

Birth-and-Death Type Systems with Parameter and Chaotic Dynamics of some Linear Kinetic Models

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Abstract. J. Banasiak and M. Lachowicz proved in [Math. Models Methods Appl. Sci. 12 (2002), 755 – 775] that, under certain conditions on the coefficients, the dynamics generated by birth-and-death type systems with proliferation was chaotic. In this paper we extend this result to systems with parameter-dependent coefficients and present an application to a linear Boltzmann equation describing inelastic collisions.

Keywords: *Topological chaos, linear dynamical systems, birth-and-death systems, linear Boltzmann equation*

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1. Introduction

The phenomenon of topological chaos in infinite-dimensional linear dynamical systems has been recently investigated in a series of papers [9, 10, 14, 15, 20, 24]. We recall below the definition of topological chaos, that was formulated in [16], first introducing basic notation.

Let $(\mathbf{x}(t, \cdot))_{t \geq 0}$ be a continuous dynamical system in a complete metric space (X, d) . By $O(\mathbf{p}) = \{\mathbf{x}(t, \mathbf{p})\}_{t \geq 0}$ we denote the orbit of $(\mathbf{x}(t, \cdot))_{t \geq 0}$, originating from \mathbf{p} . We say that $(\mathbf{x}(t, \cdot))_{t \geq 0}$ is topologically transitive if for any two non-empty open sets $U, V \subset X$ there is a $t_0 \geq 0$ such that $\mathbf{x}(t_0, U) \cap V \neq \emptyset$. A periodic point of $(\mathbf{x}(t, \cdot))_{t \geq 0}$ is any point $\mathbf{p} \in X$ satisfying $\mathbf{x}(T, \mathbf{p}) = \mathbf{p}$ for some $T > 0$.

Definition 1.1. [16] Let X be a metric space. A dynamical system $(\mathbf{x}(t, \cdot))_{t \geq 0}$ in X is said to be (topologically) *chaotic* in X if it is transitive and its set of periodic points is dense in X .

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Historically speaking, the original Devaney's definition contained the so-called sensitive dependence on initial conditions which sometimes is considered central to the idea of chaos, but it can be proved, see e.g. [10, 13], that if a system satisfies the conditions of Definition 1.1, then it is also sensitively dependent on initial conditions. This result, as well as the following ones, require X to be nondegenerate in the sense that no tail $\{\mathbf{x}(t, \mathbf{p})\}_{0 \leq t \leq t_0}$, $t_0 < \infty$, of an orbit is dense in X .

Devaney's definition was introduced with nonlinear dynamical systems in mind but it turned out that it is very closely related to the property of iterates of linear operators investigated in e.g. [18] that is called hypercyclicity: a bounded linear operator A on a Banach space X is called *hypercyclic* if for some $\mathbf{x} \in X$ we have $\overline{\{A^n \mathbf{x}\}_{n \geq 0}} = X$ or, rephrasing this in the language of dynamical systems, there is an orbit of the discrete dynamical system generated by A that is dense in X . In [18] the authors proved the theorem that A is hypercyclic if and only if it is topologically transitive and this theorem can be easily generalized to continuous systems [10, 15]. Thus, we can rephrase Devaney's definition by saying that $(\mathbf{x}(t, \cdot))_{t \geq 0}$ is chaotic if and only if it has an orbit that is dense in X and its set of periodic points is dense.

That linear continuous dynamical systems appearing in applications can be chaotic in the sense of Devaney was possibly first noticed in [24], where the authors applied the criterion, formulated in [19], to a simple kinetic type system with constant coefficients that could describe interactions of test particles with the background in which the particles can only lose energy. However, it was noticed already in [22, 26] that also first order linear hyperbolic equations can give rise to chaotic dynamics, though the definition of chaos used in these papers was slightly different, see also [27].

The results of [24] were later extended in [8, 9, 11] to more general birth-and-death type systems with non-constant coefficients that arise, e.g., in modelling the development of drug resistance in cancer cells. It turns out that similar systems appear in extended kinetic theory where they describe interactions of particles with the background in which particles can either lose or gain a quantum of energy. Models of this type arise, for instance, in semiconductor theory (electron scattering on the crystalline lattice) but also in the neutron transport in gases, see e.g. [5, 18]. In this paper, for technical reasons, we shall discuss a simplified model of this type that preserves, however, the essential features of the original ones.

The main difference between the birth-and-death system of [9] and the models discussed here is that the coefficients now depend not only on the discrete variable (representing the energy jump in each interaction) but also on a continuous parameter representing the so-called reduced energy, Section 4, and the dependence of coefficients on this parameter may be quite irregular. Thus, the main mathematical difficulty we deal with in this paper is to provide uniform

estimates of solutions to the stationary birth-and-death type infinite systems of equations with parameter dependent coefficients so that the methods developed in [9] could be applied; this is done in Section 3. Section 2 is devoted to a survey of recent results on chaotic linear systems based on [12] and in Section 4 we show how the results of the previous two sections can be applied to specific kinetic models.

2. Analytical background

Possibly the most widely used set of conditions ensuring that a strongly continuous (C_0) semigroup $(T(t))_{t \geq 0}$ generated by an operator A is chaotic is given in the following theorem.

Theorem 2.1. [15, Theorem 3.1] *Let X be a separable Banach space and let A be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X . $(T(t))_{t \geq 0}$ is chaotic if the following conditions are satisfied:*

1. *The point spectrum of A , $\sigma_p(A)$, contains an open connected set U such that $U \cap i\mathbb{R} \neq \emptyset$;*
2. *There exists a selection $U \ni \lambda \rightarrow \mathbf{x}_\lambda$ of eigenvectors of A such that the function $F_\Phi(\lambda) = \langle \Phi, \mathbf{x}_\lambda \rangle$ is analytic in U for any $\Phi \in X^*$;*
3. *$F_\Phi \equiv 0$ on U if and only if $\Phi = 0$.*

