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Nonlinear Boundary Value Problems Involving the p-Laplacian and p-Laplacian-Like Operators

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Abstract. We study nonlinear boundary value problems for systems driven by the vector p-Laplacian or p-Laplacian-like operators and having a maximal monotone term. We consider periodic problems and problems with nonlinear boundary conditions formulated in terms of maximal monotone operators. This way we achieve a unified treatment of the classical Dirichlet, Neumann and periodic problems. Our hypotheses permit the presence of Hartman and Nagumo-Hartman nonlinearities, partially extending this way some recent works of Mawhin and his coworkers.

Keywords: *Ordinary p-Laplacian, p-Laplacian-like operator, maximal monotone operator, Nagumo-Hartaman nonlinearity, fixed point, complete continuity*

MSC 2000: 34B15, 34C25

1. Introduction

In this paper we study the following two nonlinear boundary value problems in $\mathbb{R}^N$:

$$\begin{cases} \big(\alpha(x'(t))\big)' \in A(x(t)) + F\big(t, x(t), x'(t)\big) \quad \text{a.e. on } T = [0, b] \\ x(0) = x(b),\ x'(0) = x'(b), \end{cases} \tag{1}$$

and

$$\begin{cases} \big(\|x'(t)\|^{p-2}x'(t)\big)' \in A(x(t)) + F\big(t, x(t), x'(t)\big) \quad \text{a.e. on } T = [0, b] \\ \big(\varphi_p(x'(0)), -\varphi_p(x'(b))\big) \in \xi\big(x(0), x(b)\big),\ 1 < p < \infty. \end{cases} \tag{2}$$

Here $a : \mathbb{R}^N \to \mathbb{R}^N$ is a suitable homeomorphism which is not in general homogeneous, $A : D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map, $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to$

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$2^{\mathbb{R}^N} \setminus \{\varnothing\}$ is a multivalued in general nonlinearity satisfying Caratheodory type conditions, $\varphi_p : \mathbb{R}^N \to \mathbb{R}^N$ is the homeomorphism defined by

$$\varphi_p(r) = \begin{cases} \|r\|^{p-2}r & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}$$

and $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map.

Boundary value problems involving the ordinary p-Laplacian have been the focus of attention of many researchers in the last decade. Most of the works deal with the scalar problem. We refer to the works of Boccardo-Drabek-Giachetti-Kucera [2], De Coster [4], Del Pino-Manasevich-Murua [5], Fabry-Fayyad [8], Guo [11] and the references therein. We also mention the work of Dang-Oppenheimer [3], where the ordinary scalar p-Laplacian is replaced by a one-dimensional possibly nonhomogeneous nonlinear differential operator.

Recently in a series of interesting papers, Mawhin and coworkers studied systems driven by the ordinary vector p-Laplacian or p-Laplacian like operators and having primarily periodic boundary conditions. We refer to the papers of Manasevich-Mawhin [16] Mawhin [18, 19] and Mawhin-Urena [20]. As the Nagumo-Hartman condition used here is distinct from the one used by Mawhin-Urena [20] we provide a partial extension of the works by Mawhin [18] and Mawhin-Urena [20], where the authors employ nonlinearities of the Hartman and Nagumo-Hartman type. Also in these works the ordinary vector p-Laplacian with periodic boundary conditions is used, $A \equiv 0$ and the nonlinearity is single-valued.

The problems that we study here are more general since they involve the maximal monotone operator A, which in the case of Problem (1) is not necessarily defined everywhere (see hypotheses $H(A)_1$). This way we incorporate in our framework differential variational inequalities. Moreover, in the case of Problem (2), the nonlinear multivalued boundary conditions used here achieve a unified treatment of the Dirichlet, Neumann and periodic problems and go beyond them (see Section 5). This way we extend the semilinear works (i.e., $p = 2$) of Erbe-Krawcewicz [7], Frigon [9], Kandilakis-Papageorgiou [14] and Halidias-Papageorgiou [12] and the recent nonlinear works of Kyritsi-Matzakos-Papageorgiou [15] and Papageorgiou-Papageorgiou [21]. Our approach is based on nonlinear operator theory and fixed point arguments.

