Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 24 (2005), No. 4, 763–789

Dirichlet and Hardy Spaces of Harmonic and Monogenic Functions

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Abstract. In this paper we obtain a characterization of the Dirichlet $\mathbf{D}_p$-spaces of monogenic Clifford algebra valued functions in the unit ball in $\mathbb{R}^{m+1}$ by the coefficients of a homogeneous series expansion.

Keywords: *Holomorphic functions, monogenic functions, function spaces, Dirichlet spaces, Hardy spaces*

MSC 2000: Primary 30C45, secondary 30G35

1. Introduction

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ be the class of analytic functions on U. For $p \in \mathbb{R}$, consider the well-known fractional *Dirichlet space* D_p, defined by

$$D_p = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A} \,:\, \sum_{n=1}^{\infty} n^{1-p} |a_n|^2 < \infty \right\}.$$

For $p = 0$, we obtain the well-known classical Dirichlet space and it is easy to show that (see [7, p. 28])

$$\sum_{n=1}^{\infty} n|a_n|^2 = \iint_U |f'(z)|^2 dx\, dy$$

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ISSN 0232-2064 / $ 2.50

and this integral is equal to the area of the image $f(U)$, counting multiplicities. In 1980 Stegenga proved (see [11, p. 114]) the next important result.

Theorem 1.1. *Let $f : U \to \mathbb{C}$ be an analytic function and let $-1 < p$, then $f \in D_p$ if and only if*

$$\iint_U |f'(z)|^2 \big(1 - |z|^2\big)^p dx\, dy < \infty.$$

The space D_p is a Banach space under the norm

$$\|f\|_{D_p} := |f(0)| + \left(\iint_U |f'(z)|^2 (1 - |z|^2)^p dx\, dy \right)^{\frac{1}{2}}.$$

D_p-spaces belong to the family of the so-called weighted function spaces and they are intensively studied in the recent years by several authors as R. Aulaskari [2], K. Stroethoff [12], K. Zhu [15] among others. These spaces constitute an important tool to clarify and explore the behaviour of functions near to the boundary. In a parallel way M.V. Shapiro et al. [8], Malonek et al. [9] and Cnops et al. [6] have worked on generalizations of these scales of spaces to Clifford algebra valued functions defined in $\mathbb{R}^n, n > 2$.

Due to the double characterization by series expansions and by integral expressions given by Theorem 1.1, D_p-spaces become important because through these characterizations it is possible to give a precise answer to an elementary question: To which spaces belong derivatives and primitives of D_p-functions? It is easy to prove (see [4] that if $f^{(k)}$ is the k-th derivative and $F_{(k)}$ the k-th primitive of $f \in D_p$, then $f^{(k)} \in D_{p+2k}$ and $F_{(k)} \in D_{p-2k}$.

In this paper we present a generalization of this concept to D_p-spaces of harmonic or monogenic Clifford algebra valued functions. Thus in Sections 2 and 3 we introduce the harmonic Dirichlet spaces and prove that Theorem 1.1 can be extended to these kind of functions. In Section 4 we prove that for different values of $p > -1$ these spaces form a scale and the inlusions are strict. Finally, in Section 5 we find that if f is a monogenic function in D_p and admits a harmonic primitive, then the results for analytic functions extend to this case, i.e., the primitive belongs to D_{p-2}. It is important to remark an essential difference between analytic and monogenic functions. Monogenic functions can have several primitives. In [5] and explicitly in [9] it is proved that any monogenic function has a monogenic primitive. But two monogenic primitives of a monogenic function can differ by functions and not only by a constant, and they can have very different regularity. The same is true for harmonic primitives of monogenic functions.

Remark 1.2. We have chosen the denotations for the different situations that occur in our paper in the following way: Not to confuse the reader we denote by $D_p(B)$ the Dirichlet spaces of holomorphic functions in the unit disk, the Dirichlet spaces of *harmonic* functions in the unit ball $B \subset \mathbb{R}^{m+1}$ are denoted by $\mathbf{D}_p(B)$ and the Dirichlet spaces of *harmonic Clifford-valued* functions in the unit ball by $\mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$. The Bloch spaces B^α, the Bergman spaces b^2 and the Hardy spaces h^2 are denoted in the same manner. Opposite to the other denotations the Dirichlet spaces of *monogenic Clifford valued* functions in the unit ball are denoted by $\mathbf{D}_p(\mathcal{M}, \mathbf{Cl}_{0,n}, B)$.

1.1. Clifford algebras. Let $e_1, \dots, e_n$ be the elements of an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ and $\mathbf{Cl}_{0,n}$ the 2^n-dimensional universal Clifford algebra over $\mathbb{R}$ generated modulo the relation $x^2 = -|x|^2 e_0$, where $x \in \mathbb{R}^n$ and e_0 is the identity of $\mathbf{Cl}_{0,n}$. The corresponding multiplication rules are given by $e_i e_j + e_j e_i = -2\delta_{ij}, i, j = 1, \dots, n$. Then the set $\{e_A : A \subset \{1, \dots, n\}\}$ with $e_A = e_{h_1}, \dots e_{h_r}$, $1 \le h_1 < \cdots < h_r \le n$, $e_\varphi = e_0 = 1$, forms a basis of $\mathbf{Cl}_{0,n}$, and therefore each element $a \in \mathbf{Cl}_{0,n}$ can be represented in the form $a = \sum_A a_A e_A$, where a_A are real numbers. A conjugate element to a is defined by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{h_r} \dots \overline{e}_{h_r}$, $\overline{e}_k = -e_k$, $k = 1, \dots, n$, $\overline{e}_0 = e_0 = 1$. Given a Clifford number $a = \sum_A a_A e_A$ its norm is given by $|a|_0 = (\sum_A a_A^2)^{\frac{1}{2}}$.

1.2. Clifford analysis. Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set. For $m \le n$, let D denote the generalized Cauchy-Riemann operator and $\overline{D}$ its conjugate operator:

$$
\begin{aligned}
D &= \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + \cdots + e_m \frac{\partial}{\partial x_m} \\
\overline{D} &= \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - \cdots - e_m \frac{\partial}{\partial x_m} .
\end{aligned}
$$

A function $f \in C^1(\Omega; \mathbf{Cl}_{0,n})$ is said to be left monogenic if Df=0. We denote by $\mathcal{M}(\mathbf{Cl}_{0,n})$ the set of left monogenic functions. Consider the real Euclidean space $\mathbb{R}^{m+1}$ with norm $|x| = \sqrt{x_0^2 + x_1^2 + \cdots + x_m^2}$ for $x = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$. As above, we define the ball of radius $r > 0$ $B_r = \{ x \in \mathbb{R}^{m+1} : |x| < r \}$. We denote $B_1 = B$ and its boundary by S. For $-1 < p$, consider the fractional *Dirichlet space* $\mathbf{D}_p(\mathcal{M}, \mathbf{Cl}_{0,n}, B)$ of left monogenic functions, defined by

$$
\mathbf{D}_p(\mathcal{M}, \mathbf{Cl}_{0,n}, B) = \left\{ f \in \mathcal{M}(\mathbf{Cl}_{0,n}) : \int_B |\overline{D} f|_0^2 \big(1 - |x|^2\big)^p \, dV < \infty \right\}.
$$

We observe that $\overline{D} f = \sum_{i,\ A} \partial_i f_A \overline{e_i} e_A$, and then

$$
|\overline{D} f|_0^2 \le (m+1) \sum_{i,A} (\partial_i f_A)^2 = (m+1) \sum_A |\nabla f_A|^2, \tag{1.1}
$$

where ∇ denotes the gradient. We have then

$$\int_B |\overline{D}f|_0^2 \big(1-|x|^2\big)^p \, dV \le (m+1) \sum_A \int_B |\nabla f_A|^2 \big(1-|x|^2\big)^p \, dV \,.$$

It is well known that $D\overline{D}f = \Delta f = 0$, then each $f_A : B \to \mathbb{R}$ is a harmonic function. Therefore it is natural to characterize at first the Dirichlet spaces of *harmonic* scalar-valued functions.

2. Harmonic Dirichlet spaces

We begin with the study of spaces of harmonic functions. Following the notations of classical harmonic analysis we denote by $n > 2$ the dimension of the real Euclidean space $\mathbb{R}^n$ with norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We define the ball of radius $r > 0$ by $B_r = \{x \in \mathbb{R}^n \, : \, |x| < r\}$. Let S be the boundary of the unit ball B_1. Let $\Omega \subset \mathbb{R}^n$ be an open set, a twice continuously differentiable, complex-valued function u defined on Ω is harmonic on Ω if $\Delta u \equiv 0$, where $\Delta = \partial_1^2 + \cdots + \partial_n^2$ and ∂_j^2 denotes the second partial derivative with respect to the $j-th$ variable. We recall Green‘s identity

$$\int_\Omega (u\Delta v - v\Delta u) \, dV = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds \,,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, and u and v are C^2-functions on a neighborhood of $\overline{\Omega}$; the symbol $\frac{\partial u}{\partial n}$ denotes differentiation with respect to the outward unit normal $\mathbf{n}$. Thus for $\zeta \in \partial\Omega$, $(\frac{\partial u}{\partial n})(\zeta) = (\nabla u)(\zeta) \cdot \mathbf{n}(\zeta)$, where $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ denotes the gradient of u and $\cdot$ denotes the usual Euclidean inner product.

Let u, v be twice differentiable scalar-valued functions, the product rule of the Laplacian is

$$\Delta(uv) = u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u \,. \tag{2.1}$$

We recall also the polar coodinates formula for integration on $\mathbb{R}^n$ for a Borel measurable and integrable function f on $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} f \; dV = n \operatorname{vol}(B) \int_0^\infty r^{n-1} \int_S f(r\zeta) \; d\sigma(\zeta) \, dr \,. \tag{2.2}$$

The constant $n \operatorname{vol}(B)$ arises from the normalization of σ, the normalized surface area measure on S, such that $\sigma(S) = 1$. The measure σ is the unique Borel probability measure on S that is invariant under rotations (that is $\sigma(T(E)) = \sigma(E)$ for every Borel set $E \subset S$ and every orthogonal transformation T).

Let us denote by $\mathcal{P}_m(\mathbb{R}^n)$ the complex vector space of all homogeneous polynomials on $\mathbb{R}^n$ of degree m. Let $\mathcal{H}_m(\mathbb{R}^n)$ denote the subspace of $\mathcal{P}_m(\mathbb{R}^n)$ consisting of all homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree m. We recall the following results.