For further development it is important to understand some details of the proof of this theorem. It uses the observation [15] that for $(T(t))_{t \geq 0}$ to be hypercyclic it is sufficient that the following two spaces

$$X_0 = \{ \mathbf{x} \in X; \lim_{t \rightarrow \infty} T(t)\mathbf{x} = 0 \}$$

$$X_\infty = \{ \mathbf{w} \in X; \forall \epsilon > 0 \exists \mathbf{x} \in X, t > 0 \|\mathbf{x}\| < \epsilon \text{ and } \|T(t)\mathbf{x} - \mathbf{w}\| < \epsilon \}$$

are dense in X . Thus, density of these two spaces together with the density of the set of periodic points X_p gives chaoticity of $(T(t))_{t \geq 0}$ in X . Equivalence of concepts of weak and strong analyticity of a function, e.g. [25], yields that $\lambda \rightarrow \mathbf{x}_\lambda$ is an analytic function. Condition 3 is used through the following argument. If U' is any subset of U having an accumulation point in U and if $\Phi \in X^*$ is any functional that annihilates $\{ \mathbf{x}_\lambda; \lambda \in U' \}$, that is, $\langle \Phi, \mathbf{x}_\lambda \rangle = 0$ for $\lambda \in U'$, then from the principle of isolated zeros the analytic function $F_\Phi(\lambda) = \langle \Phi, \mathbf{x}_\lambda \rangle$ vanishes everywhere in U which, by Condition 3, is possible only if $\Phi = 0$. This in turn shows that $\overline{\text{Span}\{ \mathbf{x}_\lambda; \lambda \in U' \}} = X$. Now, it is easy to see that the sets $U_- = U \cap \{ \lambda; \Re \lambda < 0 \}, U_+ = U \cap \{ \lambda; \Re \lambda > 0 \}, U_0 = U \cap \{ \lambda; \Re \lambda = 0, \Im \lambda \text{ is rational} \}$ have accumulation points in U . Moreover $\text{Span}\{ \mathbf{x}_\lambda; \lambda \in U_- \} \subset X_0$, by $\mathbf{x}_\lambda = T(t)e^{-\lambda t} \mathbf{x}_\lambda$ we see that $\text{Span}\{ \mathbf{x}_\lambda; \lambda \in U_+ \} \subset X_\infty$

and $\text{Span}\{\mathbf{x}_\lambda; \lambda \in U_0\} \subset X_p$ so that if Condition 3 is satisfied, X_0, X_∞ and X_p are dense in X and therefore $(T(t))_{t \geq 0}$ is chaotic.

A closer look at the above analysis shows that if \mathbf{x}_λ is analytic in some open connected U , then $\overline{\text{Span}\{\mathbf{x}_\lambda; \lambda \in U'\}}$ is the same for any U' having an accumulation point in U . The following result was proved in [12].

Lemma 2.2. *If A is a closed operator in X and for some function \mathbf{x}_λ that is analytic in an open connected set U we have*

$$A\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda,$$

then, denoting by a_{n,λ_0} the n -th coefficient of Taylor's expansion of x_λ at $\lambda_0 \in U$, the set

$$Z = Z_{\lambda_0} = \overline{\text{Span}\{a_{n,\lambda_0}; n \in \mathbb{N}_0\}}$$

is independent of λ_0 . Moreover, for any $U' \subset U$ having an accumulation point in U we have

$$Z = \overline{\text{Span}\{\mathbf{x}_\lambda; \lambda \in U'\}} = \overline{\text{Span}\{\mathbf{x}_\lambda; \lambda \in U\}}.$$

An extensive and detailed discussion of the above result is given in [12]. Here we analyse its implications that are relevant to this paper.

Theorem 2.3. *Assume that the point spectrum $\sigma_p(A)$ of the generator A of a C_0 -semigroup $(T(t))_{t \geq 0}$ contains an open connected subset U of \mathbb{C} on which there exists an analytic selection $\lambda \rightarrow \mathbf{x}_\lambda$ of eigenvectors of A . Denote $Y = \overline{\text{Span}\{\mathbf{x}_\lambda; \lambda \in U\}}$. Then*

1. *if $i\mathbb{R} \cap U \neq \emptyset$, then $(T(t))_{t \geq 0}$ is chaotic in Y ;*
2. *if $U \subset \overline{\mathbb{C}_-}$ with $i\mathbb{R} \cap \partial U \neq \emptyset$; then the dynamics is unstable in the sense that an arbitrarily small perturbation ϵf of the generator, $\epsilon > 0$, makes the system chaotic in Y ;*
3. *if $U \subset \overline{\mathbb{C}_+}$ with $i\mathbb{R} \cap \partial U \neq \emptyset$, then the dynamics is unstable in the sense that an arbitrarily small perturbation $-\epsilon f$ of the generator, $\epsilon > 0$, makes the system chaotic in Y ;*
4. *there is $a \in \mathbb{R}$ such that $(A + aI)|_Y$ generates a chaotic semigroup in Y ;*
5. *if $i\mathbb{R} \cap \sigma_p(A) \subset U$, then any periodic point of $(T(t))_{t \geq 0}$ is unstable in the sense that in each neighbourhood of each periodic point there are points producing orbits that are converging to zero or unbounded.*

Proof. Point 1 follows directly from Lemma 2.2 as Y is invariant under $(T(t))_{t \geq 0}$, see [12, Criterion 3.3]. For 2. we observe that there is an ϵ_0 such that for any $0 < \epsilon < \epsilon_0$, $i\mathbb{R} \cap U + \epsilon \neq \emptyset$. Indeed, otherwise we could find a sequence $\epsilon_n \rightarrow 0$ such that $U \cap i\mathbb{R} - \epsilon_n = \emptyset$. Since U is connected, there would be ϵ' such that $U \subset \{z; \Re z \leq -\epsilon_0\}$, contradicting $i\mathbb{R} \cap \partial U \neq \emptyset$. Next, we

observe that the set of eigenvectors $\{\mathbf{x}_\lambda\}_{\lambda \in U}$ of an operator A for eigenvalues in a set U is the same as the set of eigenvectors $\{\mathbf{x}_\mu\}_{\mu \in U+\epsilon}$ of $A + \epsilon I$ in the domain $\epsilon + U$ with $\mathbf{x}_\lambda = \mathbf{x}_{\lambda+\epsilon}^\epsilon$: $\lambda \mathbf{x}_\lambda = A_\lambda \mathbf{x}_\lambda$ if and only if $(\lambda + \epsilon)\mathbf{x}_\lambda = (A + \epsilon I)\mathbf{x}_\lambda$ and $U \ni \lambda \rightarrow \mathbf{x}_\lambda$ is analytic if and only if $U + \epsilon \ni \mu \rightarrow \mathbf{x}_\mu^\epsilon$ is analytic. Therefore, for given ϵ , $\overline{\text{Span}\{\mathbf{x}_\lambda; \lambda \in U\}} = \overline{\text{Span}\{\mathbf{x}_\mu^\epsilon; \mu \in U + \epsilon\}}$. Since the open set $U + \epsilon \subset \sigma_p(A + \epsilon I)$ satisfies the assumptions of point 1, the statement is proved.