2. Mathematical background

Let (Ω, Σ) be a measurable space and X a separable Banach space. We introduce the notations

$$P_{f(c)}(X) = \big\{A \subseteq X : A \text{ is nonempty, closed (and convex)}\big\}$$
$$P_{(w)k(c)}(X) = \big\{A \subseteq X : A \text{ is nonempty, (weakly) compact (and convex)}\big\}.$$

A multifunction $F : \Omega \to P_f(X)$ is said to be *measurable*, if for all $x \in X$ $\omega \to d(x, F(\omega)) = \inf[\|x - u\| : u \in F(\omega)]$ is measurable. Also we say that $F : \Omega \to 2^X \setminus \{\varnothing\}$ is *graph measurable*, if $\mathrm{Gr}F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For multifunctions with values in $P_f(X)$ measurability implies graph measurability, while the converse holds if Σ is complete. Next let (Ω, Σ, μ) be a finite measure space and $F : \Omega \to 2^X \setminus \{\varnothing\}$ a multifunction. For $1 \le p \le \infty$ we introduce the set

$$S_F^p = \big\{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \ \ \mu - \text{a.e. on } \Omega\big\}.$$

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \setminus \{\varnothing\}$ is said to be *upper semicontinuous* (usc for short) (respectively *lower semicontinuous* (lsc for short)), if for every closed set $C \subseteq Z$, the set $G^-(C) = \{y \in Y : G(y) \cap C \neq \varnothing\}$ (respectively the set $G^+(C) = \{y \in Y : G(y) \subseteq C\}$) is closed in Y. If Z is regular and F is $P_f(Z)$-valued and usc, then it has a closed graph, i.e., $\mathrm{Gr}G = \{(y, z) \in Y \times Z : z \in G(y)\}$ is closed in $Y \times Z$. The converse is true if G is locally compact.

Now let X be a reflexive Banach space and X^* its topological dual. Recall that a monotone, demicontinuous operator $A : X \to X^*$ is maximal monotone. Also a maximal monotone coercive operator, is surjective. When $X = H$ (Hilbert space) and $A : D(A) \subseteq H \to 2^H$ is a maximal monotone operator, then for every $\lambda > 0$ we introduce the well-known operators

$$J_\lambda = (I + \lambda A)^{-1} \quad \text{(resolvent of } A\text{)}$$
$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda) \quad \text{(Yosida approximation of } A\text{)}.$$

Both operators are single-valued and defined on all of H. Moreover, J_λ is nonexpansive, while A_λ is Lipschitz continuous with constant $\frac{1}{\lambda}$ (hence A_λ is maximal monotone).

We return to the general case of X being a reflexive Banach space. An operator $A : X \to 2^{X^*}$ is said to be *pseudomonotone*, if

(a) for all $x \in X$, $A(x) \in P_{wkc}(X^*)$;
(b) A is usc from every finite dimensional subspace Z of X into X_w^*;
(c) if $x_n \xrightarrow{w} x$ in X, $x_n^* \in A(x_n)$ and $\limsup_{n\to\infty} \langle x_n^*, x_n - x\rangle \le 0$, then for every $y \in X$, there exists $x^*(y) \in A(x)$ such that $\langle x^*(y), x - y\rangle \le \liminf_{n\to\infty} \langle x_n^*, x_n - y\rangle$.

We say that $A : D(A) \subseteq X \to 2^{X^*}$ is *generalized pseudomonotone*, if for all $x_n^* \in A(x_n)$ such that $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* and $\limsup_{n\to\infty} \langle x_n^*, x_n - x\rangle \le 0$, we have $x^* \in A(x)$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$. A maximal monotone operator is generalized pseudomonotone and a pseudomonotone operator is generalized pseudomonotone. A generalized pseudomonotone operator is pseudomonotone, if it is everywhere defined and bounded. A pseudomonotone coercive operator is surjective and the sum of pseudomonotone operators is again a pseudomonotone operator. For details on multifunctions and nonlinear operators of monotone type, we refer to the books of Hu-Papageorgiou [13] and Zeidler [22].