Proposition 2.1. [3, Proposition 5.9] *If p, q are polynomials on $\mathbb{R}^n$ and q is harmonic and homogeneous with degree higher than the degree of p, then*

$$\int_S p\,q\,d\sigma = \int_S p\,\overline{q}\,d\sigma = 0\,.$$

Proposition 2.2. [3, Theorem 5.14] *Let $p = \sum_\alpha a_\alpha x^\alpha$ and $q = \sum_\beta b_\beta x^\beta$ be harmonic polynomials on $\mathbb{R}^n$, then*

$$\int_S p\,\overline{q}\,d\sigma = \sum_\alpha a_\alpha \overline{b_\alpha} c_\alpha\,,$$

where $c_\alpha = \frac{\alpha}{n(n+2)\cdots(n+2|\alpha|-2)}$.

The following corollary is immediate.

Corollary 2.3. *If $p = \sum_{|\alpha|=m} a_\alpha x^\alpha$ is a harmonic polynomial on $\mathbb{R}^n$, then*

$$\int_S |p|^2\,d\sigma = \frac{1}{n(n+2)\cdots(n+2m-2)} \sum_{|\alpha|=m} |a_\alpha|^2 \alpha!\ .$$

The following lemma gives an idea about the behaviour of ∇p and p on the sphere.

Lemma 2.4. *Let $p \in \mathcal{H}_m(\mathbb{R}^n)$ and $q \in \mathcal{H}_s(\mathbb{R}^n)$ be. Then*

$$\int_S \nabla p \cdot \nabla q\,d\sigma = \begin{cases} m(n+2m-2)\int_S pq\,d\sigma & \text{if } m = s \\ 0 & \text{if } m \neq s\,. \end{cases}$$

Proof. Let $p \in \mathcal{H}_m(\mathbb{R}^n)$ and $q \in \mathcal{H}_s(\mathbb{R}^n)$. For $\sigma \in S$, the normal derivative of pq on S is

$$(\partial_{\mathbf{n}})pq(\sigma) = \nabla(pq)(\sigma)\cdot\mathbf{n}(\sigma) = \frac{d}{dr}pq(r\sigma)|_{r=1} = \frac{d}{dr}r^{m+s}p(\sigma)q(\sigma)\Big|_{r=1}\,,$$

therefore $p(\sigma)q(\sigma) = \frac{1}{m+s}\nabla(pq)(\sigma)\cdot\mathbf{n}(\sigma)$. Then by the divergence theorem

$$\int_S pq\ d\sigma = \frac{1}{(m+s)n\mathrm{vol}(B)}\int_S \nabla(pq)\cdot\mathbf{n}\,d\sigma = \frac{1}{(m+s)n\mathrm{vol}(B)}\int_B \Delta(pq)\,dV.$$

Convert the last integral into polar coordinates, apply the product rule of Laplacian (2.1), and use the homogeneity of ∇p and ∇q to get

$$\begin{aligned}\int_S pq\ d\sigma &= \frac{2}{(m+s)}\int_0^1 r^{n+m+s-3}\int_S \nabla(p)\cdot\nabla q\,d\sigma\,dr \\ &= \frac{2}{(m+s)(n+m+s-2)}\int_S \nabla(p)\cdot\nabla q\,d\sigma\ .\end{aligned}$$

Then we obtain

$$\int_S \nabla(p)\cdot\nabla q\ d\sigma = \frac{(m+s)(n+m+s-2)}{2}\int_S pq\,d\sigma\,,$$

and we conclude applying the Proposition 2.1. ∎

Let $-1 < p$. We define $\mathbf{D}_p(B)$ to be the class of *harmonic* functions $u : B \to \mathbb{C}$ for which $\int_B |\nabla u|^2(1-|x|^2)^p\,dV < \infty$. In particular for $p = 0$ we have the classical Dirichlet space of harmonic functions $\int_B |\nabla u|^2\ dV < \infty$. We note that if $-1 < p < q$ then $\mathbf{D}_p \subset \mathbf{D}_q$.

We recall the following result.

Theorem 2.5. [3, Corollary 5.34] *Let $u : B \to \mathbb{C}$ be a harmonic function on B. Then there exist $p_k \in \mathcal{H}_k(\mathbb{R}^n)$ such that*

$$u(x) = \sum_{k=0}^{\infty} p_k(x) \tag{2.3}$$

for all $x \in B$ and the series is converging absolutely and uniformly on compact subsets of B.

Because the series (2.3) converges absolutely and uniformly on compact subsets of B we can differentiate it term by term to obtain $\nabla u = \sum_{k=0}^{\infty}\nabla p_k = \sum_{k=1}^{\infty}\nabla p_k$, and observe that this series also converges absolutely and uniformly on compact subsets of B. Then

$$|\nabla u|^2 = \nabla u\overline{\nabla u} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\nabla p_k\cdot\overline{\nabla p_l} \tag{2.4}$$

with the same kind of convergence. Let $0 < R < 1$. Then $\overline{B_R} \subset B$, by (2.2), Lemma 2.4 and due to the uniform convergence of the series (2.4) we have

$$\begin{aligned}
&\int_{B_R} |\nabla u|^2\big(1-|x|^2\big)^p\,dV \\
&\qquad = n\,\mathrm{vol}(B)\int_0^R r^{n-1}\big(1-r^2\big)^p\int_S |\nabla u(r\zeta)|^2\ d\sigma(\zeta)\,dr \\
&\qquad = n\,\mathrm{vol}(B)\int_0^R \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} r^{n+k+l-3}\big(1-r^2\big)^p\int_S \nabla p_k(\zeta)\cdot\overline{\nabla p_l(\zeta)}d\sigma(\zeta)\,dr \\
&\qquad = n\,\mathrm{vol}(B)\sum_{k=1}^{\infty}\int_0^R r^{n+2k-3}\big(1-r^2\big)^p\,dr\int_S |\nabla p_k(\zeta)|^2\,d\sigma(\zeta)\,.
\end{aligned}$$

Now by Abel's theorem we have

$$\begin{aligned}
&\int_B |\nabla u|^2 \big(1-|x|^2\big)^p \, dV \\
&\qquad = \lim_{R\to 1^-} \int_{B_R} |\nabla u|^2 \big(1-|x|^2\big)^p \, dV \\
&\qquad = n \operatorname{vol}(B) \lim_{R\to 1^-} \sum_{k=1}^{\infty} \int_0^R r^{n+2k-3} \big(1-r^2\big)^p \, dr \int_S |\nabla p_k(\zeta)|^2 \, d\sigma(\zeta) \\
&\qquad = n \operatorname{vol}(B) \sum_{k=1}^{\infty} \int_0^1 r^{n+2k-3} \big(1-r^2\big)^p \, dr \int_S |\nabla p_k(\zeta)|^2 \, d\sigma(\zeta) \\
&\qquad = \frac{n \operatorname{vol}(B)}{2} \sum_{k=1}^{\infty} \frac{\Gamma(\frac{n}{2}+k-1)\Gamma(p+1)}{\Gamma(\frac{n}{2}+k+p)} k(n+2k-2)\|p_k\|_2^2 \,,
\end{aligned}$$

where $\|\cdot\|$ denotes the L^2 norm on S. Now, we use the well known approximation for the function Γ, $\Gamma(az+b) \approxeq \sqrt{2\pi} e^{-az}(az)^{az+b-\frac{1}{2}}$ for $|\arg z| < \pi,\ a > 0$ and $|z| \to \infty$. Therefore

$$\frac{\Gamma(\frac{n}{2}+k-1)\Gamma(p+1)}{\Gamma(\frac{n}{2}+k+p)} \approxeq k^{-1-p}$$

and then

$$\sum_{k=1}^{\infty} \frac{\Gamma(\frac{n}{2}+k-1)\Gamma(p+1)}{\Gamma(\frac{n}{2}+k+p)} k(n+2k-2)\|p_k\|_2^2$$

is a convergent series if and only if $\sum_{k=1}^{\infty} k^{1-p}\|p_k\|_2^2$ is a convergent series. Finally we have proved

Theorem 2.6. *Let $u : B \to \mathbb{C}$ be a harmonic function with series expansion in harmonic polynomials given by $u(x) = \sum_{k=0}^{\infty} p_k(x)$. Let $-1 < p$. Then u belongs to the Dirichlet $\mathbf{D}_p(B)$ if and only if*

$$\sum_{k=1}^{\infty} k^{1-p}\|p_k\|_2^2 < \infty \;. \tag{2.5}$$

Now we can extend our definition of a Dirichlet space for arbitrary real values of p. Let $p \in \mathbb{R}$. We say that $u : B \to \mathbb{C}$ belongs to the space $\mathbf{D}_p(B)$ if the series (2.5) is convergent.

We observe that this theorem is equivalent to Theorem 1.1, were homogeneous harmonic expansions behaves like power series expansions in the classical case, and as is usual we observe that homogeneous expansions are better behaved than multiple power series in higher dimensions.

In terms of coefficients the previous theorem can be written as

Corollary 2.7. *Let $u : B \to \mathbb{C}$ be a harmonic function with series expansion in harmonic polynomials given by $u(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha x^\alpha$. Then u belongs to the Dirichlet $\mathbf{D}_p(B)$, $p \in \mathbb{R}$, if and only if*

$$\sum_{k=1}^{\infty} \frac{k^{1-p}}{n(n+2)\cdots(n+2k-2)} \sum_{|\alpha|=k} |a_\alpha|^2 \alpha! < \infty .$$

Example 2.8. Let $\zeta \in S$. We consider the harmonic function $u : \mathbb{R}^n \setminus \{\zeta\} \to \mathbb{R}$ given by $u(x) = \frac{1}{|x-\zeta|^{n-2}}$. Then

$$|\nabla u(x)|^2 = \frac{(2-n)^2}{|x-\zeta|^{2n-2}} .$$

Because $n > 2$ it is clear that $u \notin \mathbf{D}$, the classical Dirichlet space. However

$$\begin{aligned}
\int_B |\nabla u(x)|^2 \big(1-|x|^2\big)^p \, dV &= (2-n)^2 \int_B \frac{\big(1-|x|^2\big)^p}{|x-\zeta|^{2n-2}} \, dV \\
&\le 2^p (2-n)^2 \int_B \frac{(1-|x|)^p}{(1-|x|)^{2n-2}} \, dV \\
&= 2^p (2-n)^2 n \operatorname{vol}(B) \int_0^1 r^{n-1} \int_S \frac{(1-|r\zeta|)^p}{(1-|r\zeta|)^{2n-2}} d\sigma(\zeta) dr \\
&= 2^p (2-n)^2 n \operatorname{vol}(B) \int_0^1 \frac{r^{n-1}\, dr}{(1-r)^{2n-2-p}} ,
\end{aligned}$$

and this last integral is finite if and only if $2n-3 < p$. Thus $u \in \mathbf{D}_p(B)$ if $2n-3 < p$. We will improve this result in the next section.