The proof of point 3 is the same as of 2., and the proof of 4. follows the same lines with a fixed a such that $a + U \cap i\mathbb{R} \neq \emptyset$. Finally, 5. follows from the spectral mapping theorem for semigroups for the point spectrum, [23], and from the fact that $T(t)u$ is periodic with period τ if and only if $1 \in \sigma_p(T(\tau))$ with the corresponding eigenvector u . ■

3. Birth-and-death type problems with parameter dependent coefficients

In this section we shall discuss the existence of solutions of the following system, that are in some sense behaving uniformly with respect to the parameter so that they are analytic functions of λ valued in $X = L_1(\Omega, l^1)$:

$$\begin{aligned} \lambda F_0(\vartheta) &= -a_0(\vartheta)F_0(\vartheta) + d_1(\vartheta)F_1(\vartheta) \\ \lambda F_n(\vartheta) &= -a_n(\vartheta)F_n(\vartheta) + b_{n-1}(\vartheta)F_{n-1}(\vartheta) + d_{n+1}(\vartheta)F_{n+1}(\vartheta) \quad (n \geq 1), \end{aligned} \tag{1}$$

where Ω is a bounded measurable subset of \mathbb{R}^d , $\vartheta \in \Omega$, $n \in \mathbb{N}_0$, a_n, b_n, c_n are measurable, almost everywhere finite and non-negative functions on Ω . Further, we assume that there exists a (possibly empty) set K of isolated points of Ω such that $a_n, b_n \in L_{1,loc}(\Omega')$ and $1/d_n \in L_{\infty,loc}(\Omega')$, where $\Omega' = \Omega \setminus K$. We assume that for almost every $\vartheta \in \Omega'$ there exist the limits

$$\lim_{n \rightarrow \infty} a_n(\vartheta) = a(\vartheta), \quad \lim_{n \rightarrow \infty} b_n(\vartheta) = b(\vartheta), \quad \lim_{n \rightarrow \infty} d_n(\vartheta) = d(\vartheta),$$

with $b(\vartheta) > 0$ and $d(\vartheta) < +\infty$ for a.e. ϑ . We shall also introduce the limit equation to (1)

$$\begin{aligned} \lambda F_0(\vartheta) &= -a(\vartheta)F_0(\vartheta) + d(\vartheta)F_1(\vartheta) \\ \lambda F_n(\vartheta) &= -a(\vartheta)F_n(\vartheta) + b(\vartheta)F_{n-1}(\vartheta) + d(\vartheta)F_{n+1}(\vartheta) \quad (n \geq 1). \end{aligned} \tag{2}$$

Before we start, it is advantageous to take a quick look at the aims of the analysis to follow. System (1) arises as the eigenvalue problem for an evolution equation in X with the generator given by the right-hand side of (1) (or is somehow related to it). To prove that the corresponding evolution is chaotic we should prove, by Theorem 2.3, that there are solutions to (1) that form an X -analytic

function in some open connected set. In the following sequence of lemmas we shall prove a weaker but still sufficient result that there is a family of X -analytic solutions $(F_n(\vartheta, \lambda))_{n \in \mathbb{N}}$ to (1) with varying domains of analyticity $U_\vartheta \ni \lambda$ that, nevertheless, has the property that the closed linear envelopes of elements of this family with $\Re \lambda \geq 0$ are equal. Thus, the closed linear envelope of $(F_n(\vartheta, \lambda))_{n \in \mathbb{N}}$ can be taken as the chaoticity space Y of Theorem 2.3. An important additional consequence of the construction is that the subspace $Y \subseteq X$ is determined by point-wise in ϑ properties of $(F_n(\vartheta, \lambda))_{n \in \mathbb{N}}$ in l^1 : roughly speaking, the system is chaotic in X if and only if it is chaotic in l^1 for almost each ϑ .

Lemma 3.1. *Assume that for almost every $\vartheta \in \Omega'$, $b(\vartheta) < d(\vartheta)$ and $a(\vartheta) < b(\vartheta) + d(\vartheta)$ and $a(\vartheta) \neq 2\sqrt{b(\vartheta)d(\vartheta)}$. Then, for a.e. $\vartheta \in \Omega'$ there exists an open connected set U_ϑ , $0 \in U_\vartheta \subset \mathbb{C}$ such that any solution to (2) is an l^1 -analytic function of $\lambda \in U_\vartheta$.*

Proof. Since the coefficients of (2) are constant for each ϑ , any solution $\mathbf{F}(\vartheta, \lambda) = (F_n(\vartheta, \lambda))_{n \in \mathbb{N}}$ to (2) is a linear combination of $(\omega_1^n(\vartheta, \lambda))_{n \in \mathbb{N}}$ and $(\omega_2^n(\vartheta, \lambda))_{n \in \mathbb{N}}$, where

$$\omega_{1,2}(\vartheta, \lambda) = \frac{\lambda + a(\vartheta) \pm \sqrt{(\lambda + a(\vartheta))^2 - 4b(\vartheta)d(\vartheta)}}{2d(\vartheta)},$$

so it is sufficient to show that $\max\{\omega_{1,2}(\vartheta, \lambda)\} \leq q(\vartheta) < 1$ for λ from some complex neighbourhood of 0. Let us drop the dependence on ϑ and focus on real λ . To simplify, we denote $\frac{\lambda}{d} = z$, $\frac{a}{d} = \bar{a}$ and $\frac{b}{d} = \bar{b}$ and note that the branching points of $\omega_{1,2}$ are at $z_\pm = -\bar{a} \pm 2\sqrt{\bar{b}}$, with $z_- = -\bar{a} - 2\sqrt{\bar{b}} < 0$; while z_+ may be larger than zero. Let first $-\bar{a} + 2\sqrt{\bar{b}} > 0$ and take $-\bar{a} - 2\sqrt{\bar{b}} < z < -\bar{a} + 2\sqrt{\bar{b}}$. In this case $|\omega_{1,2}(z)| = |(z + \bar{a} \pm \frac{i}{2}\sqrt{-(z + \bar{a})^2 + 4\bar{b}})| = \sqrt{\bar{b}} < 1$ independently of z . Next, if $-\bar{a} + 2\sqrt{\bar{b}} < 0$, then for $z > -\bar{a} + 2\sqrt{\bar{b}}$ we have $|\omega_2(z)| = \frac{1}{2}(z + \bar{a} + \sqrt{(z + \bar{a})^2 - 4\bar{b}})$ as both terms are positive. Differentiating, we check that $\omega_2(z)$ is a strictly increasing function for $z + \bar{a} > 0$, with $\omega_2(z) = 1$ attained at $z = \bar{b} + 1 - \bar{a} > 0$. Hence, if the assumptions are satisfied, there is a closed interval $I \subset (-\bar{a} - 2\sqrt{\bar{b}}, -\bar{a} + 2\sqrt{\bar{b}})$ in the first case, and $I \subset (-\bar{a} - 2\sqrt{\bar{b}}, \bar{b} + 1 - \bar{a})$ in the second case, that contains zero and over which $\omega_2(z) \leq q < 1$ for some constant q . Taking now $\omega_1(z) = \frac{1}{2}(z + \bar{a} - \sqrt{(z + \bar{a})^2 - 4\bar{b}})$ for $z > -\bar{a} - 2\sqrt{\bar{b}}$, we see that $0 < \omega_1(z) < \omega_2(z)$ and therefore there is a closed interval I containing 0 such that $\max\{|\omega_{1,2}(z)|\} \leq q < 1$ on this interval. Since both $|\omega_{1,2}(z)|$ are continuous functions of complex z (as we excluded the branching points at $-\bar{a} \pm 2\sqrt{\bar{b}}$), for each $z \in I$ there is a complex neighbourhood in which $\max\{|\omega_{1,2}(z)|\} \leq q' < 1$ for some fixed q' . Taking U to be the union of these neighbourhoods we obtain a connected open complex neighbourhood of zero, U , where the estimate $z = \max\{|\omega_{1,2}(z)|\} \leq q' < 1$ is valid. Hence the solution $(F_n(\lambda))_{n \in \mathbb{N}}$ to (2) is the uniform in U limit in l^1 norm

of l^1 -analytic (entire) functions $\lambda \rightarrow (F_0(\lambda), F_1(\lambda), \dots, F_n(\lambda), 0, 0, \dots)$ and thus it is an analytic function in U . ■