Recall that if V, Z are Banach spaces and $K : V \to Z$, we say that K is *completely continuous*, if $v_n \xrightarrow{w} v$ in V implies that $K(v_n) \to K(v)$ in Z. In our analysis of problems (1) and (2) we shall use the following multivalued nonlinear alternative theorem due to Bader [1] which improves a result of Dugundji-Granas [6, p. 98].

Proposition 2.1. *If X, Y are Banach spaces with Y reflexive, W is a bounded open subset of X with $0 \in W$, $G : \overline{W} \to P_{wkc}(Y)$ is usc from $\overline{W}$ into Y_w, bounded, and $K : Y \to X$ is completely continuous, then one of the following alternatives holds:*

- **(a)** *there exist $x_0 \in \partial W$ and $s \in (0,1)$ such that $x_0 \in s(K \circ G)(x_0)$; or*
- **(b)** *$\Phi = G \circ K$ has a fixed point (i.e., there exist $\overline{x} \in \overline{W}$ such that $\overline{x} \in \Phi(\overline{x})$).*

3. Problems with p-Laplacian–like operators

In this section we deal with Problem (1) and we do not require that $D(A) = \mathbb{R}^N$. Our analysis of Problem (1) starts with the study of the auxiliary periodic problem

$$\begin{cases} -(\alpha(x'(t)))' + A_\lambda(x(t)) + \|x(t)\|^{p-2}x(t) = g(t) \quad \text{a.e. on } T = [0,b] \\ x(0) = x(b),\ x'(0) = x'(b), \end{cases} \tag{3}$$

where $1 < p < \infty, g \in L^q(T, \mathbb{R}^N), \frac{1}{p} + \frac{1}{q} = 1$ and $\lambda > 0$. We introduce the following hypotheses on the maps a and A:

H(a)$_1$: $a : \mathbb{R}^N \to \mathbb{R}^N$ is continuous, strictly monotone and there exists a function $\gamma : [0, +\infty) \to [0, +\infty)$ such that $\gamma(r) \to +\infty$ as $r \to +\infty$ and for all $x \in \mathbb{R}^N$ we have $\gamma(\|x\|)\|x\| \leq (a(x), x)_{\mathbb{R}^N}$.

H(A)$_1$: $A : D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $0 \in A(0)$.

Remark 3.1. We emphasize that we do not require that $D(A) = \mathbb{R}^N$.

In what follows we shall use the two spaces $C^1_{per}(T, \mathbb{R}^N) = \{x \in C^1(T, \mathbb{R}^N) : x(0) = x(b), x'(0) = x'(b)\}$ and $W^{1,p}_{per}(T, \mathbb{R}^N) = \{x \in W^{1,p}(T, \mathbb{R}^N) : x(0) = x(b)\}$.

Proposition 3.2. *If hypotheses* H(a)$_1$ *and* H(A)$_1$ *hold, then Problem* (3) *has a unique solution $x \in C^1_{per}(T, \mathbb{R}^N)$ such that $a(x') \in W^{1,q}_{per}(T, \mathbb{R}^N)$.*

Proof. Let $f : T \times \mathbb{R}^N \to \mathbb{R}^N$ be defined by $f(t,x) = A_\lambda(x) + \|x\|^{p-2}x - g(t)$. Evidently f is a Caratheodory function. Also let $\eta : \mathbb{R}^N \to \mathbb{R}^N$ be defined by $\eta(x) = x$. Then if $h(t) = h_1^+(t)$ where $h_1(t) = \sup_{r>0}[-r^p + r^{p-1} + \frac{1}{\lambda}r + \|g(t)\|r + \|g(t)\|]$, and $R_0 > \max\{1, \|\overline{g}\|\}$ where $\overline{g} = \frac{1}{b}\int_0^b g(t)dt$, with all the above data we can apply Corollary 3.1 of Manasevich-Mawhin [16] and obtain a solution for (3). The uniqueness follows at once from hypotheses $\mathrm{H}(\alpha)_1$ and the monotonicity of A_λ and strict monotonicity of φ_p. ∎