It is well known that for analytic functions $D_1 = h^2$, where h^2 is the Hardy space. We prove now that it is true for harmonic functions on $\mathbb{R}^n$, too. We need some notation.

Given a function u on B, we denote by u_r the function on S defined by $u_r(\zeta) = u(r\zeta)$ with $0 \le r < 1$. We recall that the Hardy space $\mathbf{h}^2(B)$ consists of the class of harmonic functions on B for which $|u\|_{\mathbf{h}^2} = \sup_{0 \le r < 1} \|u_r\|_2 < \infty$, where $\|\cdot\|_2$ denotes the L^2 norm on S. Let $0 < r < s < 1$, then it is well known that $\|u_r\|_2 \le \|u_s\|_2$ and consequently

$$\|u\|_{\mathbf{h}^2} = \lim_{r \to 1^-} \|u_r\|_2 ,$$

for each $u \in \mathbf{h}^2(B)$. We have the following theorem.

Theorem 2.9. *The Dirichlet space* $\mathbf{D}_1(B)$ *coincides with the Hilbert space* $\mathbf{h}^2(B)$. *Moreover if* $u \in \mathbf{h}^2(B)$ *is given by* $u = \sum_{k=0}^{\infty} p_k$ *where* $p_k \in \mathcal{H}_k(\mathbb{R}^n)$, *then* $\|u\|_{\mathbf{h}^2}^2 = \sum_{k=0}^{\infty} \|p_k\|_2^2$.

Proof. It is known that $\mathbf{h}^2(B)$ is a Hilbert space. Let $0 < r < 1$ and $u \in \mathbf{h}^2(B)$ as in the statement. By the absolute and uniform convergence of the series (2.3) and by Proposition 2.1

$$\|u_r\|_2^2 = \int_S |u(r\zeta)|^2 \, d\sigma(\zeta) = \sum_{k=0}^{\infty} r^k \int_S |p_k(\zeta)|^2 \, d\sigma(\zeta) \ .$$

By Abel's theorem we have

$$\|u\|_{h_2}^2 = \lim_{r\to 1^-} \|u_r\|_2^2 = \lim_{r\to 1^-} \sum_{k=0}^{\infty} r^k \int_S |p_k(\zeta)|^2 \, d\sigma(\zeta) = \sum_{k=0}^{\infty} \int_S |p_k(\zeta)|^2 \, d\sigma(\zeta).$$

By Theorem 2.6 we obtain the result. ∎

Observation 2.10. If $u : B \to \mathbb{C}$ is a harmonic function with the series expansion $u = \sum_{k=0}^{\infty} p_k$, where $p_k \in \mathcal{H}_k(\mathbb{R}^n)$, then $u \in \mathbf{D}_p(B)$ if and only if $\sum_{k=1}^{\infty} k^{1-p} \|p_k\|_{\mathbf{h_2}}^2 < \infty$.

Theorem 2.11. *The Dirichlet space* $\mathbf{D}_2(B)$ *coincides with the Bergman space (Hilbert space)* $\mathbf{b}^2(B)$. *Moreover, if* $u \in \mathbf{b}^2(B)$ *is given by* $u = \sum_{k=0}^{\infty} p_k$, *where* $p_k \in \mathcal{H}_k(\mathbb{R}^n)$, *then*

$$\|u\|_{\mathbf{b}^2}^2 = n \operatorname{vol}(B) \sum_{k=0}^{\infty} \frac{1}{n+2k} \|p_k\|_2^2 \ .$$

Proof. Recall that $u \in \mathbf{b}^2(B)$ means that $\int_B |u(x)|^2 \, dV < \infty$. Let $0 < R < 1$. Then $\overline{B_R} \subset B$ and by (2.2), Lemma 2.4, and the uniform convergence of the series (2.4) we have

$$\begin{aligned}
\int_{B_R} |u(x)|^2 \, dV &= n \operatorname{vol}(B) \int_0^R r^{n-1} \int_S |u(rx)|^2 \, d\sigma(\zeta) \, dr \\
&= n \operatorname{vol}(B) \int_0^R r^{n-1} \int_S \Big| \sum_{k=0}^{\infty} p_{A,k}(r\zeta) \Big|^2 \, d\sigma(\zeta) \, dr \\
&= n \operatorname{vol}(B) \int_0^R r^{n-1} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} r^{k+l} \int_S p_{A,k}(\zeta) \overline{p_{A,l}(\zeta)} \, d\sigma(\zeta) \, dr \\
&= n \operatorname{vol}(B) \sum_{k=0}^{\infty} \int_0^R r^{n+2k-1} \|p_{A,k}(\zeta)\|_2^2 \, dr \\
&= n \operatorname{vol}(B) \sum_{k=0}^{\infty} \frac{R^{n+2k}}{n+2k} \|p_{A,k}(\zeta)\|_2^2 \ .
\end{aligned}$$

If now $u \in \mathbf{D}_2$, then we can take the limit as $R \to 1^-$ in the previous formula to obtain by Abel's theorem

$$\int_B |u(x)|^2 \, dV = n \operatorname{vol}(B) \sum_{k=0}^{\infty} \frac{1}{n+2k} \|p_{A,k}(\zeta)\|_2^2 . \qquad \blacksquare$$

Let $\alpha > 0$. We define the Bloch space $\mathbf{B}^\alpha$ as the set of harmonic funtions $u : B \to \mathbb{C}$ such that $\sup_{x\in B}(1-|x|)^\alpha |\nabla u(x)| < \infty$.

Theorem 2.12. *Let $\alpha > 0$. Then $\mathbf{B}^\alpha \subset \bigcap_{2\alpha-1<p} \mathbf{D}_p(B)$.*

Proof. Let $\alpha > 0$, $p > 2\alpha - 1$ and $M > 0$ be such that $|\nabla u(x)| < \frac{M}{(1-|x|)^\alpha}$, then by (2.2)

$$\begin{aligned}\int_B |\nabla u(x)|^2 \big(1-|x|^2\big)^p \, dV &\le 2^p M^2 \int_B (1-|x|)^{p-2\alpha} \, dV \\ &= 2^p M^2 n \operatorname{vol}(B) \int_0^1 (1-r)^{p-2\alpha} \, dr \\ &= 2^p M^2 n \operatorname{vol}(B)\end{aligned}$$

that completes the proof. $\blacksquare$

In a similar way we can prove that if $p > -1$, then $\bigcup_{0<\alpha<\frac{p+1}{2}} \mathbf{B}^\alpha \subset \mathbf{D}_p(B)$.

3. Strict inclusions

The next proposition improves the result of Example 2.8

Lemma 3.1. *Let $u : \mathbb{R}^n \setminus \{\zeta\} \to \mathbb{R}$, $\zeta \in S$, given by $u(x) = \frac{1}{|x-\zeta|^{n-2}}$. Then $u \in \big(\bigcap_{n-2<p} \mathbf{D}_p\big) \setminus \mathbf{D}_{n-2}$.*

Proof. Let $-1 < p$

$$\frac{1}{(2-n)^2} \int_B |\nabla u(x)|^2 \big(1-|x|^2\big)^p dx = \int_B \frac{\big(1-|x|^2\big)^p}{|x-\zeta|^{2n-2}} \, dx = \int_B \frac{\big(1-|x|^2\big)^p}{|x-e_1|^{2n-2}} \, dx \,,$$

where $e_1 = (1, 0 \ldots, 0) \in S$. Now we translate and change to spherical coordinates

$$\begin{aligned}\int_B \frac{\big(1-|x|^2\big)^p}{|x-e_1|^{2n-2}} dx &= \int_{B(e_1,1)} \frac{\big(1-|x+e_1|^2\big)^p}{|x|^{2n-2}} \, dx \\ &= \Gamma \int_{\frac{\pi}{2}}^{\pi} \sin^{n-2}\theta_1 \, d\theta_1 \int_0^{-2\cos\theta_1} \frac{\big(-r^2 - 2r\cos\theta_1\big)^p}{r^{2n-2}} r^{n-1} \, dr \,,\end{aligned}$$

where $\Gamma = \int_0^{2\pi} d\theta_{n-1} \int_0^{\pi} \sin\theta_{n-2} d\theta_{n-2} \cdots \int_0^{\pi} \sin^{n-3}\theta_2 d\theta_2 < \infty$. Now

$$\int_{\frac{\pi}{2}}^{\pi} \sin^{n-2}\theta_1 \, d\theta_1 \int_0^{-2\cos\theta_1} \frac{\big(-r^2 - 2r\cos\theta_1\big)^p}{r^{2n-2}} r^{n-1} \, dr = \int_{\frac{\pi}{2}}^{\pi} \sin^{n-2}\theta_1 \, d\theta_1 \int_0^{-2\cos\theta_1} \frac{(-r - 2\cos\theta_1)^p}{r^{n-1-p}} \, dr \,,$$

and this last integral exists if and only if $n-1-p<1$, that is $n-2<p$. ∎

Lemma 3.2. *There exists a harmonic function $u : B \to \mathbb{C}$ such that $u \in \mathbf{D}_1(B) \setminus \bigcup_{q<1} \mathbf{D}_q(B)$.*

Proof. We know from Theorem 2.9 that $\mathbf{D}_1(B) = \mathbf{h}^2(B)$ and therefore there exists a linear isometry T from the classic Hilbert space l^2 onto $\mathbf{D}_1(B)$. It is well known (see [13, p. 130]) that

$$A = \left\{ \frac{1}{\sqrt{k}\log k} \right\}_{k=2}^{\infty} \in l^2 \setminus \bigcup_{q<2} l^q \,.$$

Consider a sequence of harmonic polynomials $\{q_k\}_{k=2}^{\infty}$ with norm $\|q_k\|_{\mathbf{h}^2} = k^{-\frac{1}{2}} \log^{-2} k$. Let $a_k = \{a_n^k\}_{n=1}^{\infty} \in l^2$ such that $T(a_k) = q_k$. Define $s_l = \sum_{k=2}^{l} a_k$ then $T(s_l) = \sum_{k=2}^{l} q_k$ and by Proposition 2.1 and Theorem 2.9 for $m \geq l$

$$\|s_m - s_l\|_{l^2}^2 = \Big\| \sum_{k=l+1}^{m} q_k \Big\|_{\mathbf{h}^2}^2 = \sum_{k=l+1}^{m} \|q_k\|_{\mathbf{h}^2}^2 = \sum_{k=l+1}^{m} \frac{1}{k\log^2 k} \,.$$

