapping from the countably compact space $(\hat{B},wS)$ onto the metric space $(\hat{B},d)$. Furthermore, we can prove that $(\hat{B},wS)$ is also a Hausdorff space, so, by Lemma 1, $(\hat{B},wS)$ is a countably compact metric space and, hence, $(\hat{B},wS)$ is sequentially compact. Now, the sequence $\{x_j\}$ in S is subseries $w\hat{B}$-convergent and, by Step 1, every subseries of the series $\sum\hat{f}(x_j)$ converges uniformly with respect to $\hat{f}\in\hat{B}$. This shows that every subseries of $\sum f(x_j)$ converges uniformly with respect to $f\in B$.

`Step 3.` Suppose the group topology on G is generated by a group seminorm $|\cdot|$. Define an equivalence relation on G by $g\sim h$ iff $|g-h|=0$. Let Q be the quotient set and $q:G\to Q$ the quotient map. If $g\sim h$, then $|g|=|g-h+h|\leqslant|g-h|+|h|$ so $|g|=|h|$. Now define a function $\|\cdot\|$ on Q by $\|q(g)\|=|g|$ and let $q(g)+q(h)=q(g+h)$, then $\|\cdot\|$ is a group norm on the quotient group $(Q,+)$ and q is an isometric homomorphism from $(G,|\cdot|)$ onto $(Q,\|\cdot\|)$.

Suppose B is a wE-countably compact subset of F. Let $q(F)=\{q\circ f: f\in F\}$ and $q(B)=\{q\circ f: f\in B\}$. Then $(E,q(F))$ is a Q-valued duality pair and $q(B)$ is wE-countably compact because $|f_n(x)-f(x)|\to 0$ implies $\|q(f_n(x))-q(f(x))\|\to 0$ for each $x\in E$, i.e., $(q(B),wE)$ is a continuous image of the countably compact space (B,wE). Now let $\triangle=\{j_1,j_2,\ldots\}$ be a strictly increasing sequence in $\mathbb{N}$. Since $\sum_{k=1}^{\infty}f(x_{j_k})=f(x_\triangle)$,

$$\begin{aligned}\Big\|q\big(f(x_\triangle)\big)-\sum_{k=1}^{n}q\big(f(x_{j_k})\big)\Big\| &= \Big\|q\Big(f(x_\triangle)-\sum_{k=1}^{n}f(x_{j_k})\Big)\Big\|\\ &=\Big|f(x_\triangle)-\sum_{k=1}^{n}f(x_{j_k})\Big|\longrightarrow 0\end{aligned}$$

as $n \to \infty$ for each $f \in F$, so $\{x_j\}$ is subseries $wq(F)$-convergent. By Step 2, for $\triangle = \{j_1, j_2, \ldots\}$ and $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$\Big| f(x_\triangle) - \sum_{k=1}^{n} f(x_{j_k}) \Big| = \Big\| q\big(f(x_\triangle)\big) - \sum_{k=1}^{n} q\big(f(x_{j_k})\big) \Big\| < \varepsilon$$

for all $n > n_0$ and all $f \in B$, i.e., each subseries of $\sum f(x_j)$ converges uniformly with respect to $f \in B$.

`Step 4.` Any group topology is generated by a family of group seminorms. Suppose the group topology on G is generated by the family $\{|\cdot|_\alpha : \alpha \in I\}$ of group seminorms. Then G is homeomorphic to a subspace of the product space $\prod_{\alpha \in I}(G, |\cdot|_\alpha)$ and sets in the shape of $\{g \in G : |g|_{\alpha_i} < \varepsilon_i, 1 \le i \le n\}$ make a neighborhood base at 0 in G. Note that this kind of neighborhood depends on finitely many of group seminorms. Thus, by Step 3, $\{x_j\}$ is subseries convergent in the topology of uniform convergence on wE-countably compact subsets of F.

`Step 5.` Let B be a wE-compact subset of F. Then (B, wE) is compact and, hence, countably compact so the desired assertion holds by Step 4. The theorem is proved. ∎

For every $H \subseteq G^E$, let

$$\overline{H}^s = \Big\{ f \in G^E : \exists \{f_k\} \subseteq H \text{ such that } f(x) = \lim_k f_k(x) \, \forall x \in E \Big\},$$

be the sequential wE-closure of H. Clearly, $\overline{G^E}^s = G^E$ and $\overline{H}^s \supseteq H$, so we have the smallest subfamily $[F]$ of G^E satisfying $F \subseteq [F]$ and $\overline{[F]}^s = [F]$.