Let $\widehat{D} = \{x \in C^1_{per}(T,\mathbb{R}^N) : a(x') \in W^{1,q}_{per}(T,\mathbb{R}^N)\}$. For $\lambda > 0$, let $S_\lambda : \widehat{D} \subseteq L^p(T,\mathbb{R}^N) \to L^q(T,\mathbb{R}^N)$ be the nonlinear operator defined by $S_\lambda(x) = -(a(x'))' + \widehat{A}_\lambda(x)$, where for every $x \in \widehat{D}$, $\widehat{A}_\lambda(x)(\cdot) = A_\lambda(x(\cdot))$. Note that if $x \in \widehat{D}$, then $A_\lambda(x(\cdot)) \in C(T,\mathbb{R}^N)$.

Proposition 3.3. *If the hypothesis* H(a)$_1$ *holds and* $\lambda > 0$, *then* $S_\lambda : \widehat{D} \subseteq L^p(T,\mathbb{R}^N) \to L^q(T,\mathbb{R}^N)$ *is maximal monotone.*

Proof. Let $J : L^p(T,\mathbb{R}^N) \to L^q(T,\mathbb{R}^N)$ be the continuous, strictly monotone (thus maximal monotone) operator defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$. From Proposition 3.2 we know that $R(S_\lambda + J) = L^q(T,\mathbb{R}^N)$. We will show that S_λ is maximal monotone. Indeed first note that S_λ is monotone. Suppose that for some $y \in L^p(T,\mathbb{R}^N)$ and some $v \in L^q(T,\mathbb{R}^N)$, we have

$$(S_\lambda(x) - v, x - y)_{qp} \geq 0 \quad \text{for all } x \in \widehat{D}. \tag{4}$$

Hereafter by $(\cdot,\cdot)_{qp}$ we denote the duality brackets for the pair $(L^q(T,\mathbb{R}^N), L^p(T,\mathbb{R}^N))$. Since $S_\lambda + J$ is surjective, we can find $x_1 \in \widehat{D}$ such that $S_\lambda(x_1) + J(x_1) = v + J(y)$. Using this in (4) with $x = x_1 \in \widehat{D}$, we obtain $y = x_1 \in \widehat{D}$ since J is strictly monotone and $v = S_\lambda(x_1)$. ∎

Next we study of the following regular approximation of Problem (1):

$$\begin{cases} \big(\alpha(x'(t))\big)' \in A_\lambda(x(t)) + F\big(t, x(t), x'(t)\big) & \text{a.e. on } T = [0,b] \\ x(0) = x(b),\ x'(0) = x'(b), \end{cases} \tag{5}$$

where $\lambda > 0$. Our hypotheses on the data of (5) are the following:

H(a)$_2$: $a : \mathbb{R}^N \to \mathbb{R}^N$ is a monotone map such that $a(y) = c(y)y$ or $a(y) = (c_k(y_k)y_k)_{k=1}^N$ for all $y = (y)_{k=1}^N \in \mathbb{R}^N$, with $c : \mathbb{R}^N \to \mathbb{R}_+$ and $c_k : \mathbb{R} \to \mathbb{R}_+$, $k \in \{1,...,N\}$, continuous maps and for all $y \in \mathbb{R}^N$ we have $(a(y),y)_{\mathbb{R}^N} \geq c_0\|y\|^p$ for some $c_0 > 0$.