Lemma 3.2. *Under the assumptions of Lemma 3.1 there exists a nested sequence $(\mathcal{E}_N)_{N \in \mathbb{N}}$ of measurable sets such that $\bigcup_{N=1}^\infty \mathcal{E}_N = \Omega'$ and for each \mathcal{E}_N there is an open complex neighbourhood of 0, U_N , and $q < 1$ such that for all $\lambda \in U_N, \vartheta \in \mathcal{E}_N, \max\{|\omega_{1,2}(\lambda, \vartheta)|\} \leq q$.*

Proof. First let $-a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} < 0$. Define

$$\mathcal{E}_N = \left\{ \vartheta \in \Omega'; \max\{|\omega_{1,2}(\lambda, \vartheta)|\} \leq 1 - \frac{1}{N}, |\lambda| < \frac{1}{N}, -a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} \leq -\frac{2}{N} \right\}.$$

Since $\vartheta \rightarrow \max\{|\omega_{1,2}(\lambda, \vartheta)|\}$ is measurable for each λ , the sets

$$\left\{ \vartheta \in \Omega'; \max\{|\omega_{1,2}(\lambda, \vartheta)|\} \leq 1 - \frac{1}{N}, -a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} \leq -\frac{2}{N} \right\}$$

are measurable for each λ . Thus,

$$\bigcap_{\lambda \in D_{1/N, \mathbb{Q}}} \left\{ \vartheta \in \Omega'; \begin{array}{l} \max\{|\omega_{1,2}(\lambda, \vartheta)|\} \leq 1 - \frac{1}{N}, \\ |\lambda| < \frac{1}{N}, -a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} \leq -\frac{2}{N} \end{array} \right\},$$

where $D_{1/N, \mathbb{Q}}$ is the intersection of the radius $\frac{1}{N}$ disc $D_{1/N}$ with the set of complex numbers with rational real and imaginary parts, is measurable as the intersection is countable. But since the only discontinuity of $\omega_{1,2}(\lambda, \vartheta)$ occurs at the branching points, we see that if $|\vartheta| < \frac{1}{N}$ and $-a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} \leq -\frac{2}{N}$ the function $\lambda \rightarrow \max\{|\omega_{1,2}(\lambda, \vartheta)|\}$ is continuous ($\lambda = -a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)}$ is impossible). Thus, for a given $\vartheta, \max\{|\omega_{1,2}(\lambda, \vartheta)|\} \leq 1 - \frac{1}{N}$ for all λ if and only if it holds for λ with rational coefficients and we can replace the rational lattice in the above intersection with $D_{1/N}$ so that \mathcal{E}_N are measurable. Next, it is clear that if $N < M$, then $\mathcal{E}_N \subset \mathcal{E}_M$. Finally, if $\vartheta \in \Omega'$, then, from the previous lemma and the assumptions, there are M, N, R such that $\max\{|\omega_{1,2}(\lambda, \vartheta)|\} \leq 1 - \frac{1}{M}, |\lambda| < \frac{1}{N}$ and $-a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} \leq -\frac{1}{R}$, that is $\vartheta \in \mathcal{E}_{\max\{N, M, R\}}$, which shows that the family $(\mathcal{E}_N)_{N \in \mathbb{N}}$ exhausts Ω' .

If let $-a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} > 0$, then the argument is similar, with the only difference that we have to restrict ϑ with $-a(\vartheta) + 2\sqrt{b(\vartheta)d(\vartheta)} > \frac{2}{N}$ and $-a(\vartheta) - 2\sqrt{b(\vartheta)d(\vartheta)} < -\frac{2}{N}$. ■

Lemma 3.3. *Under the assumptions and notations of the previous lemmas, let us fix an arbitrary \mathcal{E}_N with the corresponding neighbourhood of zero U_N . Then for every $r > 0$ there exists a measurable set E_r such that $\mu(\mathcal{E}_N \setminus E_r) < \frac{1}{r}$ and a solution $\mathbf{F}^{E_r}(\lambda) = (F_n^{E_r}(\lambda))_{n \in \mathbb{N}}$ of (1) that is an analytic X -valued function of $\lambda \in U_N$. Moreover, $E_{r_1} \subset E_{r_2}$ for $r_1 > r_2, \bigcup_{r>0} E_r = \mathcal{E}_N, \mathbf{F}^{E_r}(\lambda, \vartheta) = 0$ for $\vartheta \notin E_r$.*

Proof. First, let us fix $\vartheta \in \mathcal{E}_N$ and drop dependence on ϑ in this part of the proof. System (1) can be written as the following first order system of difference equations:

$$F_{n+1} = \mathcal{A}_n F_n \quad (n \geq 1), \quad F_0 = \begin{bmatrix} \frac{\lambda+a_0}{d_1} \\ 1 \end{bmatrix} F_0$$

where

$$F_{n+1} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}, \quad \mathcal{A}_n = \begin{bmatrix} \frac{\lambda+a_n}{d_{n+1}} & -\frac{b_{n-1}}{d_{n+1}} \\ 1 & 0 \end{bmatrix} \quad (n \geq 1).$$

Due to the assumptions, \mathcal{A}_n can be written as $\mathcal{A}_n = \mathcal{A} + \mathcal{B}_n$ where

$$\mathcal{A} = \begin{bmatrix} \frac{\lambda+a}{d} & -\frac{b}{d} \\ 1 & 0 \end{bmatrix}, \quad \mathcal{B}_n = \begin{bmatrix} \alpha_n(\lambda) & \beta_n \\ 0 & 0 \end{bmatrix},$$

with $\alpha_n(\lambda) = (\lambda(d - d_{n+1}) + da_n - ad_{n+1})/dd_{n+1}$ and $\beta_n = bd_{n+1} - db_{n-1}$. It is clear that $\alpha_n \rightarrow 0$ uniformly for $\lambda \in U_N$ (as U_N is bounded) and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$ so $\|\mathcal{B}_n\| \rightarrow 0$ uniformly in λ (for any matrix norm). From Lemma 3.2 there is a $0 < \delta < 1$, independent of $\lambda \in U_N$ for which $\|\mathcal{A}^n\| \leq c\delta^n$ for all $n \geq 1$. Also, for any c_1 there is an n_0 such that $\|\mathcal{B}_n\| \leq c_1$ for $n \geq n_0$, uniformly in $\lambda \in U_N$. Following the proof of the stability theorem for difference systems, [1, Theorem 5.2.3], we obtain $\|F_n\| \leq C(n_0)(\delta(1 + cc_1/\delta))^{n-n_0}$ for some constant $C(n_0)$ independent of λ so that one can pick c_1 such that $\delta(1 + cc_1/\delta) = q < 1$. Hence $|F_n(\lambda)| \leq Cq^n$ for $n > n_0$ and arguing as in Lemma 3.1, we see that $\lambda \rightarrow \mathbf{F}(\lambda) = (F(\lambda)_n)_{n \in \mathbb{N}}$ is an l^1 -analytic in U_N .