Then $\{s_l\}_{l=2}^{\infty}$ is a Cauchy sequence in l^2. Let $s \in l^2$, with $s = \lim_{k\to\infty} s_k$. Let $T(s) = u \in \mathbf{h}^2(B)$. Observe that we only know that

$$\lim_{l\to\infty} \|T(s_l) - u\|_{\mathbf{h}^2} = \lim_{l\to\infty} \Big\| \sum_{k=2}^{l} q_k - u \Big\|_{\mathbf{h}^2} = \lim_{l\to\infty} \|s_k - s\|_{l^2} = 0 \,.$$

Our task is of course, to prove that $u = \sum_{k=2}^{\infty} q_k$. As u is a harmonic function on B u can be represented as $u = \sum_{k=0}^{\infty} p_k$, where $p_k \in \mathcal{H}_k(\mathbb{R}^n)$ for $k = 0,\ 1,\ 2, \ldots$ and

$$\sum_{k=0}^{\infty} \|p_k\|_{\mathbf{h}^2}^2 = \|u\|_{\mathbf{h}^2}^2 = \sum_{k=2}^{\infty} \frac{1}{k\log^2 k}.$$

We set $q_0 = q_1 = 0$. Let m be a fix index. We will prove that $p_m = q_m$. Let $\varepsilon > 0$. There exists an $N > 0$ such that $\sum_{k=l}^{\infty} \|p_k\|_{\mathbf{h}^2}^2 < \frac{\varepsilon}{2}$ and $\| \sum_{k=2}^{l} q_k - u\|_{\mathbf{h}^2}^2 < \frac{\varepsilon}{2}$ for all $l \geq N$. Then for all $l \geq N+m$ we have the estimation

$$\Big\| \sum_{k=2}^{l} q_k - \sum_{k=0}^{l} p_k \Big\|_{\mathbf{h}^2}^2 = \Big\| \sum_{k=2}^{l} q_k - \sum_{k=0}^{l} p_k - \sum_{k=l+1}^{\infty} p_k + \sum_{k=l+1}^{\infty} p_k \Big\|_{\mathbf{h}^2}^2$$
$$\leq \Big\| \sum_{k=2}^{l} q_k - u \Big\|_{\mathbf{h}^2}^2 + \sum_{k=l}^{\infty} \|p_k\|_{\mathbf{h}^2}^2 < \varepsilon\,.$$

Then

$$\int_S |q_m(\zeta) - p_m(\zeta)|^2\, d\sigma(\zeta) \leq \sum_{k=0}^{l} \int_S |q_k(\zeta) - p_k(\zeta)|^2\, d\sigma(\zeta) = \sum_{k=2}^{l} \|q_k - p_k\|_{\mathbf{h}^2}^2 < \varepsilon\,.$$

As $\varepsilon > 0$ is arbitrary we have by continuity of the harmonic polynomials $q_m = p_m$, for all $m = 0,\ 1, 2, \ldots$. In particular now we know that $\|p_k\|_{\mathbf{h}^2} = k^{-\frac{1}{2}} \log^{-2} k$. In case of $\alpha > 0$ we obtain

$$\sum_{k=0}^{\infty} k^{\alpha} \|p_k\|_{\mathbf{h}^2}^2 = \sum_{k=2}^{\infty} \frac{k^{\alpha}}{k \log^2 k} = \infty$$

and therefore $u \notin \mathbf{D}_q$ for all $q < 1$. ∎

We recall that if $u : B \to \mathbb{C}$ is a harmonic function with series expansion in harmonic polynomials given by (2.3), then in particular for each $\alpha \in \mathbb{R}$, $v(x) = \sum_{k=0}^{\infty} k^{\alpha} p_k(x)$ converges absolutely and uniformly on each compact set of B and therefore defines a harmonic function on B.

Theorem 3.3. *Let $q \in \mathbb{R}$ be fixed. Then* $\bigcup_{p<q} \mathbf{D}_p(B) \subsetneq \mathbf{D}_q(B) \subsetneq \bigcap_{q<r} \mathbf{D}_r(B)$.

Proof. Let $u : B \to \mathbb{R}$ given by $u(x) = |x - \zeta|^{2-n}$ with series expansion in harmonic homogeneous polynomials $u(x) = \sum_{k=0}^{\infty} p_k(x)$. Define a harmonic function $v : B \to \mathbb{C}$ by

$$v(x) = \sum_{k=1}^{\infty} \frac{p_k(x)}{k^{\frac{n-q-2}{2}}}\,.$$

By Lemma 3.1 and Theorem 2.6 it is easy to see that $v \in \big(\bigcap_{q<r} \mathbf{D}_r(B)\big) \backslash \mathbf{D}_q(B)$. Let $\tilde{u} : B \to \mathbb{C}$ be the harmonic function in Lemma 3.2 with the series expansion $\tilde{u} = \sum_{k=2}^{\infty} q_k$. Define $g : \Delta \to \mathbb{C}$ by

$$g(x) = \sum_{k=2}^{\infty} \frac{q_k(x)}{k^{1-\frac{q}{2}}}\,.$$

Then

$$\sum_{n=2}^{\infty} k^{1-q} \left\| \frac{q_k(x)}{k^{1-\frac{q}{2}}} \right\|_{\mathbf{h}^2} = \sum_{k=2}^{\infty} \frac{1}{k \log^2 k} < \infty\,.$$

Let $p < q$. Then

$$\sum_{k=2}^{\infty} k^{1-p} \left\| \frac{q_k(x)}{k^{1-\frac{q}{2}}} \right\|_{\mathbf{h}^2} = \sum_{k=2}^{\infty} \frac{1}{k^{1+p-q} \log^2 k} = \infty \,,$$

therefore $g \in \mathbf{D}_q(B) \setminus \bigcup_{p<q} \mathbf{D}_p(B)$. ∎

4. The spaces $\mathbf{D}_p(S)$, $0 < p < 1$

Let $f \in L^2(S)$. Then f is an element of $\mathbf{D}_p(S)$, $0 < p < 1$, if

$$\|f\|^2_{\mathbf{D}_p} = \sup_{I \subset S} \frac{1}{|I|^p} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n-p}} \, d\sigma(x) \, d\sigma(y) < \infty \,,$$

where I is a surface ball (i.e., $I = S \cap B_{r_0}(x_0)$ where $x_0 \in S$). For $n = 2$ the spaces $\mathbf{D}_p(S)$, $0 < p < 1$, form a scale: $\mathbf{D}_p(S) \subseteq \mathbf{D}_q(S)$, $0 < p < q < 1$. Unfortunately, we cannot prove such a property for $n > 2$ which is caused by the fact that $|I| \sim C r_I^{n-1} \sim r_I$ if and only if $n = 2$.

For $f \in L^1(S)$, let F be the harmonic extension of f to B, i.e.,

$$F(x) = \int_B f(\xi) \, \frac{1 - |x|^2}{|x - \xi|^n} \, d\sigma(\xi).$$

The following theorem characterizes the spaces $\mathbf{D}_p(S)$ also in term of the harmonic extensions.

Theorem 4.1. *Let $0 < p < 1$ and let F be a harmonic function in B with boundary values $f \in \mathbf{D}_p(S)$. Then the quantities*

$$M = \|f\|^2_{\mathbf{D}_p} = \sup_{I \subset S} \frac{1}{|I|^p} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n-p}} \, d\sigma(x) \, d\sigma(y)$$

$$N = \sup_{I \subset S} \frac{1}{|I|^p} \int_{Q_h(x_0)} |\nabla F(x)|^2 (1 - |x|^2)^p \, dx$$

are equivalent $M \approx N$, where $I = S \cap B_h(x_0)$ and $Q_h(x_0) = B \cap B_h(x_0)$.

Remark 4.2. Here, "$M \approx N$" means that there exist constants C_1, $C_2 > 0$ such that $C_1 \, M \le N \le C_2 \, M$. The measure of a surface ball I with radius r_I is equivalent to r_I^{n-1}.

Remark 4.3. In the classic theory the following property is well known. Let $p \in (0,1)$ and $f \in H^2$. Then $f \in D_p$ if and only if

$$\sup_{I \subseteq T} |I|^{-p} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{2-p}} |d\zeta||d\eta| < \infty \,,$$

where $T = \partial U$ and the supremum is taken over all arcs $I \subseteq T$ (see [14, Lemma 6.1.1]). Then Theorem 4.1 is the corresponding counterpart.

For the proof of this theorem we will need some additional lemmas and Theorem 4.6 which is a generalization of a well-known lemma by Stegenga [11, Lemma 3.2].

Lemma 4.4. *Let* $0 < \alpha \le \frac{1}{2}$, *then*

$$\left\|(1-|x|)^{\frac{1-2\alpha}{2}}|\nabla F|\right\|_{L^2(B)} + \|F\|_{L^2(B)} \approx \|f\|_{L^2_\alpha(S)}$$

for any function F *harmonic in* B.

Proof. What we have to prove is the following:

$$\begin{aligned}\|f\|_{L^2_\alpha(S)} &\le C\|F\|_{W^2_{\alpha+\frac{1}{2}}} \\ &\le C\left(\left\|(1-|x|^2)^{\frac{1-2\alpha}{2}}|\nabla F|\right\|_{L^2(B)} + \|F\|_{L^2(B)}\right) \\ &\le C\|f\|_{L^2_\alpha(S)}\,,\end{aligned}$$

where $\|f\|_{L^2_\alpha(S)} = \|\operatorname{tr} F\|_{L^2_\alpha(S)}$. The first estimation is valid due to the trace theorem for Sobolev spaces [1]. One part of the last estimation can be proven by using the representation of a harmonic function and the mapping properties of the Poisson integral plus the trace theorem. The remaining estimation is a special case of a general result from [10], which implies that for $0 < s < 1$,

$$\|F\|_{W^2_s(B)} \approx \|(1-|x|)^{1-s}|\nabla F| + |F|\,\|_{L^2(B)}$$

We take $s = \frac{1}{2} + \alpha$. ∎

An immediate consequence is that under the assumptions of the previous lemma

$$\|f\|_{L^2_\alpha(S)} \approx \left\|(1-|x|)^{\frac{1-2\alpha}{2}}|\nabla F|\right\|_{L^2(B)}.$$

We abbreviate $a := \|f\|_{L^2_\alpha(S)}$, $b := \left\|(1-|x|)^{\frac{1-2\alpha}{2}}|\nabla F|\right\|_{L^2(B)}$ and $c := \|\, F\,\|_{L^2(B)}$. Then Lemma 4.4 immediately implies that there exist positive constants c_1, c_2 such that $c_1\, a \le b + c \le c_2\, a$ and $b \le c_2 a$. To obtain the other inequality we remark that $b = 0$ implies that $F = const$ and the inequality is fulfilled. If $b > 0$, there exists a natural number k such that $c_2 a \le k\, b$ and we get $a \le c_1^{-1} c_2 a \le c_1^{-1}\, k\, b$. The next implication is

Lemma 4.5. *Let be* $f \in L_2(S)$ *the boundary values of a harmonic function* F *and* $0 < p < 1$, *then*

$$\int_B |\nabla F(x)|^2 (1-|x|^2)^p\, dx \approx \int_S \int_S \frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}\, d\sigma(x)\, d\sigma(y).$$

Proof. We set $p = 1 - 2\alpha$, i.e., $\alpha \in (0, \frac{1}{2})$. ∎

We are now able to prove the following theorem.