In fact, $[F] = \bigcap \{H \subseteq G^E : H \supseteq F, \overline{H}^s = H\}$. $[F]$ is called the Baire family generated by F ([7]). Now we can strengthen Theorem 1 as follows.

Theorem 2. *Let G be an Abelian topological group, E a nonempty abstract set, F a family of mappings from E into G and $[F]$ the Baire family generated by F. Then there exists the largest subfamily Φ of G^E satisfying the conditions*

- **(1)** $\Phi \supseteq [F]$;
- **(2)** $\overline{\Phi}^s = \Phi$, *i.e., Φ is sequentially wE-closed in G^E;*
- **(3)** *subseries wF-convergent sequences in E are subseries convergent in the topology of uniform convergence on wE-compact subsets of Φ, wE-countably compact subsets of Φ and wE-sequentially compact subsets of Φ. In particular, subseries wF-convergent sequences in E are subseries $w\Phi$-convergent.*

Proof. Let $\mathcal{E}$ be the collection of all sequences in E which are subseries wF-convergent. For $\{x_j\} \in \mathcal{E}$ and $\Delta \subseteq \mathbb{N}$, let x_Δ denote an element in E satisfying $\sum_{j \in \Delta} f(x_j) = f(x_\Delta)$ for all $f \in F$.

Consider the family

$$\Phi = \left\{ f \in G^E : \sum_{j\in\Delta} f(x_j) = f(x_\Delta), \{x_j\} \in \mathcal{E}, \Delta \subseteq \mathbb{N} \right\}.$$

Clearly, $\Phi \supseteq F$ and each $\{x_j\} \in \mathcal{E}$ is subseries $w\Phi$-convergent so (3) holds by Theorem 1. Suppose $f_k \in \Phi$ and $\lim_k f_k(x) = f(x)$ exists at each $x \in E$. Let $\{x_j\} \in \mathcal{E}$ and $\Delta \subseteq \mathbb{N}$. Then $\{x_j\}$ is subseries $w\Phi$-convergent and $\{f_k : k \in \mathbb{N}\}$ is a conditionally wE-sequentially compact subset of Φ. By Theorem 1, the series $\sum_{j\in\Delta} f_k(x_j)$ converges to $f_k(x_\Delta)$ uniformly with respect to $k \in \mathbb{N}$ and

$$\sum_{j\in\Delta} f(x_j) = \sum_{j\in\Delta} \lim_k f_k(x_j) = \lim_k \sum_{j\in\Delta} f_k(x_j) = \lim_k f_k(x_\Delta) = f(x_\Delta).$$

Thus, $f \in \Phi$ and (2) holds. The theorem is proved. ∎

Note that the main part $[F]$ of Φ has a clear structure. In fact, set $F_1 = F$ and for every ordinal number α define $F_\alpha = \overline{\cup_{\gamma<\alpha} F_\gamma}^s$. This transfinite induction is similar to the definition of Baire functions ([7], §43; [10], p.236). It is easy to see that if $\overline{F_{\alpha_0}}^s = F_{\alpha_0}$ for some ordinal number α_0, then $[F] = F_{\alpha_0}$.

If $\overline{F}^s = F$, then $[F] = F$. This is a very special case even if (E, F) is a scalar valued dual pair of vector spaces. In fact, for a locally convex space X and its dual X', $\overline{X'}^s = X'$ iff every closed graph linear map from X into the sequence space $(c, \|\cdot\|_\infty)$ is weakly continuous ([24], p.206). It is difficult to limit the exact expansion scale for $\overline{X'}^s$ but there is a result which shows that $\overline{X'}^s \subseteq (X, \beta(X, X'))'$, the dual with respect to the strongest (X, X')-admissible topology $\beta(X, X')$ ([13]).

3. Special cases of general subseries convergence theorems

Theorem 1 and Theorem 2 present two fundamental principles in analysis. Many of Orlicz–Pettis type results become immediately special cases of our general version, e.g., let (E, F) be a scalar valued dual pair of vector spaces, then Theorem 1 becomes the famous Orlicz–Pettis–Bennet–Kalton–Dierolf theorem, immediately. It is a satisfaction to see that our general results give a series of substantial improvements of many important special results. We would like to show a few of typical results.