For the next part of the proof we return to the dependence on ϑ . By the above, for almost any $\vartheta \in \mathcal{E}_N$ there are $C(\vartheta)$ and $q(\vartheta) < 1$ such that $|F_n(\vartheta, \lambda)| \leq C(\vartheta)q^n$ for all $n \geq 1$, uniformly for $\lambda \in U_N$, thus we have a family of l^1 -analytic functions $\mathbf{F}(\lambda, \vartheta)$. To prove the second part of the lemma, let us take a sequence of non-negative numbers $(q_r)_{r \in \mathbb{N}}$ such that $q_r \nearrow 1$. Consider

$$E_r = \{\vartheta \in \mathcal{E}_N; |F_n(\vartheta, \lambda)| \leq rq_r^n, \lambda \in U_N, n \geq 1\}$$

First, the sets E_r form a nested sequence with $\mu(\mathcal{E}_N \setminus \bigcup_{r=1}^\infty E_r) = 0$. Since $E_r = \bigcap_{n \geq 1, \lambda \in U_N} \{\vartheta \in \mathcal{E}_N; |F_n(\vartheta, \lambda)| \leq rq_r^n\}$ and from the recurrence formula, the functions F_n are measurable with respect to ϑ provided F_0 and all the coefficients are measurable, we see, as in the proof of Lemma 3.2, that each E_r is measurable with $\mu(\mathcal{E}_N \setminus E_r) \rightarrow 0$.

Next we observe that each F_n is uniquely determined by F_0 so that we can write $F_n^{F_0}$, moreover, for a given F_0 and a characteristic function χ_E of any $E \subset \Omega'$ we have $F_n^{\chi_E F_0} = \chi_E F_n^{F_0}$. This can be immediately checked for $n = 0, 1$ and then by induction for arbitrary n as each F_n is a linear combination of F_{n-1} and F_{n-2} . Thus, starting with $F_0 = \chi_{\mathcal{E}_N}$, we find a collection $(E_r)_{r \in \mathbb{N}}$ of sets on

which we have $|F_n^{\chi_{E_r}}(\vartheta, \lambda)| = |\chi_{E_r} F_n^{\chi_{\mathcal{E}_N}}(\vartheta, \lambda)| = |\chi_{E_r} F_n^{\chi_{\Omega'}}(\vartheta, \lambda)| \leq r q_r^n$ which means that $\lambda \rightarrow \mathbf{F}^{\chi_{E_r}}(\lambda) = (F_n^{\chi_{E_r}}(\lambda))_{n \in \mathbb{N}}$ is an analytic $L_1(\Omega)$ -valued function for $\lambda \in U_N$. ■

Denote $\mathcal{F}_N = \text{Span}\{\mathbf{F}^{\chi_{E_r}}(\lambda); E_r \subset \mathcal{E}_N, \lambda \in U_N\}$, where E_r are the sets constructed in the previous lemma and let $\mathcal{F} = \overline{\{\mathcal{F}_N; N \in \mathbb{N}\}}$, the closure in X . By $\mathcal{E}_{\vartheta, \lambda \in U_\vartheta}$ we denote the set of all l^1 -analytic solutions in U_ϑ to (1) for a fixed $\vartheta \in \Omega'$ and put

$$X_\vartheta = \overline{\text{Span } \mathcal{E}_{\vartheta, \lambda \in U_\vartheta}}. \tag{3}$$

Lemma 3.4. $\Phi \in X^*$ annihilates \mathcal{F} if and only if $\Phi(\vartheta)$ annihilates $\mathcal{E}_{\vartheta, \lambda \in U_\vartheta}$ for almost all $\vartheta \in \Omega'$.

Proof. Using the fact that the space X is of type (L), [21, pp. 69-70], and the Cauchy formula, we obtain that if \mathbf{A}_k is the k -th coefficient of the Taylor expansion of $\mathbf{F}^{\chi_{E_r}}(\lambda)$ at $\lambda = 0$, and $A_k(\vartheta)$ is the k -th coefficient of the Taylor expansion of $F^{\chi_\Omega}(\vartheta, \lambda)$ at $\lambda = 0$ for a.a. $\vartheta \in \Omega'$, then $[\mathbf{A}_k](\vartheta) = \chi_{E_r} A_k(\vartheta)$.

Passing now to the proof of the lemma, $\Phi \in X^*$ annihilates \mathcal{F} if and only if Φ annihilates \mathcal{F}_N for any N . By Proposition 2.2 it is equivalent to

$$\int_{\Omega} \chi_{E_r} \langle \Phi(\vartheta), A_k(\vartheta) \rangle d\vartheta = 0$$

for any k and any $E_r \subset \mathcal{E}_N$. Since sets E_r and \mathcal{E}_N exhaust Ω' , we see that if Φ annihilates \mathcal{F} , then for almost every ϑ , $\Phi(\vartheta)$ annihilates $\mathcal{E}_{\vartheta, \lambda \in U_\vartheta}$. The converse is immediate. ■

For further use it is convenient to rephrase this result in a slightly different way. Define

$$\mathcal{A}_N = \{\mathbf{F}^{\chi_{E_r}}(\lambda); E_r \subset \mathcal{E}_N, \lambda \in U_N\}$$

and denote by $\mathcal{A}_N^{\pm, 0}$ subsets of \mathcal{A}_N containing elements with $\Re \lambda$ positive, negative and 0, respectively. Further, denote $\mathcal{A} = \bigcup_{N=1}^{\infty} \mathcal{A}_N$ and $\mathcal{A}^{\pm, 0} = \bigcup_{N=1}^{\infty} \mathcal{A}_N^{\pm, 0}$.

Corollary 3.5. $\overline{\text{Span } \mathcal{A}^{\pm, 0}} = \mathcal{F}$.