Theorem 4.6. *Let I and J be surface balls with center x_0 and radii $r_J \geq 3r_I$, where r_J is the radius of J and r_I is the radius of I respectively, and F a harmonic function in B with boundary values $f \in L^1_{loc}(S)$. For $p \in (0,1)$ there exists a constant $C > 0$ (independent of I and J) such that*

$$\int_{Q_{r_I}(x_0)} |\nabla F(x)|^2 (1-|x|)^{1-2\alpha} dx \leq C \Bigg[\int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n-1+2\alpha}}\, d\sigma(x)\, d\sigma(y) \\ + r_I^{n+1-2\alpha} \bigg(\int_{S\setminus \frac{2}{3}J} |f(x)-f_J| \frac{d\sigma(x)}{|x-x_0|^n} \bigg)^2 \Bigg].$$

Proof. Let φ be a continuous function such that $0 \leq \varphi(x) \leq 1$, where $\varphi(x) = 1$ on $\frac{2}{3}J$, $\operatorname{supp}\varphi \subset \frac{3}{4}J$, and $|\varphi(x)-\varphi(y)| \leq Cr_J^{-1}|x-y|$, $x, y \in S$. Let $\tilde{\varphi}(x) = 1-\varphi(x)$, then

$$f = f_J + (f-f_J)\varphi + (f-f_J)\tilde{\varphi} = f_1 + f_2 + f_3.$$

In the same way we get for the Poisson integral $Pf = Pf_1 + Pf_2 + Pf_3$. Because of $f_1 = const = f_J$ we get $|\nabla Pf_1(x)| = 0$ and this part contributes nothing. Let us consider the next part: We estimate

$$\int_{Q_{r_I}(x_0)} |\nabla Pf_2(x)|^2 (1-|x|)^{1-2\alpha}\, dB_x \leq \int_B |\nabla Pf_2(x)|^2 (1-|x|)^{1-2\alpha}\, dB_x \\ \leq C \int_S \int_S \frac{|f_2(x)-f_2(y)|}{|x-y|^{n-1+2\alpha}}\, d\sigma(x)\, d\sigma(y)$$

and due to Lemma 4.4,

$$\leq \int_{x\in J} \int_{y\in J} \ldots + \int_{x\notin J} \int_{y\in \frac{3}{4}J} \ldots + \int_{x\in \frac{3}{4}J} \int_{y\notin J} \ldots = B_1 + B_2 + B_3\,.$$

Now, we estimate these three integrals. For B_1 we have

$$\begin{aligned} |f_2(x)-f_2(y)| &= |(f(x)-f_J)\varphi(x) - (f(y)-f_J)\varphi(y)| \\ &= |(f(x)-f(y))\varphi(x) + (f(y)-f_J)(\varphi(x)-\varphi(y))| \\ &\leq |f(x)-f(y)| + C\, r_j^{-1}|x-y|\, |f(y)-f_J|\,, \end{aligned}$$

the term B_2 gives

$$|f_2(x)-f_2(y)| = |(f(y)-f_J)(\varphi(x)-\varphi(y))| \leq C\, r_J^{-1}|x-y|\, |f(y)-f_J|\,,$$

and for the last term B_3 we get

$$|f_2(x)-f_2(y)| = \big|[f(x)-f(y)+f(y)-f_J]\varphi(x)\big| \leq |f(x)-f_J|\,.$$

It remains to estimate

$$
\begin{aligned}
r_J^{-2} \int_J \int_J & \frac{|f(y)-f_J|^2}{|x-y|^{n-2+2\alpha}}\, d\sigma(x)\, d\sigma(y) \\
&= r_J^{-2} \int_J |f(y)-f_J|^2 \left[\int_J |x-y|^{2-n-2\alpha}\, d\sigma(x) \right] d\sigma(y) \\
&\le r_J^{-2} \int_J |f(y)-f_J|^2 \int_0^{r_J} \frac{r^{n-1}}{r^{n+2\alpha-2}}\, dr \\
&\le C r_J^{-2\alpha} \int_J |f(y)-f_J|^2\, d\sigma(y) \\
&\le C r^{-2\alpha+1-n} \int_J \int_J |f(y)-f(x)|^2\, d\sigma(x)\, d\sigma(y) \\
&\le C \int_J \int_J \frac{|f(y)-f(x)|^2}{|x-y|^{n-1+2\alpha}}\, d\sigma(x)\, d\sigma(y)
\end{aligned}
$$

$$
\begin{aligned}
\int_{x\notin J} \int_{y\in \frac{3}{4}J} & r_J^{-2} \frac{|f(x)-f(y)|^2}{|x-y|^{n-3+2\alpha}}\, d\sigma(x)\, d\sigma(y) \\
&\le r_J^{-2} \int_{y\in\frac{3}{4}J} |f(y)-f_J|^2 \int_{x\notin J} \frac{1}{|x-y|^{n-3+2\alpha}}\, d\sigma(x)\, d\sigma(y) \\
&\le r_J^{-2}\, \mathrm{dist}\left(S\setminus J, \frac{3}{4}J\right)^{3-2\alpha-n} \int_{y\in\frac{3}{4}J} |f(y)-f_J|^2\, d\sigma(y) \\
&\le C \int_J \int_J \frac{|f(y)-f(x)|^2}{|x-y|^{n-1+2\alpha}}\, d\sigma(x)\, d\sigma(y)\,.
\end{aligned}
$$

We now consider the third part. We consider the Poisson kernel $P_a(x)$ where $a \in Q_{r_I(x_0)}$ and $x \in S \setminus \frac{2}{3}J$, then

$$
|\nabla_a P_a(x)| = \left| \nabla_a \left(\frac{1-|a|^2}{|x-a|^n} \right) \right| \le \frac{C_1}{|x-a|^n} \le \frac{C}{|x-x_0|^n}
$$

because of $|x-a| = |x-x_0+x_0-a| \ge \big| |x-x_|-|x_0-a| \big| \ge c|x-x_0|$ if $a \in Q_{r_I(x_0)}$ and $x \in S\setminus \frac{2}{3}J$ and thus

$$
|\nabla P f_3(x)| \le \int_S |\nabla P_a(x-y)|\, f_3(y)\, d\sigma(y) \le C \int_{S\setminus\frac{2}{3}J} \frac{|f(y)-f_J|}{|y-x_0|^n}\, d\sigma(y).
$$

Hence,

$$
\begin{aligned}
\int_{Q_{r_I(x_0)}} & |\nabla P f_3(x)|^2 \left(1-|x|^2\right)^{1-2\alpha} dx \\
&\le \left(\int_{Q_{r_I(x_0)}} \left(1-|x|^2\right)^{1-2\alpha} dx \right) \left(\int_{S\setminus\frac{2}{3}J} \frac{|f(x)-f_J|}{|x-x_0|^n}\, d\sigma(x) \right)^2 \\
&\le C\, r_I^{n+1-2\alpha} \left(\int_{S\setminus\frac{2}{3}J} \frac{|f(x)-f_J|}{|x-x_0|^n}\, d\sigma(x) \right)^2.
\end{aligned}
$$

This completes the proof. ∎

Remark 4.7. For $n = 2$ the space D_1 coincides with the space of analytic (i.e. holomorphic) functions of bounded mean oscillation BMOA and hence BMOA is contained in all D_p spaces with $0 < p < 1$. We are not able to prove this property in higher dimensions but we will use the norm of functions of bounded mean oscillation (BMO) for estimations. A function $f \in L^2(S)$ is of bounded mean oscillation, i.e. $f \in BMO(S)$ if

$$\sup_{I \in S} \frac{1}{|I|} \int_I |f - f_I| \, d\sigma \quad \text{with} \quad f_I = \frac{1}{|I|} \int_I f(x) \, d\sigma(x) < \infty. \tag{4.1}$$

We define an equivalent norm to the standard BMO-norm.

Lemma 4.8. *The standard BMO-norm* (4.1) *is equivalent to*

$$\sup_{I \in S} \left(\frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)|^2 \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{2}}.$$

Proof. From the well-known John-Nirenberg theorem we get that the BMO-norm (4.1) is equivalent to $\left(\sup_{I \in S} \frac{1}{|I|} \int_I |f - f_I|^2 \, d\sigma\right)^{\frac{1}{2}}$. It is easily seen that

$$\begin{aligned}
&\frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)|^2 \, d\sigma(x) \, d\sigma(y) \\
&\qquad \le \frac{2}{|I|^2} \left(\int_I \int_I |f(x) - f_I|^2 \, d\sigma(x) \, d\sigma(y) + \int_I \int_I |f(y) - f_I|^2 \, d\sigma(x) \, d\sigma(y) \right) \\
&\qquad \le \frac{4}{|I|} \int_I |f(x) - f_I|^2 \, d\sigma(x) \\
&\qquad \le 4 \, \|f\|^2_{BMO} \, .
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\frac{1}{|I|} \int_I |f(x) - f_I| \, d\sigma(x) \\
&\qquad \le \frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)| \, d\sigma(x) \, d\sigma(y) \\
&\qquad \le \frac{1}{|I|^2} \left(\int_I \int_I 1 \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{2}} \left(\int_I \int_I |f(x) - f(y)|^2 \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{2}} \\
&\qquad \le \left(\frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)|^2 \, d\sigma(x) \, d\sigma(y) \right)^{\frac{1}{2}}.
\end{aligned}$$

∎

We will also need the following result which shows why we are not able to include the $\mathbf{D}_p(S)$ spaces in BMO but is a useful tool for norm estimations we will do later on.