3.1. Vitali-Hahn-Saks type results. Let $\sum$ be a σ-algebra of subsets of a set. The classical Vitali–Hahn–Saks theorem says that if $\{f_n\}$ is a sequence in $ca(\sum, \mathbb{C})$ such that $\lim_n f_n(A)$ exists at each $A \in \sum$, then $\{f_n\}$ is uniformly countably additive. So this is a result on a conditionally $w\sum$-sequentially compact families of scalar valued measures. In 1980, W. H. Graves and W. Ruess [6] have proved that any $w\sum$-compact family $K \subseteq ca(\sum, X)$ is uniformly countably additive, where $ca(\sum, X)$ is the family of countably additive measures valued in a locally convex space X ([6], Lemma 6, Theorem 7). However, as well known, $w\sum$-compact families must be $w\sum$-countably compact and, in general, the converse is not true ([24], Theorem 14.1.9). Now we can improve the Vitali–Hahn–Saks–Graves–Ruess theorem as follows.

Theorem 3. *Let G be an Abelian topological group and $\sum$ a σ-algebra of subsets of a set. If a subset K of $ca(\sum, G)$ is conditionally $w\sum$-sequentially compact or $w\sum$-countably compact or, in particular, $w\sum$-compact, then K is uniformly countably additive.*

Proof. $(\sum, ca(\sum, G))$ forms a G-valued duality pair and each disjoint sequence in $\sum$ is subseries $w[ca(\sum, G)]$-convergent. ∎

As a special case of Theorem 2 or, an immediate consequence of Theorem 3, we have the following Nikodym convergence theorem.

Theorem 4. *Let G be an abelian topological group and $\sum$ a σ-algebra of subsets of a set. If $f_n \in ca(\sum, G)$ and $f_n(A) \to f(A)$ at each $A \in \sum$, then the limit measure f belongs to $ca(\sum, G)$.*

Theorem 1 and Theorem 2 also imply the Vitali-Hahn-Saks-Nikodym theorem for strongly additive vector measures ([2], [4]), but we omit it.

3.2. Hahn-Schur type results. As an application of the matrix methods in analysis, P. Antosik and C. Swartz gave a general version of the Hahn-Schur theorem ([1], Theorem 8.1). We strengthen this version as follows.

Theorem 5. *Let G be an Abelian topological group and $x_{ij} \in G$ for all $i, j \in \mathbb{N}$. Suppose for each i the series $\sum_{j=1}^{\infty} x_{ij}$ is subseries convergent and $\lim_i \sum_{j\in\Delta} x_{ij}$ exists for each $\Delta \subseteq \mathbb{N}$. Let $x_j = \lim_i x_{ij}$ for all j. Then*

- **(a)** $\sum_{j=1}^{\infty} x_{ij}$ *is uniformly subseries convergent for $i \in \mathbb{N}$;*
- **(b)** $\sum x_j$ *is subseries convergent;*
- **(c)** $\lim_i \sum_{j\in\Delta} x_{ij} = \sum_{j\in\Delta} x_j$ *uniformly for $\Delta \subseteq \mathbb{N}$.*

To see this, for each i define $f_i : 2^N \to G$ by $f_i(\Delta) = \sum_{j\in\Delta} x_{ij}$ and consider the G-valued duality pair $(2^N, \{f_i\})$. It is easy to see that the singleton sequence $\{1\}$,$\{2\}$,$\{3\}$,... in 2^N is subseries $w\{f_i\}$-convergent and $\{f_i : i \in \mathbb{N}\}$ is conditionally $w[2^N]$-sequentially compact by hypothesis. Thus, Theorem 1 and Theorem 2 imply (a) and (b). Condition (c) can be obtained by (a) and (b). A more general version of Theorem 5 can be found in [12].

A series of useful results in functional analysis and measure theory follows immediately from the abstract version above ([1], Chapter 8). Especially, Theorem 5 can be used to obtain versions of the Nikodym convergence theorem for certain finitely additive measures which need not be countably additive.

3.3. Thomas-Swartz type results. For a compact space Ω and $C(\Omega)$, G. Thomas [22], C. Swartz [19], Li Ronglu and C. Swartz [11] have established an Orlicz–Pettis type result. As was stated above, compactness is stronger than countable compactness. So we would like to establish the same conclusion for spaces which are countably compact or sequentially compact.