Proof. Follows as the previous proof by Proposition 2.2. ■

4. Chaos in linear kinetic models

This section is devoted to the possibility of chaos to occur in linear kinetic equations describing inelastic collisions in the extended kinetic theory. Here we consider a gas of test particles of mass m endowed only with translational degrees of freedom propagating through a three-dimensional host medium of much heavier particles that may have a quite complicated internal structure

and thus non-negligible internal degrees of freedom. Test particles collide with host medium losing or gaining a unit of energy ΔE at each collision. The particle distribution function in the space homogeneous case is governed by the linear Boltzmann equation, see e.g. [5, 18],

$$\frac{\partial f}{\partial t} = C^i f, \tag{4}$$

where

$$\begin{aligned} [C^i f](v\boldsymbol{\omega}) &= \int_{S^2} \left[n_1 \frac{v_+}{v} \sigma(v_+, \boldsymbol{\omega} \cdot \boldsymbol{\omega}') f(v_+ \boldsymbol{\omega}') \right. \\ &\quad \left. + n_2 \frac{v_-}{v} \nu(v_-, \boldsymbol{\omega} \cdot \boldsymbol{\omega}') H(v - \delta) f(v_- \boldsymbol{\omega}') \right] d\boldsymbol{\omega}' \\ &\quad - f(v\boldsymbol{\omega}) \int_{S^2} [n_1 \sigma(v, \boldsymbol{\omega} \cdot \boldsymbol{\omega}') H(v - \delta) + n_2 \nu(v, \boldsymbol{\omega} \cdot \boldsymbol{\omega}')] d\boldsymbol{\omega}' \end{aligned} \tag{5}$$

is the inelastic collision operator. Here, $\mathbf{v} = v\boldsymbol{\omega}$ is the velocity variable, with modulus v and direction $\boldsymbol{\omega} \in S^2$ (the unit sphere in \mathbb{R}^3), $v_{\pm} = \sqrt{v^2 \pm \delta^2}$, $\delta^2 = 2\Delta E/m$. Also σ and ν are the inelastic collision frequencies (for the endothermic and exothermic process, respectively), n_1 and n_2 are the number densities of the particles in the ground and excited state, respectively (which are assumed constant) related by $n_2/n_1 = e^{-\Delta E/KT} < 1$, where K is the Boltzmann constant and T is the background temperature, and H is the Heaviside function.

We shall consider an isotropic medium, eliminating thus the angle dependence from the scattering cross-sections. This will allow to use the recent results [9, 10, 12] on chaos in birth-and-death models to identify cases when (5) with arbitrary small production of particles gives rise to chaotic dynamics.

In what follows it will be more convenient to use the normalized kinetic energy $\zeta = \frac{1}{2}v^2$. Since the original equation is posed in the space $X = L_1(\mathbb{R}^3, d\mathbf{v})$, in $(\zeta, \boldsymbol{\omega})$ variables we have $X = L_1(\mathbb{R}_+ \times S^2, \sqrt{\zeta} d\zeta d\boldsymbol{\omega})$. We also normalize the energy jump to 1. Thus, in the isotropic case (5) can be written as

$$\begin{aligned} \frac{\partial f}{\partial t}(\zeta, \boldsymbol{\omega}, t) &= n_1 \sqrt{\frac{\zeta+1}{\zeta}} \sigma(\zeta+1) \int_{S^2} f(\zeta+1, \boldsymbol{\omega}', t) d\boldsymbol{\omega}' \\ &\quad + n_2 \sqrt{\frac{\zeta-1}{\zeta}} \nu(\zeta-1) H(\zeta-1) \int_{S^2} f(\zeta-1, \boldsymbol{\omega}', t) d\boldsymbol{\omega}' \\ &\quad - 4\pi f(\zeta, \boldsymbol{\omega}, t) (n_2 \nu(\zeta) + n_1 H(\zeta-1) \sigma(\zeta)). \end{aligned} \tag{6}$$

To eliminate the weight $\sqrt{\zeta}$ from the space, we shall define $F(\zeta, \boldsymbol{\omega}) = \sqrt{\zeta} f(\zeta, \boldsymbol{\omega})$

getting

$$\begin{aligned} \frac{\partial F}{\partial t}(\zeta, \boldsymbol{\omega}, t) &= [BF(\cdot, \cdot, t)](\zeta, \boldsymbol{\omega}) - N(\zeta, \boldsymbol{\omega})F(\zeta, \boldsymbol{\omega}, t) \\ &= n_1\sigma(\zeta + 1) \int_{S^2} F(\zeta + 1, \boldsymbol{\omega}', t) d\boldsymbol{\omega}' \\ &\quad + n_2\nu(\zeta - 1)H(\zeta - 1) \int_{S^2} F(\zeta - 1, \boldsymbol{\omega}', t) d\boldsymbol{\omega}' \\ &\quad - 4\pi F(\zeta, \boldsymbol{\omega}, t)(n_2\nu(\zeta) + n_1H(\zeta - 1)\sigma(\zeta)), \end{aligned} \tag{7}$$

where the meaning of the operator B and the function N follows from the above formula. It is advantageous to write this problem in the form of an infinite system of equations by introducing the reduced energy $\xi \in [0, 1[$, $\zeta = \xi + n$ and define $F_n(\xi, \boldsymbol{\omega}) = F(\xi + n, \boldsymbol{\omega})$. We shall adopt the same convention to all other functions of ζ that appear in the problem. Note that the norm changes now to

$$\|F\| = \sum_{n=0}^{\infty} \|F_n\|_{L_1([0,1] \times S^2)} \tag{8}$$

and the equation (7) takes the form

$$\begin{aligned} \partial_t F_0(\xi, \boldsymbol{\omega}, t) &= -4\pi n_2\nu_0(\xi)F_0(\xi, \boldsymbol{\omega}, t) + n_1\sigma_1(\xi) \int_{S_2} F_1(\xi, \boldsymbol{\omega}', t) d\boldsymbol{\omega}' \\ \partial_t F_n(\xi, \boldsymbol{\omega}, t) &= -4\pi(n_2\nu_n(\xi) + n_1\sigma_n(\xi))F_n(\xi, \boldsymbol{\omega}, t) \\ &\quad + n_1\sigma_{n+1}(\xi) \int_{S_2} F_{n+1}(\xi, \boldsymbol{\omega}', t) d\boldsymbol{\omega}' \\ &\quad + n_2\nu_{n-1}(\xi) \int_{S_2} F_{n-1}(\xi, \boldsymbol{\omega}', t) d\boldsymbol{\omega}' \quad (n \geq 1). \end{aligned} \tag{9}$$