Lemma 4.9. *Let $f \in \mathbf{D}_p(S)$, then*

$$\left(\frac{1}{|I|}\int_I |f(x)-f_I|\,d\sigma(x)\right)^2 \le C\, r_I^{(n-2)(p-1)}\|f\|^2_{\mathbf{D}_p}.$$

Proof. We estimate the equivalent BMO-norm:

$$\begin{aligned}
&\left(\frac{1}{|I|}\int_I |f(x)-f_I|\,d\sigma(x)\right)^2\\
&\qquad\le \frac{1}{|I|^2}\int_I\int_I |f(x)-f(y)|^2\,d\sigma(x)\,d\sigma(y)\\
&\qquad= \frac{|I|^{p-2}}{|I|^p}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}|x-y|^{n-p}\,d\sigma(x)\,d\sigma(y)\\
&\qquad\le C\, r_I^{(p-2)(n-1)+n-p}\frac{1}{|I|^p}\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}\,d\sigma(x)\,d\sigma(y)\\
&\qquad\le C\, r_I^{(n-2)(p-1)}\|f\|^2_{\mathbf{D}_p}.
\end{aligned}$$

∎

Now, we are able to prove Theorem 4.1.

Proof of Theorem 4.1. First, we assume

$$N = \sup_{I\subset S}\frac{1}{|I|^p}\int_{Q_h(x_0)} |\nabla F(x)|^2\big(1-|x|^2\big)^p\,dx$$

to be finite. We restrict F to $Q_h(x_0) = B\cap B_h(x_0)$. Now, F is also a harmonic function in $Q_h(x_0)$ with boundary values on $\partial Q_h(x_0) = I\cup T$, where I is the surface ball with center x_0 and radius h and $T = (\partial B_h(x_0)\cap B)$. Moreover, $\operatorname{tr} F\,|_{Q_h(x_0)}(x) = \operatorname{tr} F(x)$ for $x\in I$ and thus

$$\begin{aligned}
\int_I\int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}\,d\sigma(x)\,d\sigma(y) &\le \int_{I\cup T}\int_{I\cup T}\frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}\,d\sigma(x)\,d\sigma(y)\\
&\le C_1\int_{Q_h(x_0)} |\nabla F(x)|^2(1-|x|^2)^p\,dx\\
&\le C_1\,N\,h^{(n-1)p} \le C\,N|I|^p.
\end{aligned}$$

and thus

$$M = \|f\|^2_{\mathbf{D}_p} = \sup_{I\subset S}\frac{1}{|I|^p}\int_I\int_I\frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}\,d\sigma(x)\,d\sigma(y)\le C\,N. \tag{4.2}$$

Now, let $M = \|f\|^2_{\mathbf{D}_p}$ be finite. We estimate with Lemma 4.6

$$\begin{aligned}
\int_{Q_{r_I}(x_0)} |\nabla F|^2(1-|x|)^p\,d\sigma(x) \le C\Bigg(&\int_I\int_I\frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}\,d\sigma(x)d\sigma(y)\\
&+ r_I^{n+p}\left(\int_{S\setminus\frac{2}{3}J}|f(x)-f_J|\,\frac{dx}{|x-x_0|^n}\right)^2\Bigg).
\end{aligned}$$

For the first expression we immediately get

$$\int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n-p}}\, d\sigma(x)\, d\sigma(y) \le C\, \|f\|^2_{\mathbf{D}_p}\,.$$

To estimate the second expression we consider a nested sequence of surface balls J_k, where all surface balls have the same center x_0 but differ by their radii: $J_0 := J$, i.e., $r_{J_0} = r_J$, and $r_{J_k} = 3^k\, r_J$. Obviously there exists a smallest number n_J such that $J_{n_J-1} \subsetneqq S$ but $J_{n_J} \supseteqq S$ and hence $S \subseteqq \bigcup_{k=0}^{n_J} \big(J_k \setminus J_{k-1}\big)$, where $J_{-1} := \emptyset$. We have

$$\begin{aligned}
&\int_{S\setminus \frac{2}{3}J} |f(x)-f_J|\, \frac{d\sigma(x)}{|x-x_0|^n} \\
&\le \sum_{k=0}^{n_J} \int {-}J_k \setminus J_{k-1} |f(x)-f_J|\, \frac{d\sigma(x)}{|x-x_0|^n} \\
&\le C \sum_{k=0}^{n_J} [3^k r_J]^{-n} \int_{J_k} |f(x)-f_{J_k}|\, d\sigma(x) + [3^k r_J]^{-1} |f_{J_k} - f_J| \\
&\le \frac{C}{r_I} \sum_{k=0}^{n_J} \left(3^{-k} [3^k r_J]^{-(n-1)} \int_{J_k} |f(x)-f_{J_k}|\, d\sigma(x) + 3^{-k} |f_{J_k} - f_J| \right) \\
&\le \frac{C}{r_I} \sum_{k=0}^{n_J} \left(3^{-k} \frac{1}{|J_k|} \int_{J_k} |f(x)-f_{J_k}|\, d\sigma(x) + C \sum_{l=1}^{k} 3^{-k} |f_{J_l} - f_{J_{l-1}}| \right) \\
&\le \frac{C}{r_I} \sum_{k=0}^{n_J} \left(\frac{3^{-k}}{|J_k|} \int_{J_k} |f(x)-f_{J_k}| d\sigma(x) + C \sum_{l=1}^{k} \frac{3^{-k}}{|J_{l-1}|} \int_{J_{l-1}} |f(x)-f_{J_l}|\, d\sigma(x) \right) \\
&\le \frac{C}{r_I} \sum_{k=0}^{n_J} \left(\frac{3^{-k}}{|J_k|} \int_{J_k} |f(x)-f_{J_k}|\, d\sigma(x) + C \sum_{l=1}^{k} \frac{3^{-k}|J_l|}{|J_{l-1}||J_l|} \int_{J_l} |f(x)-f_{J_l}|\, d\sigma(x) \right) \\
&\le \frac{C}{r_I} \sum_{k=0}^{n_J} \left(\frac{3^{-k}}{|J_k|} \int_{J_k} |f(x)-f_{J_k}| dS_x + C \sum_{l=1}^{k} \frac{3^{-k+n-1}}{|J_l|} \int_{J_l} |f(x)-f_{J_l}|\, d\sigma(x) \right) \\
&\le \frac{C}{r_I} \sum_{k=0}^{n_J} \left(3^{-k} \frac{1}{|J_k|} \left(\int_{J_k} \int_{J_k} |f(x)-f(y)|^2\, d\sigma(x)\, d\sigma(y) \right)^{\frac{1}{2}} \right. \\
&\quad \left. + C \sum_{l=1}^{k} 3^{-k} 3^{n-1} \frac{1}{|J_l|} \left(\int_{J_l} \int_{J_l} |f(x)-f(y)|^2\, d\sigma(x) d\sigma(y) \right)^{\frac{1}{2}} \right),
\end{aligned}$$

and by Lemma 4.9 we estimate

$$\le \frac{C}{r_I} \sum_{k=0}^{\infty} \left(3^{-k} (3^{k+1} r_J)^{(n-2)(p-1)/2} \|f\|_{\mathbf{D}_p} + C \sum_{l=1}^{k} 3^{-k+n-1} (3^l r_J)^{(n-2)(p-1)/2}\, \|f\|_{\mathbf{D}_p} \right)$$

hence

$$\int_{S\setminus\frac{2}{3}J} |f(x) - f_J|\,\frac{d\sigma(x)}{|x-x_0|^n} \le r_I^{((n-2)(p-1)-2)/2}\,\|f\|_{\mathbf{D}_p} \sum_{k=0}^{\infty} \frac{1}{3^k}(1 + k\,3^{n-1})\,,$$

thus

$$\begin{aligned} r_I^{n+p}\left(\int_{S\setminus\frac{2}{3}J} |f(x) - f_J|\,\frac{d\sigma(x)}{|x-x_0|^n}\right)^2 &\le C\,r_I^{(n-2)(p-1)-2+n+p}\,\|f\|^2_{\mathbf{D}_p} \\ &\le C r_I^{(n-1)p}\,\|f\|^2_{\mathbf{D}_p} \le \tilde{C}\,|I|^p\,\|f\|^2_{\mathbf{D}_p}\,. \end{aligned}$$

and

$$N = \sup_{I\subset S} \frac{1}{|I|^p} \int_{Q_h(x_0)} |\nabla F(x)|^2 \big(1 - |x|^2\big)^p\,dx \le \tilde{C}\|f\|^2_{\mathbf{D}_p} \le \tilde{C}\,M. \tag{4.3}$$

Now, equations (4.2) and (4.3) show that $M \approx N$ and hence Theorem 4.1 is proven. ∎

5. Harmonic space $\mathcal{H}(\Omega, \mathbf{Cl}_{0,n})$

Let $\Omega \subset \mathbb{R}^{m+1}$ be an open set. In this section we introduce the space of harmonic Clifford valued functions $\mathcal{H}(\Omega, \mathbf{Cl}_{0,n})$ defined by

$$\mathcal{H}(\Omega, \mathbf{Cl}_{0,n}) = \big\{u : \Omega \to \mathbf{Cl}_{0,n} \;:\; D\overline{D}u = \overline{D}Du = 0\big\}\,.$$

If $u = \sum_A u_A e_A$, then it is clear that $u \in \mathcal{H}(\Omega, \mathbf{Cl}_{0,n})$ if and only if $u_A : \Omega \to \mathbb{R}$ is a harmonic function for all $A \subset \{1, \ldots, n\}$. Then the space of left monogenic functions on Ω is a subset of $\mathcal{H}(\Omega, \mathbf{Cl}_{0,n})$, that is $\mathcal{M}(\mathbf{Cl}_{0,n}) \subset \mathcal{H}(\mathbf{Cl}_{0,n})$. Now we present some classical results in the context of harmonic Clifford valued functions.

We recall the Poisson kernel

$$P(x,\zeta) = \frac{1-|x|^2}{|x-\zeta|^{m+1}}\,, \tag{5.1}$$

then if $u = \sum_A u_A e_A \in \mathcal{H}(\mathbf{Cl}_{0,n}, B)$, we have $u(x) = \sum_A \int_B u_A(\zeta) P(x,\zeta) d\sigma(\zeta) e_A$. The following results are immediate and give us some idea on how to translate results of the classic harmonic theory. The proof for scalar-valued functions is given in [3]. The generalization to Clifford valued functions is easily seen from the definition of the norm and the estimation (1.1). The first two are two distortion theorems.