Theorem 6. *Let Ω be a countably compact or sequentially compact or, in particular, a compact space and $C(\Omega, G)$ the family of continuous functions from Ω into an abelian topological group G. If a series $\sum f_j$ in $C(\Omega, G)$ is subseries convergent in the topology of pointwise convergence on Ω, then $\sum f_j$ is subseries convergent in the topology of uniform convergence on Ω, i.e., for every subsequence $\{f_{j_k}\}$ of $\{f_j\}$ there exists an $f \in C(\Omega, G)$ such that $\sum_{k=1}^{\infty} f_{j_k}(w)$ converges to $f(w)$ uniformly for $w \in \Omega$.*

Proof. If $w_\alpha \to w$ in Ω, then $f(w_\alpha) \to f(w)$ for each $f \in C(\Omega, G)$. This shows that $(\Omega, wC(\Omega, G))$ is a continuous image of Ω with its original topology. Hence, for example, if Ω is countably compact in the original topology, then $(\Omega, wC(\Omega, G))$ is also countably compact, and Theorem 1 is available for the pair $(C(\Omega, G), \Omega)$. ∎

3.4. An improvement of the Stiles theorem. Stiles type results [18] require metrizability of spaces and continuity of coordinate functionals, now, we can drop these conditions.

Theorem 7. *Let X be a sequentially complete topological vector space with a basis $\{e_k\}$ and $\{f_k\}$ the coordinate functionals with respect to $\{e_k\}$. If a series $\sum x_j$ in X is subseries $w\{f_k\}$-convergent, then $\sum x_j$ is subseries convergent in the original topology of X.*

Proof. Define $P_n : X \to X$ by $P_n x = \sum_{k=1}^{n} f_k(x) e_k$. Then $P_n x \to x$ for each $x \in X$, i.e., $\{P_n : n \in \mathbb{N}\}$ is conditionally wX-sequentially compact. If $\{y_i\}$ is a subsequence of $\{x_j\}$, then let $\sum_{i=1}^{\infty} y_i$ denote its $w\{f_k\} - sum$. For each $n \in \mathbb{N}$ and $\{y_i\} \subseteq \{x_j\}$, $\sum_{i=1}^{\infty} f_n(y_i) e_n = \lim_m \sum_{i=1}^{m} f_n(y_i) e_n = \left[\lim_m \sum_{i=1}^{m} f_n(y_i)\right] e_n = \left[\sum_{i=1}^{\infty} f_n(y_i)\right] e_n = f_n(\sum_{i=1}^{\infty} y_i) e_n$ and

$$\sum_{i=1}^{\infty} P_n(y_i) = \sum_{i=1}^{\infty} \sum_{k=1}^{n} f_k(y_i) e_k = \sum_{k=1}^{n} f_k\Big(\sum_{i=1}^{\infty} y_i\Big) e_k = P_n\Big(\sum_{i=1}^{\infty} y_i\Big),$$

i.e., the subseries $w\{f_k\}$-convergent sequence $\{x_j\}$ is subseries $w\{P_n\}$-convergent. Thus, for every $\Delta \subseteq \mathbb{N}$ the series $\sum_{j\in\Delta} P_n(x_j)$ converges in X uniformly with respect to $n \in \mathbb{N}$.

We claim that $\{\sum_{i=1}^k y_i\}_{k=1}^\infty$ is Cauchy in X for each $\{y_i\} \subseteq \{x_j\}$ and, therefore, convergent since X is sequentially complete. Let U be a closed neighborhood of 0 in X. There exists an integer N such that

$$\sum_{i=m}^n P_k y_i = P_k(\sum_{i=m}^n y_i) \in U \quad \forall k \in \mathbb{N},\ n > m \geq N.$$

Hence, $\sum_{i=m}^n y_i = \lim_k P_k(\sum_{i=m}^n y_i) \in U$ for $n > m \geq N$ since U is closed. The theorem is proved. ∎

3.5. A generalization of Kalton's theorem. Let (X, Y) be a scalar valued dual pair of vector spaces and τ a polar topology on X. The Kalton's theorem says that if (X, τ) is separable, then subseries wY-convergent sequences in X are subseries τ-convergent [9].