H(F)$_1$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x, y \in \mathbb{R}^N$, $t \to F(t,x,y)$ is graph measurable;

(ii) for almost all $t \in T$, $(x,y) \to F(t,x,y)$ has closed graph;

(iii) for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t,x,y)$ we have

$$(v,x)_{\mathbb{R}^N} \geq -c_1\|x\|^p - c_2\|x\|^r\|y\|^{p-r} - c_3(t)\|x\|^s$$

with $c_1, c_2 > 0$, $c_3 \in L^1(T)_+$, $1 \leq r, s < p$;

(iv) there exists $M > 0$ such that if $\|x_0\| = M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, we can find a $\delta > 0$ such that for almost all $t \in T$, we have

$$\inf\left[(v,x)_{\mathbb{R}^N} + c_0\|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta, v \in F(t,x,y)\right] \geq 0\,;$$

(v) for almost all $t \in T$, all $\|x\| \leq M$, all $y \in \mathbb{R}^N$ and all $v \in F(t,x,y)$, we have

$$\|v\| \leq c_4(t) + c_5\|y\|^{p-1}$$

with $c_4(t) \in L^\eta(T)_+$, $\eta = \max\{2, q\}$, $c_5 > 0$.

Remark 3.4. Hypothesis $\mathrm{H(F)}_1$(iv) is a suitable extension to the present setting of the so-called "Hartman condition" (see Mawhin [19]).

Proposition 3.5. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(A)}_1$ *and* $\mathrm{H(F)}_1$ *hold, then Problem* (5) *has a solution* $x \in C^1_{per}(T, \mathbb{R}^N)$ *with* $a(x') \in W^{1,q}_{per}(T, \mathbb{R}^N)$.

Proof. First we do the proof by assuming the following stronger version of hypothesis $\mathrm{H(F)}_1$(iv):

"(iv)" there exists an $M > 0$ such that if $\|x_0\| = M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, we can find $\delta > 0$ and $c_6 > 0$ such that for almost all $t \in T$ we have

$$\inf\left[(v,x)_{\mathbb{R}^N} + c_0\|y\|^p : \|x - x_0\| + \|y - y_0\| < \delta, v \in F(t,x,y)\right] \geq c_6 > 0. \quad (6)$$

Let $S_\lambda : \widehat{D} \subseteq L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be the maximal monotone operator introduced earlier in this section (see Proposition 3.3). Also as before let $J : L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$. This operator is maximal monotone. Set $V_\lambda = S_\lambda + J$. Then V_λ is maximal monotone. Also let $U : \widehat{D} \subseteq L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be the nonlinear differential operator defined by $U(x) = -(a(x'))'$, $x \in \widehat{D}$. From Proposition 3.3 we have that U is maximal monotone. Clearly V_λ is coercive. So $R(V_\lambda) = L^q(T, \mathbb{R}^N)$. Moreover, V_λ is also injective. So we can define the map

$$K_\lambda = V_\lambda^{-1} : L^q(T, \mathbb{R}^N) \to \widehat{D} \subseteq W^{1,p}_{per}(T, \mathbb{R}^N).$$

`Claim 1:` $K_\lambda : L^q(T, \mathbb{R}^N) \to W^{1,p}_{per}(T, \mathbb{R}^N)$ is completely continuous.

Suppose that $u_n \xrightarrow{w} u$ in $L^q(T, \mathbb{R}^N)$. Set $x_n = K_\lambda(u_n)$, $n \geq 1$. We have

$$\|x_n\|^{p-1}_{1,p} \leq c_8\|u_n\|_q \quad \text{with } c_8 > 0\,,$$

hence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^N)$ is bounded. Therefore we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T, \mathbb{R}^N)$ and $x_n \to x$ in $L^p(T, \mathbb{R}^N)$. Because $u_n = V_\lambda(x_n)$, $n \geq 1$, it follows that $u = V_\lambda(x) = S_\lambda(x) + J(x) = U(x) + \widehat{A}_\lambda(x) + J(x)$. For every $n \geq 1$, $x_n \in \widehat{D}$ and so $a(x_n') \in W^{1,q}_{per}(T, \mathbb{R}^N)$. Hence $a(x_n') = \overline{a}_n + \widehat{a}_n$, with $\overline{a}_n \in \mathbb{R}^N$ and $\widehat{a}_n \in V = \{v \in W^{1,q}_{per}(T, \mathbb{R}^N) : \int_0^b v(t)dt = 0\}$. From the equation $U(x_n) + \widehat{A}_\lambda(x_n) + J(x_n) = u_n$, if follows that $\{(a(x_n'))'\}_{n\geq1} \subseteq L^q(T, \mathbb{R}^N)$ is bounded, hence it follows that $\{\widehat{a}_n\}_{n\geq1} \subseteq C(T, \mathbb{R}^N)$ is relatively compact. For every $n \geq 1$ and every $t \in T$, we have