Theorem 5.1 (Harnacks Inequality for the ball). *Let $u : B \to \boldsymbol{Cl}_{0,n}$ be a harmonic function on B, with u_A positive, then*

$$\frac{1-|x|}{(1+|x|)^m}|u(0)|_0 \le |u(x)|_0 \le \frac{1+|x|}{(1-|x|)^m}|u(0)|_0$$

for all $x \in B$.

Theorem 5.2 (Harnacks Inequality). *Suppose that $\Omega \subset \mathbb{R}^{m+1}$ is an open and connected set and that K is a compact subset of Ω. Then there is a constant $C \in (1, \infty)$ such that*

$$\frac{1}{C} \le \frac{|u(y)|_0}{|u(x)|_0} \le C$$

for all points x and y in K and all harmonic function $u \in \mathcal{H}(\boldsymbol{Cl}_{0,n}, \Omega)$ with u_A positive.

We denote $S^+ = \{\zeta \in S : \zeta_m > 0\}$, $S^- = \{\zeta \in S : \zeta_m < 0\}$ and $\mathbf{N} = (0, \ldots, 0, 1) \in S$ the north pole of S.

Theorem 5.3 (Harmonic Schwarz Lemma). *Let $u : B \to \boldsymbol{Cl}_{0,n}$ be a harmonic function on B, $|u| < 1$ on B, and $u(0) = 0$. Then*

$$|u(x)|_0 \le 2^{\frac{n}{2}} U(|x|\boldsymbol{N})$$

for every $x \in B$, where $U = P[\chi_{S^+} - \chi_{S^-}]$ is the Poisson integral of the function $\chi_{S^+} - \chi_{S^-}$. Equality holds for some $0 \ne x \in B$ if and only if $u_A = U \circ T_A$, T_A an orthogonal transformation of S, for each $A \subset \{1, \ldots, n\}$.

Observation 5.4. Observe that $2^{\frac{n}{2}} U(|x|\mathbf{N}) = \left|\sum_A U(|x|\mathbf{N})e_A\right|_0$.

The following result is for the operator $\overline{D}$.

Theorem 5.5. *Let $u : B \to \boldsymbol{Cl}_{0,n}$ be a harmonic function on B, $|u| < 1$ on B. Then*

$$|\overline{D}u(0)|_0 \le (m+1)^{\frac{1}{2}} 2^{\frac{n+2}{2}} \frac{\mathrm{vol}(B_m)}{\mathrm{vol}(B_{m+1})}\ ,$$

where $\mathrm{vol}(B_i)$ *denotes the volume of the i-dimensional unit ball. Equality holds if and only if $u_A = U \circ T_A$, where T_A is an orthogonal transformation of S, for each $A \subset \{1, \ldots, n\}$.*

6. The spaces $\mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$ and $\mathbf{h}^2(\mathbf{Cl}_{0,n}, B)$

We consider in this section the unit ball $B \subset \mathbb{R}^m$, with $m \le n$. Now we give in the next theorem a characterization of Dirichlet spaces $\mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$, $-1 < p$, defined by

$$\mathbf{D}_p(\mathbf{Cl}_{0,n}, B) = \left\{ u \in \mathcal{H}(\mathbf{Cl}_{0,n}, B) : \int_B |\nabla u|_0^2 \big(1 - |x|^2\big)^p \, dV < \infty \right\},$$

where ∇u means now the vector gradient. Generalizing the results on $\mathbf{D_p}$ spaces of scalar valued harmonic functions we get immediately the next theorem.

Theorem 6.1. *Let $u : B \to \boldsymbol{Cl}_{0,n}$ be a harmonic function given by $u = \sum_A u_A e_A = \sum_A \big(\sum_{k=0}^{\infty} p_{k,A}\big) e_A$, where $p_{k,A} \in \mathcal{H}_k(\mathbb{R}^n)$. Let $-1 < p$. Then the following statements are equivalent:*

- **i)** *u belongs to the Dirichlet space $\mathbf{D}_p(\boldsymbol{Cl}_{0,n}, B)$;*
- **ii)** *each $u_A : B \to \mathbb{R}$ belongs to the Dirichlet space $\mathbf{D}_p(B)$;*
- **iii)** $\sum_A \sum_{k=0}^{\infty} k^{1-p} \|p_{k,A}\|_{\mathbf{h}^2}^2 < \infty$.

Let $0 < p$. We say that $u \in \mathcal{H}(\mathbf{Cl}_{0,n})$ belongs to the Bergman space $\mathbf{b}^p(\mathbf{Cl}_{0,n}, B)$ if $\|u\|_{\mathbf{b}^p,\mathrm{Cl}}^p = \int_B |u|_0^p \, dV < \infty$. Let $\alpha > 0$. We define the Bloch space $\mathbf{B}^\alpha(\mathbf{Cl}_{0,n}, B)$ as the set of harmonic functions $u : B \to \mathbf{Cl}_{0,n}$ such that $\sup_{x\in B}(1 - |x|)^\alpha |\nabla u(x)|_0 < \infty$. The following corollaries are an immediate consequence of Theorem 2.11 and Theorem 6.1.

Corollary 6.2. *The Dirichlet space $\mathbf{D}_2(\boldsymbol{Cl}_{0,n}, B)$ coincides with the Bergman space $\mathbf{b}^2(\boldsymbol{Cl}_{0,n}, B)$. Moreover, if $u \in \mathbf{b}^2(\boldsymbol{Cl}_{0,n}, B)$ is given by $u = \sum_A \sum_{k=0}^{\infty} p_{A,k} e_A$ where $p_{A,k} \in \mathcal{H}_k(\mathbb{R}^n)$, then*

$$\|u\|_{\mathbf{b}^2,\mathrm{Cl}}^2 = n \,\mathrm{vol}(B) \sum_A \sum_{k=0}^{\infty} \frac{1}{n+2k} \|p_{A,k}\|_2^2$$

for each $A \subset \{1, \ldots, n\}$.

Corollary 6.3. *Let $\alpha > 0$. Then $\mathbf{B}^\alpha(\boldsymbol{Cl}_{0,n}, B) \subset \bigcap_{2\alpha-1<p} \mathbf{D}_p(\boldsymbol{Cl}_{0,n}, B)$.*

In a similar way we can prove that if $p > -1$ then $\bigcup_{0<\alpha<\frac{p+1}{2}} \mathbf{B}^\alpha(\mathbf{Cl}_{0,n}, B) \subset \mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$. Let $f,\ g : S \to \mathbf{Cl}_{0,n}$ be Clifford valued functions. We define an inner product by $(f, g) = \int_S \overline{f} g \, d\sigma = \sum_{A,B} \int_S f_A g_B \, d\sigma \overline{e}_A e_B$. This inner product defines the right linear Hilbert space $L^2(\mathbf{Cl}_{0,n}, S)$. We say that f belongs to the class $L^2(\mathbf{Cl}_{0,n}, S)$ if $\|f\|_{2,\mathrm{Cl}} = \sqrt{(f, f)} = \big(\int_S |f|_0^2 \, d\sigma\big)^{\frac{1}{2}} < \infty$. Observe that

$$\|f_A\|_{L_2} \le \|f\|_{2,\mathrm{Cl}} \le \sum_A \|f_A\|_{L_2} \ . \tag{6.1}$$

We define the Hardy space $\mathbf{h}^2(\mathbf{Cl}_{0,n}, B)$ of harmonic Clifford valued functions to be the class of harmonic functions $u : B \to \mathbf{Cl}_{0,n}$ for which $\|u\|_{\mathbf{h}_{2,\mathrm{Cl}}} = \sup_{0\le r<1} \|u_r\|_{2,\mathrm{Cl}} < \infty$.

Proposition 6.4. *Let $u : B \to \boldsymbol{Cl}_{0,n}$ be a harmonic function given by $u = \sum_a u_A e_A$. Then $\|u_r\|_{2,\mathrm{Cl}} \le \|u_s\|_{2,\mathrm{Cl}}$ for all $0 \le r < s < 1$. Therefore if $u \in \mathbf{h}^2(\boldsymbol{Cl}_{0,n}, B)$, then*

$$\|u\|_{h_{2,\mathrm{Cl}}} = \lim_{r\to 1^-} \|u_r\|_{2,\mathrm{Cl}} \ .$$

Proof. Let $0 < r < s < 1$. Then it is known from the classic real harmonic case that $\|u_{A,r}\|_2 \le \|u_{A,s}\|_2$. By definition we have

$$\|u_r\|_{2,\mathrm{Cl}} = \Big(\sum_A \|u_{A,r}\|_2^2\Big)^{\frac{1}{2}} \le \Big(\sum_A \|u_{A,s}\|_2^2\Big)^{\frac{1}{2}} = \|u_s\|_{2,\mathrm{Cl}} .$$ ∎

Proposition 6.5. *Let $f \in L^2(\boldsymbol{Cl}_{0,n})$, $u = P[f]$ that is*

$$u(x) = \sum_A \int_B f_A(\zeta)P(x,\zeta)d\sigma(\zeta)e_A ,$$

where P is the Poisson kernel given by (5.1). *Then $\|u_r\|_{2,\mathrm{Cl}} \le \|f\|_{2,\mathrm{Cl}}$ for all $0 \le r < 1$.*

Proof. Let $0 < r < 1$. It follows immediately from the real classical estimation $\|u_{A,r}\|_2 \le \|f_A\|_2$ in the harmonic case. ∎

Theorem 6.6. *The map $f \to P[f] = u$ is a linear isometry of $L^2(\boldsymbol{Cl}_{0,n}, S)$ onto $\mathbf{h}^2(\boldsymbol{Cl}_{0,n}, B)$.*

Proof. It is clear that the map is linear. By Proposition 6.5 we have

$$0 \le \|f\|_{2,\mathrm{Cl}} - \|u_r\|_{2,\mathrm{Cl}} \le \|f - u_r\|_{2,\mathrm{Cl}} \to 0 \qquad \text{as } r \to 1^{-1}$$

since for each A, $\|f_A - u_{A,r}\| \to 0$ as $r \to 1^-$. Therefore by Proposition 6.4 $\|f\|_{2,\mathrm{Cl}} = \lim_{r\to 1^-} \|u_r\|_{2,\mathrm{Cl}} = \|u\|_{h_{2,\mathrm{Cl}}}$. It is clear from (6.1) that $u \in \mathbf{h}^2(\mathbf{Cl}_{0,n}, B)$ if and only if $u_A \in \mathbf{h}^2(B)$, then there exists $f_A \in L^2$ such that $u_A = P[f_A]$. We define $f = \sum_A f_A e_A$. Then $f \in L^2(\mathbf{Cl}_{0,n})$ and $u = P[f]$. ∎