Note that the transformation $F \rightarrow (F_n)_{n \in \mathbb{N}}$ described above is an isomorphism between X and the space \mathcal{X} of functional sequences defined by the finiteness of the norm (8). Thus, in the sequel we shall move between these two descriptions without additional explanations. The solvability of the Cauchy problem for (5) falls into the domain of substochastic semigroups and was treated recently in several papers, see e.g. [4, 2, 17], where in general if $\sigma, \nu \in L_{1,loc}(S^2 \times [0, +\infty))$, then there is an extension $(\mathcal{K}, D(\mathcal{K}))$ of $(K, D(N)) = (-N + B, D(N))$, where $D(N) = \{f \in L_1(S^2 \times [0, \infty)); Nf \in L_1(S^2 \times [0, \infty))\}$ that generates a semigroup $(G_K(t))_{t \geq 0}$ solving a realization of (7). We shall need a relation between \mathcal{K} and the maximal operator \mathcal{K}_{max} defined by the same expression $-N + B$ but on the domain

$$\begin{aligned} D_{max} &= \{f \in L_1(S^2 \times [0, \infty)); [BF(\cdot, \cdot)](\zeta, \boldsymbol{\omega}), N(\zeta, \boldsymbol{\omega})F(\zeta, \boldsymbol{\omega}) \\ &\quad \text{are finite a.e, and} \\ &\quad (\zeta, \boldsymbol{\omega}) \rightarrow [BF(\cdot, \cdot)](\zeta, \boldsymbol{\omega}) - N(\zeta, \boldsymbol{\omega})F(\zeta, \boldsymbol{\omega}) \in L_1(S^2 \times [0, \infty))\}. \end{aligned} \tag{10}$$

In general, [6], $\mathcal{K} \neq \mathcal{K}_{max}$. However, we have

Proposition 4.1. *If $N \in L_{1,loc}(S^2 \times [0, \infty)) \cap L_\infty(S^2 \times [\delta, \infty))$ for any $\delta > 0$, then $\mathcal{K} = \mathcal{K}_{max}$.*

Proof. By [3, 21], if $D(K) \subsetneq D_{max}$, then for any $\lambda > 0$ there is a solution $F \in L_1(S^2 \times [0, \infty))$ satisfying

$$\begin{aligned} \lambda F_0(\xi, \omega) &= -4\pi n_2 \nu_0(\xi) F_0(\xi, \omega) + n_1 \sigma_1(\xi) \int_{S^2} F_1(\xi, \omega') d\omega' \\ \lambda F_n(\xi, \omega) &= -4\pi(n_2 \nu_n(\xi) + n_1 \sigma_n(\xi)) F_n(\xi, \omega) \\ &\quad + n_1 \sigma_{n+1}(\xi) \int_{S^2} F_{n+1}(\xi, \omega') d\omega' \\ &\quad + n_2 \nu_{n-1}(\xi) \int_{S^2} F_{n-1}(\xi, \omega') d\omega' \quad (n \geq 1). \end{aligned} \tag{11}$$

Putting $F_n = F_{n0} + F_{n1}$, where $F_{n0}(\xi) = \frac{1}{4\pi} \int_{S^2} F_n(\xi, \omega) d\omega$ and $\int_{S^2} F_{n1}(\xi, \omega) d\omega = 0$, we split (11) as

$$\begin{aligned} \bar{\lambda} F_{00}(\xi) &= -n_2 \nu_0(\xi) F_{00}(\xi) + n_1 \sigma_1(\xi) F_{10}(\xi) \\ \bar{\lambda} F_{n0}(\xi) &= -(n_2 \nu_n(\xi) + n_1 \sigma_n(\xi)) F_{n0}(\xi) + n_1 \sigma_{n+1}(\xi) F_{n+1,0}(\xi) \\ &\quad + n_2 \nu_{n-1}(\xi) F_{n-1,0}(\xi) \quad (n \geq 1), \end{aligned} \tag{12}$$

where we denoted $\bar{\lambda} = \frac{\lambda}{4\pi}$, and

$$\begin{aligned} \bar{\lambda} F_{01}(\xi, \omega) &= -n_2 \nu_0(\xi) F_{01}(\xi, \omega) \\ \bar{\lambda} F_{n1}(\xi, \omega) &= -(n_2 \nu_n(\xi) + n_1 \sigma_n(\xi)) F_{n1}(\xi, \omega) \quad (n \geq 1). \end{aligned} \tag{13}$$

In both systems the equality is in $L_1([0, 1) \times S^2)$ so that the equality in each component is satisfied almost everywhere. Since we have two countable systems of equations, we can find a set of full measure $A \subset [0, 1)$ for (12) and $B \subset [0, 1) \times S^2$ for (13) on which the equality in all the components hold simultaneously. Hence for a fixed $\xi \in A$, (12) can be viewed as the equation for eigenvectors and eigenvalues for a birth-and-death system with bounded coefficients. For such a system we know that the generator is the maximal operator as it is bounded. Thus, the only solution to such equation in l^1 for $\lambda > 0$ is the zero solution. Consequently, $F_0(\xi) = 0$ a.e. on $[0, 1)$. For (13), $F_1(\xi, \omega) = 0$ a.e. on $[0, 1) \times S^2$. Thus $F(\xi, \omega) = F_0(\xi) + F_1(\xi, \omega) = 0$ a.e. on $[0, 1) \times S^2$. ■

To investigate possible chaotic behaviour of solutions of (7) we consider the related eigenvalue problem (11). Thanks to Proposition 4.1, we see that the integrable solutions to (11) are the eigenfunctions of the generator. Splitting

this system as above, we observe that (11) is irrelevant. The first one is the same as (1) with $\vartheta = (\xi, \boldsymbol{\omega})$ and $\Omega = [0, 1] \times S^2$ (we must keep $\boldsymbol{\omega}$ to be able to work in the whole space). Let us assume that $\lim_{n \rightarrow \infty} \sigma_n(\xi) = \sigma(\xi) > 0$ and $\lim_{n \rightarrow \infty} \nu_n(\xi) = \nu(\xi) > 0$ for almost any ξ . Then $a(\xi) = 4\pi(n_2\nu(\xi) + n_1\sigma(\xi))$, $b(\xi) = 4\pi n_2\nu(\xi)$ and $d(\xi) = 4\pi n_1\sigma(\xi)$. To be able to apply the theory of the previous section, we require for a.e. ξ

$$\frac{n_2 \nu(\xi)}{n_1 \sigma(\xi)} < 1. \tag{14}$$

Condition $a(\xi) < b(\xi) + d(\xi)$ is obviously not satisfied, as $a(\xi) = b(\xi) + d(\xi)$, which is not surprising as the model is conservative. However, if we introduce an arbitrarily small positive perturbation of the right-hand side of (4), say ϵf , so that

$$\frac{\partial f}{\partial t} = \epsilon f + C^i f, \tag{15}$$

then a becomes $a_\epsilon(\xi) = 4\pi(n_2\nu(\xi) + n_1\sigma(\xi)) - \epsilon$, and this assumption will be satisfied.