$$x_n'(t) = a^{-1}(\overline{a}_n + \widehat{a}_n(t)).$$

Integrating this equation over $T = [0, b]$ and since $x_n(0) = x_n(b)$, we obtain

$$\int_0^b a^{-1}\big(\overline{a}_n + \widehat{a}_n(t)\big)dt = 0.$$

Invoking Proposition 2.2 of Manasevich-Mawhin [16], we infer that $\{\overline{a}_n\}_{n\geq1} \subseteq \mathbb{R}^N$ is bounded. So we conclude that $\{a(x_n')\}_{n\geq1} \subseteq C(T, \mathbb{R}^N)$ is relatively compact. Hence $\{a(x_n')\}_{n\geq1} \subseteq W^{1,q}_{per}(T, \mathbb{R}^N)$ is bounded and so we may assume that $a(x_n') \xrightarrow{w} \beta$ in $W^{1,q}_{per}(T, \mathbb{R}^N)$. Because $x_n \to x$ in $L^p(T, \mathbb{R}^N)$ and U is maximal monotone, it follows that $\beta = U(x)$, hence $a(x_n') \xrightarrow{w} a(x')$ in $W^{1,q}_{per}(T, \mathbb{R}^N)$ and so $a(x_n') \to a(x')$ in $C(T, \mathbb{R}^N)$. So we have that $x_n' \to x'$ in $C(T, \mathbb{R}^N)$. Therefore finally we can say that $x_n \to x$ in $W^{1,p}_{per}(T, \mathbb{R}^N)$ (in fact we have shown that $x_n \to x$ in $C^1(T, \mathbb{R}^N)$). We conclude that the whole sequence $\{x_n = K_\lambda(u_n)\}_{n\geq1}$ strongly converges to $x = K_\lambda(u)$. This proves the claim.

Next let $N : C = \{x \in W^{1,p}_{per}(T, \mathbb{R}^N) : \|x(t)\| \leq M \text{ for all } t \in T\} \to L^q(T, \mathbb{R}^N)$ be the multivalued operator defined by $N(x) = S^q_{F(\cdot,x(\cdot),x'(\cdot))}$. From Hu-Papageorgiou [13, p. 236] we know that N has values in $P_{wkc}(L^q(T, \mathbb{R}^N))$ and it is usc from C with the relative $W^{1,p}_{per}(T, \mathbb{R}^N)$-norm topology into $L^q(T, \mathbb{R}^N)_w$. Set $N_1(x) = -N(x) + J(x)$. Then Problem (5) is equivalent to the abstract multivalued fixed point problem

$$x \in K_\lambda N_1(x). \tag{7}$$

Let $M_1 > 0$ be such that $M_1^p > \frac{p}{rc_0}\Big[c_1 M^p b + \frac{rc_2^{\frac{p}{r}} M^p b^{\frac{p}{r}}}{c_0 p} + \|c_3\|_1 M^s\Big]$. We consider the following set in $W^{1,p}_{per}(T, \mathbb{R}^N)$:

$$W = \big\{x \in W^{1,p}_{per}(T, \mathbb{R}^N) : \|x(t)\| < M \text{ for all } t \in T \text{ and } \|x'\|_p < M_1\big\}.$$

Set $W_1 = \{x \in W^{1,p}_{per}(T, \mathbb{R}^N) : \|x(t)\| < M \text{ for all } t \in T\}$ and $W_2 = \{x \in W^{1,p}_{per}(T, \mathbb{R}^N) : \|x'\|_p < M_1\}$. We have $W = W_1 \cap W_2$ and W_1, W_2 are open. So $W = W_1 \cap W_2$ is an open and of course bounded subset of $W^{1,p}_{per}(T, \mathbb{R}^N)$ with $0 \in W$. Note that $\overline{W} = \{x \in W^{1,p}_{per}(T, \mathbb{R}^N) : \|x(t)\| \leq M$ for all $t \in T$ and $\|x'\|_p \leq M_1\}$.