7. Primitives in $\mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$

Let $u \in \mathcal{H}(\mathbf{Cl}_{0,n}, B)$. We say that $U \in \mathcal{H}(\mathbf{Cl}_{0,n}, B)$ is a primitive of u if $\overline{D}U = u$. Then $0 = \Delta U = D\overline{D}U = Du$, that is if $u \in \mathcal{H}(\mathbf{Cl}_{0,n}, B)$ admits a primitive, then necessarily u is left monogenic. Each function $U \in \mathcal{H}(\mathbf{Cl}_{0,n}, B)$ is a primitive of at least $u = \overline{D}U$. Moreover if U is a primitive of u, then $\{U + f \,:\, f \in \mathcal{H}(\mathbf{Cl}_{0,n}, B) \text{ and } \overline{D}f = 0\}$ is the set of primitives of u. Let $\mathcal{S}(\mathbf{Cl}_{0,n}) \subset \mathcal{H}(\mathbf{Cl}_{0,n})$ define the space of primitives of S, that is

$$P(S) = \big\{U \in \mathcal{H}(\mathbf{Cl}_{0,n}) : \overline{D}U = u \in S\big\}.$$

Then $P(S) = P(S \cap \mathcal{M}(\mathbf{Cl}_{0,n}, B)) \subset P(\mathcal{M}(\mathbf{Cl}_{0,n}, B))$. In [9], Gürlebeck and Malonek proved that $P(\mathcal{M}(\mathbf{Cl}_{0,n}, B)) \neq \emptyset$, even more if $u \in \ker D \cap L^2(\Omega)$, then u admits a primitive $U \in \ker D \cap W_2^1(\Omega)$, where W_2^1 stands for the functions where all coordinates functions belong to the corresponding Sobolev space, that is we can choose as primitive a monogenic function. We want to characterize the space of primitives of $\mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$, that is

$$P\big(\mathbf{D}_p(\mathbf{Cl}_{0,n}, B)\big) = P\big(\mathbf{D}_p(\mathbf{Cl}_{0,n}, B) \cap \mathcal{M}(\mathbf{Cl}_{0,n}, B)\big).$$

We have a partial answer.

Theorem 7.1. *Let $1 < p$ be fixed. Then*

$$P(\mathbf{D}_p(\boldsymbol{Cl}_{0,n}, B)) = P(\mathbf{D}_p(\boldsymbol{Cl}_{0,n}, B) \cap \mathcal{M}(\boldsymbol{Cl}_{0,n}, B)) = \mathbf{D}_{p-2}(\boldsymbol{Cl}_{0,n}, B).$$

Proof. Let $U,\ u \in \mathcal{H}(\mathbf{Cl}_{0,n}, B)$ be with $\overline{D}U = u$ and given by $U = \sum_A U_A e_a$, $u = \sum_A u_a e_A = \sum_A (\sum_{k=0}^{\infty} p_{A,k}) e_A$, where $p_{A,k} \in \mathcal{H}_k(\mathbb{R}^n)$ for each $A \subset \{1, \ldots, n\}$. Because of $|\overline{D}U|_0 = |u|_0$ we get

$$|\overline{D}U|_0^2 = |u|_0^2 = \sum_A |u_A|^2 = \sum_A \Big| \sum_{k=0}^{\infty} p_{A,k} \Big|^2. \tag{7.1}$$

Observe that we do not assure in (7.1) that $|\nabla U_A|^2 = |\sum_{k=0}^{\infty} p_{A,k}|^2$. Let $0 < R < 1$ and $-1 < p$. Then $\overline{B_R} \subset B$, by (2.2), Lemma 2.4 and due to the uniform convergence of the series (2.4) we have

$$\begin{aligned}
&\int_{B_R} |\overline{D}U(x)|_0^2 \big(1 - |x|^2\big)^{p-2}\, dV \\
&\quad = \int_{B_R} \sum_A \Big| \sum_{k=0}^{\infty} p_{A,k}(x) \Big|^2 \big(1 - |x|^2\big)^{p-2}\, dV \\
&\quad = n \operatorname{vol}(B) \sum_A \int_0^R r^{n-1} \big(1 - r^2\big)^{p-2} \int_S \Big| \sum_{k=0}^{\infty} p_{A,k}(r\zeta) \Big|^2 d\sigma(\zeta)\, dr \\
&\quad = n \operatorname{vol}(B) \sum_A \int_0^R r^{n-1} \big(1 - r^2\big)^{p-2} \int_S \sum_{k=0}^{\infty} p_{A,k}(r\zeta) \overline{\sum_{l=0}^{\infty} p_{A,l}(r\zeta)\, d\sigma(\zeta)\, dr} \\
&\quad = n \operatorname{vol}(B) \sum_A \int_0^R r^{n-1} \big(1 - r^2\big)^{p-2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} r^{k+l} \int_S p_{A,k}(\zeta) \overline{p_{A,l}(\zeta)}\, d\sigma(\zeta)\, dr \\
&\quad = n \operatorname{vol}(B) \sum_A \sum_{k=0}^{\infty} \int_0^R r^{n+2k-1} (1 - r^2)^{p-2} \|p_{A,k}(\zeta)\|_2^2\, dr\,.
\end{aligned}$$

We observe that if $U \in \mathbf{D}_{p-2}(\mathbf{Cl}_{0,n}, B)$ or if $u \in \mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$ the limit $R \to 1^-$

exists in the previous formula and we obtain by Abel's Theorem

$$\int_B |\overline{D}U(x)|_0^2\big(1-|x|^2\big)^{p-2}\,dV$$
$$= n\,\mathrm{vol}(B)\sum_A\sum_{k=0}^{\infty}\int_0^1 r^{n+2k-1}\big(1-r^2\big)^{p-2}\|p_{A,k}(\zeta)\|_2^2\,dr$$
$$= n\,\mathrm{vol}(B)\sum_A\sum_{k=0}^{\infty}\frac{\Gamma(\frac{n}{2}+k)\Gamma(p-1)}{\Gamma(\frac{n}{2}+k+p-1)}\|p_{A,k}(\zeta)\|_2^2\,,$$

and this last series converges if and only if $\sum_{k=0}^{\infty} k^{1-p}\|p_{A,k}(\zeta)\|_2^2$ is a convergent series for all $A \in \{1,\dots,n\}$. ∎

We have the following relation between Dirichlet space and Bergman space.

Corollary 7.2. *It holds:*

$$P\big(\mathbf{b}^2(\boldsymbol{Cl}_{0,n},B)\big)=P\big(\mathbf{b}^2(\boldsymbol{Cl}_{0,n},B)\cap\mathcal{M}(\boldsymbol{Cl}_{0,n},B)\big)=\mathbf{D}_0(\boldsymbol{Cl}_{0,n},B)=\mathbf{D}(\boldsymbol{Cl}_{0,n},B).$$

Example 7.3. It is well known that the Cauchy kernel

$$E(x)=\frac{1}{\mathrm{vol}(B)}\frac{\overline{x}}{|x|^{m+1}} \qquad \text{where} \qquad \overline{x}=\textstyle\sum_{i=0}^{m} x_i e_i$$

is left monogenic. We consider its modified translation $E_\zeta(x)=\mathrm{vol}(B)E(x-\zeta)$ with $\zeta\in S$. Then

$$\overline{D}E_\zeta(x)=2\frac{\partial E_\zeta(x)}{\partial x_0}=\frac{1}{|x-\zeta|^{2m+2}}+\frac{(m+1)^2(x_0-\zeta_0)^2}{|x-\zeta|^{2m+4}}-\frac{(m+1)^2(x_0-\zeta_0)^4}{|x-\zeta|^{2m+6}}\,.$$

We claim that E_ζ belongs to $\mathbf{D}_p(\mathbf{Cl}_{0,n},B)$ for each $p>3m+3$. For example, we calculate

$$\int_B\left|\frac{(x_0-\zeta_0)^4}{|x-\zeta|^{2m+6}}\right|^2\big(1-|x|^2\big)^p dx=\int_B\frac{|x_0-\zeta_0|^8}{|x-\zeta|^{4m+12}}\big(1-|x|^2\big)^p dx$$
$$=\int_B\frac{|x_0-1|^8}{|x-e|^{4m+12}}\big(1-|x|^2\big)^p dx$$

where $e=(1,0\dots,0)\in S$. Now we translate and change to spherical coordinates

$$\int_B\frac{|x_0-1|^8}{|x-e|^{4m+12}}\big(1-|x|^2\big)^p\,dx$$
$$=\int_{B(e_1,1)}\frac{x_0^8\big(1-|x+e|^2\big)^p}{|x|^{4m+12}}\,dx$$
$$=\Gamma\int_{\frac{\pi}{2}}^{\pi}\sin^{m-1}\theta_1\,d\theta_1\int_0^{-2\cos\theta_1}\frac{r^8\cos^8\theta\big(-r^2-2r\cos\theta_1\big)^p}{r^{4m+12}}r^m dr,$$

where $\Gamma = \int_0^{2\pi} d\theta_m \int_0^{\pi} \sin\theta_{m-1} d\theta_{m-1} \cdots \int_0^{\pi} \sin^{m-2}\theta_2\, d\theta_2 < \infty$. Now

$$\int_{\frac{\pi}{2}}^{\pi} \sin^{m-1}\theta_1 d\theta_1 \int_0^{-2\cos\theta_1} \frac{r^8\cos^8\theta(-r^2-2r\cos\theta_1)^p}{r^{4m+12}} r^m\, dr$$
$$= \int_{\frac{\pi}{2}}^{\pi} \sin^{m-1}\theta_1 d\theta_1 \int_0^{-2\cos\theta_1} \frac{\cos^8\theta(-r-2\cos\theta_1)^p}{r^{3m+4-p}}\, dr\,,$$

and this last integral exists if and only if $3m+4-p<1$, that is $3m+3<p$. A similar result is obtained with the other terms of $\overline{D}E_\zeta(x)$. Then its primitive set is a subset of $\mathbf{D}_p(\mathbf{Cl}_{0,n}, B)$ for each $3m+1<p$.

Acknowledgement. The authors gratefully acknowledge the support by DFG (GU 407/4-1), COFAA-IPN and CONACYT 37542. We thank R. Aulaskari and J. Xiao for their interest in our research and for many lively discussions.

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Received 16.11.2004