For a (X, Y)-polar topology τ, there is a family $\mathcal{F}$ of wX-bounded subsets of Y such that τ is just the topology of uniform convergence on each member of $\mathcal{F}$. Since each set in $\mathcal{F}$ is wX-bounded, for every $B \in \mathcal{F}$ and $x \in X$ the set $\{f(x) : f \in B\}$ is relatively sequentially compact in the scalar field. We show that this pointwise sequential compactness is just the key point in Kalton's result and its generalization.

Definition 1. Let (E, F) be a G-valued duality pair. A subset $B \subseteq F$ is said to be *sequentially compact* at each $x \in E$ if $\{f(x) : f \in B\}$ is relatively sequentially compact in G for each $x \in E$.

There is a typical example. Let X and Y be Banach spaces and $K(X, Y)$ the family of compact operators from X into Y. Then $(K(X, Y), X)$ is an Y-valued duality pair in the sense of $x(T) = Tx$ for $x \in X, T \in K(X, Y)$, and every bounded subset of X is sequentially compact at each $T \in K(X, Y)$. Similarly, for the family $L(X, Y)$ of continuous operators and the pair $(L(X, Y), X)$, every compact subset of X is sequentially compact at each $T \in L(X, Y)$.

Theorem 8. *Let G be a sequentially complete Abelian topological group, E a nonempty abstract set and F a family of mappings from E into G. Let $\mathcal{F}$ be a family of subsets of F such that each set in $\mathcal{F}$ is sequentially compact at each $x \in E$ and τ the topology on E of uniform convergence on sets in $\mathcal{F}$. If (E, τ) is separable, then each set in $\mathcal{F}$ is conditionally wE-sequentially compact and, hence, every subseries wF-convergent sequence in E is subseries τ-convergent.*

Proof. Let $D = \{d_k : k \in \mathbb{N}\}$ be dense in $(E, \tau), B \in \mathcal{F}$ and $\{f_k\} \subseteq B$. By the diagonal procedure $\{f_k\}$ has a subsequence $\{f_{n_k}\}$ such that the sequence

$\{f_{n_k}(d)\}$ converges at each $d \in D$. Let $x \in E$. There is a net $\{d_\alpha\}$ in D such that $d_\alpha \xrightarrow{\tau} x$ so $\lim_\alpha f(d_\alpha) = f(x)$ is uniformly with respect to $f \in B$. Now let U be a neighborhood of 0 in G. Pick a symmetric neighborhood V of 0 for which $V + V + V \subseteq U$. Then there is a d_α such that $f_{n_k}(d_\alpha) - f_{n_k}(x) \in V$ for all k. But the sequence $\{f_{n_k}(d_\alpha)\}_{k=1}^\infty$ is convergent, so there is a $k_0 \in \mathbb{N}$ such that $f_{n_k}(d_\alpha) - f_{n_j}(d_\alpha) \in V$ for all $k, j > k_0$. Thus, if $k, j > k_0$, then

$$\begin{aligned} f_{n_k}(x) - f_{n_j}(x) &= f_{n_k}(x) - f_{n_k}(d_\alpha) + f_{n_k}(d_\alpha) - f_{n_j}(d_\alpha f_{n_j}(d_\alpha) - f_{n_j}(x) \\ &\in V + V + V \subseteq U. \end{aligned}$$

This shows that $\{f_{n_k}(x)\}_{k=1}^\infty$ is Cauchy and, hence, convergent since G is sequentially complete. ∎

Theorem 8 contains much useful information. For example, if the space $K(X,Y)$ of compact operators is separable in the operator norm, then every series in $K(X,Y)$ which is subseries convergent in the weak operator topology is subseries convergent in the operator norm. In fact, by Theorem 1 or the classical Orlicz–Pettis theorem, subseries convergence in the weak operator topology implies subseries convergence in the pointwise topology wX and, hence, Theorem 6 is available. Note that the condition of separability of $(K(X,Y), \|\cdot\|)$ holds for many of useful cases, e.g., if either Y or X' has the approximation property, and they have also the separability of both of these spaces, in particular, if either has a Schauder basis, then the separability of $(K(X,Y), \|\cdot\|)$ holds.

We will omit other many interesting applications of our general theorem. We would like to say that our discussions on abstract duality pairs yield a series of nice results not only in the subseries convergence problem, but also in other analysis problems [12, 14 – 16, 21, 25 – 28].

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