`Claim 2`: For every $x \in \partial W$ and every $\xi \in (0,1)$, we have $x \notin \xi(K_\lambda \circ N_1)(x)$.

Let $x \in \overline{W}$ and suppose that for some $\xi \in (0,1)$, we have $x \in \xi(K_\lambda \circ N_1)(x)$. Then $U\big(\frac{1}{\xi}x\big) + \widehat{A}_\lambda\big(\frac{1}{\xi}x\big) + J\big(\frac{1}{\xi}x\big) = -f + J(x)$ with $f \in N(x)$, and hence

$$c_0\|x'\|_p^p \le -\xi^{p-1}(f,x)_{qp} + (\xi^{p-1}-1)\|x\|_p^p \le -\xi^{p-1}(f,x)_{qp} \tag{8}$$

(since $0 < \xi < 1$). Using hypothesis $\mathrm{H(F)}_1$(iii), we obtain

$$-\xi^{p-1}(f,x)_{qp} \le \xi^{p-1}c_1\|x\|_p^p + \xi^{p-1}c_2\int_0^b \|x(t)\|^r\|x'(t)\|^{p-r}dt + \xi^{p-1}\|c_3\|_1\|x\|_\infty^s.$$

Set $\tau = p-r$, $\theta = \frac{p}{r}$ and $\theta' = \frac{p}{\tau}$ $\big(\frac{1}{\theta} + \frac{1}{\theta'} = 1\big)$. From Hölder's inequality, we have

$$-\xi^{p-1}(f,x)_{qp} \le \xi^{p-1}c_1\|x\|_p^p + \xi^{p-1}c_2\|x\|_p^r\|x'\|_p^\tau + \xi^{p-1}\|c_3\|_1\|x\|_\infty^s.$$

Using this in (8) and because $0 < \xi < 1$, we obtain (recall the choice of M_1)

$$\|x'\|_p^p \le \frac{p}{rc_0}\left[c_1M^pb + \frac{rc_2^{\frac{p}{r}}M^pb^{\frac{p}{r}}}{c_0p} + \|c_3\|_1M^s\right] < M_1^p\,.$$

To conclude that $x \in W$ it remains to show that $\|x(t)\| < M$ for all $t \in T$. We argue by contradiction. So suppose that for some $t_0 \in T$ we have $\|x(t_0)\| = M$. Since $x \in \overline{W}$, we must have that $\|x(t_0)\| = \max_{t\in T}\|x(t)\|$. Let $\theta(t) = \frac{1}{p}\|x(t)\|^p$. We see that $\theta(\cdot)$ attains its maximum on $T = [0,b]$ at the point $t_0 \in T$. If $t_0 \in (0,b)$, then $\theta'(t_0) = 0$ and so $\|x(t_0)\|^{p-2}(x(t_0),x'(t_0))_{\mathbb{R}^N} = 0$, hence $(x(t_0),x'(t_0))_{\mathbb{R}^N} = 0$. By virtue of (6), for almost all $t \in T$ we have

$$\inf\big[(v,z)_{\mathbb{R}^N} + c_0\|y\|^p : \|z - x(t_0)\| + \|y - x'(t_0)\| < \delta,\ v \in F(t,z,y)\big] \ge c_6 > 0.$$