Next, the condition $a_\epsilon(\xi) \neq 2\sqrt{b(\xi)d(\xi)}$ here takes the form $n_2\nu(\xi) + n_1\sigma(\xi) - \epsilon - 2\sqrt{n_1n_2\nu(\xi)\sigma(\xi)} \neq 0$ that is $(\sqrt{n_2\nu(\xi)} - \sqrt{n_1\sigma(\xi)})^2 \neq \epsilon$ which introduces a slightly stronger condition than (14):

$$n_1\sigma(\xi) > n_2\nu(\xi) + \epsilon. \tag{16}$$

Clearly, we could also assume $\epsilon > (\sqrt{n_2\nu(\xi)} - \sqrt{n_1\sigma(\xi)})^2$, but our main interest is in small perturbations of (4).

Let us denote by X_ξ the closed linear envelope of all analytic solutions to (12) for a fixed ξ , as in (3). We can now formulate

Theorem 4.2. *Let the assumption (16) be satisfied for some ϵ_0 . Then for any $\epsilon \in (0, \epsilon_0)$, there is a subspace \mathcal{Y} of $L_1(\mathbb{R}^3)$ such that the dynamical system generated by (15) is chaotic in \mathcal{Y} . Moreover, $f \in \mathcal{Y}$ if and only if $(f_n(\xi, \boldsymbol{\omega}))_{n \in \mathbb{N}} \in X_\xi$ for almost any $\xi \in [0, 1]$, $\boldsymbol{\omega} \in S^2$. In particular, if for almost any ξ , $X_\xi = l^1$, then the system is chaotic in $L_1(\mathbb{R}^3)$.*

Proof. The result follows immediately from Corollary 3.5 and Theorem 2.3 as the sets $\mathcal{A}^{\pm,0}$ contain eigenvectors of the generator with positive, negative and zero real parts of the corresponding eigenvalues. ■

In [9] we proved that a birth-and-death system with proliferation is chaotic in l^1 when the birth coefficient b_n is quickly decaying to zero and the coefficients a_n and b_n are close to a constant. Specifying these assumptions to the present case we obtain the following result.

Corollary 4.3. *If for almost any $\xi \in [0, 1]$*

$$\frac{|\sigma_n(\xi) - \sigma(\xi)|}{\sigma(\xi)} \leq q^{n+1}, \quad \frac{n_2 \nu_n(\xi) \sigma_n(\xi)}{n_1 \sigma(\xi) \sigma(\xi)} \leq q^{2(n+1)}, \tag{17}$$

where q is any constant smaller than $2 - \sqrt{3}$, $n_1 \sigma(\xi) \geq \epsilon_0$ for some $\epsilon_0 > 0$, then for $0 < \epsilon < \epsilon_0$ the dynamical system generated by (15) is chaotic in $L_1(\mathbb{R}^3)$.

Proof. The conditions of [9] require $\frac{n_2 \nu_n(\xi) \sigma_n(\xi)}{n_1 \sigma(\xi) \sigma(\xi)} \leq Q^{2(n+1)}$ and

$$\frac{n_2 \nu_n(\xi)}{n_1 \sigma(\xi)} + \frac{|\sigma_n(\xi) - \sigma(\xi)|}{\sigma(\xi)} \leq Q^{n+1}, \tag{18}$$

where $Q = \frac{\sqrt{3}-1}{2}$ and

$$\inf_{n \in \mathbb{N}} \sigma(\xi)^{-n} \prod_{i=1}^n \sigma_i(\xi) > 0. \tag{19}$$

We must prove that (18) is satisfied. Let us fix ξ and drop it from the notation in the proof. Denoting $\frac{n_2 \nu_n(\xi)}{n_1 \sigma(\xi)} = a_n$ and $b_n = \frac{\sigma_n(\xi)}{\sigma(\xi)}$, we have

$$a_n + |b_n - 1| \leq \frac{q^{2(n+1)}}{1 - q^{n+1}} + q^{n+1} = \frac{q^{n+1}}{1 - q^{n+1}}.$$

It is clear, that if $Q/q < 1$, then $1/(1-q) < Q/q$ yields $1/(1-q^{n+1}) < Q^{n+1}/q^{n+1}$, as the left-hand side of the last inequality decreases and the right hand side increases with n . Taking $Q = \frac{1}{2}(\sqrt{3} - 1)$ we see, that we must have $q < 2 - \sqrt{3}$. Condition (19) requires $\inf_{n \in \mathbb{N}} \prod_{i=1}^n b_i > 0$. However, by (17) and positivity of σ_n we obtain $0 < b_n < 1 + q^{n+1}$ with $q < 1$, so that $\prod_{i=1}^\infty b_i$ converges and (19) is satisfied. ■

In many applications the up-scattering and down-scattering cross-sections are related through the so-called microreversibility conditions that in the present case read $\sqrt{\zeta} \nu(\zeta) = \sqrt{\zeta + 1} \sigma(\zeta + 1)$, see e.g. [5]. Using the reduced energy formulation, we obtain $\nu_n(\xi) = D_{n+1}(\xi) \sigma_{n+1}(\xi)$, where $D_{n+1}(\xi) = \sqrt{\frac{\xi+n+1}{\xi+n}}$. System (12) takes then the form

$$\begin{aligned} \lambda F_{00}(\xi) &= -4\pi n_2 D_1(\xi) \sigma_1(\xi) F_{00}(\xi) + 4\pi n_1 \sigma_1(\xi) F_{10}(\xi) \\ \lambda F_{n0}(\xi) &= -4\pi(n_2 D_{n+1}(\xi) \sigma_{n+1}(\xi) + n_1 \sigma_n(\xi)) F_{n0}(\xi) \\ &\quad + 4\pi n_1 \sigma_{n+1}(\xi) F_{n+1,0}(\xi) + 4\pi n_2 D_n(\xi) \sigma_n(\xi) F_{n-1,0}(\xi). \end{aligned} \tag{20}$$

Since the coefficient $D_1(\xi)$ is singular at 0, we shall work in $\Omega' = (0, 1) \times S^2$. In this case $a(\xi) = -4\pi \sigma(\xi)(n_2 + n_1)$, $d(\xi) = 4\pi n_1 \sigma(\xi)$ and $b(\xi) = 4\pi n_2 \sigma(\xi)$. Condition (16) is satisfied if $\sigma(\xi) \geq \epsilon_0 > 0$ for some ϵ_0 and again we shall consider the perturbation (15) with $\epsilon > 0$. We see that this case fits into the framework of Theorem 4.2 so that the dynamical system generated by (15) is chaotic in a subspace of $L_1(\mathbb{R}^3)$ determined by the pointwise behaviour of (20).

Remark 4.4. We should note that in the discussed applications only non-negative solutions make sense and it is only fair to point that such solutions cannot display the chaotic properties described here. In fact, since the kinetic part is conservative, the L_1 norm of any non-negative solution to (15) must grow as e^{ct} and hence the solution cannot wander. On the other hand, as we are dealing with linear systems, the difference between two physical (non-negative) solutions may be of varying sign and therefore may be chaotic.

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