We can find a $\delta_1 > 0$ such that if $t \in (t_0, t_0+\delta_1]$ we have $\|x(t) - x(t_0)\| + \|x'(t) - x'(t_0)\| < \delta$ and $x(t) \ne 0$. Then for almost all $t \in (t_0, t_0+\delta_1]$,

$$(f(t),x(t))_{\mathbb{R}^N} + c_0\|x'(t)\|^p \ge c_6 > 0. \tag{9}$$

We know that a.e. on T

$$(f(t),x(t))_{\mathbb{R}^N} = \left(\left(a\Big(\frac{1}{\xi}x'(t)\Big)\right)', x(t)\right)_{\mathbb{R}^N} - \left(A_\lambda\Big(\frac{1}{\xi}x(t)\Big), x(t)\right)_{\mathbb{R}^N} + \Big(1 - \frac{1}{\xi^{p-1}}\Big)\|x(t)\|^p$$

and hence (see 9)

$$\left(\left(a\Big(\frac{1}{\xi}x'(t)\Big)\right)', x(t)\right)_{\mathbb{R}^N} + c_0\|x'(t)\|_p^p \ge c_6 > 0 \quad \text{a.e. on } (t_0, t_0+\delta_1]\,.$$

Integrating this inequality on $[t_0, t]$ with $t \in (t_0, t_0 + \delta_1]$, after integration by parts, we obtain

$$\Big(\Big(a\Big(\frac{1}{\xi}x'(t)\Big)\Big), x(t)\Big)_{\mathbb{R}^N} - \Big(\Big(a\Big(\frac{1}{\xi}x'(t_0)\Big)\Big), x'(t_0)\Big)_{\mathbb{R}^N}$$
$$- \int_{t_0}^{t} \Big(a\Big(\frac{1}{\xi}x'(s)\Big), x'(s)\Big)_{\mathbb{R}^N} ds + c_0 \int_{t_0}^{t} \|x'(s)\|^p ds \geq c_6(t - t_0) > 0.$$

Suppose that the first version of hypothesis $\mathrm{H(a)}_2$ holds, namely that $a(y) = c(y)y$. The reasoning is similar if the other version is valid. We have

$$\Big(a\Big(\frac{1}{\xi}x'(t_0)\Big), x(t_0)\Big)_{\mathbb{R}^N} = c\Big(\frac{1}{\xi}x'(t_0)\Big)\frac{1}{\xi}\big(x'(t_0), x(t_0)\big)_{\mathbb{R}^N} = 0.$$

Therefore for $t \in (t_0, t_0 + \delta_1]$ we have $(x'(t), x(t))_{\mathbb{R}^N} > 0$ (since $0 < \xi < 1$), i.e., $\vartheta'(t) > 0$ for $t \in (t_0, t_0 + \delta_1]$. So θ is strictly increasing on $(t_0, t_0 + \delta_1]$, which contradicts the choice of t_0. Therefore we infer that $\|x(t)\| < M$ for all $t \in T$.

If $t_0 = 0$, then $\theta'_+(t_0) = \theta'_+(0) \leq 0$ and $\theta'_-(b) \geq 0$ (because $\theta(0) = \theta(b)$, from the periodic boundary conditions). So we have $(x(0), x'(0))_{\mathbb{R}^N} = 0$ (since $x(0) = x(b), x'(0) = x'(b)$, recall that $x \in \widehat{D}$). So we proceed as before. Similarly if $t_0 = b$. Therefore we conclude that $\|x(t)\| < M$ for all $t \in T$ and so $x \in W$, which proves the claim.

Now we can apply Proposition 2.1 and obtain $x \in \widehat{D} \cap \overline{W}$ which solves the fixed point Problem (7). Clearly $x \in \widehat{D} \cap \overline{W}$ is a solution of (5).

Finally it remains to remove the stronger version of hypothesis $\mathrm{H(F)}_1$(iv) (see (6)). To this end let $\varepsilon_n \downarrow 0$ and set $F_n(t, x, y) = F(t, x, y) + \varepsilon_n x$. Then Problem (5) with F replaced by F_n, has a solution $x_n \in \widehat{D} \cap \overline{W}$, $n \geq 1$. Evidently we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T, \mathbb{R}^N)$. As in the proof of Claim 1, we have $x_n \to x$ in $W^{1,p}_{per}(T, \mathbb{R}^N)$ and in the limit as $n \to \infty$ we obtain $U(x) + \widehat{A}_\lambda(x) \in N(x)$. Therefore $x \in \widehat{D} \cap \overline{W}$ is a solution of (5). ∎

Now that we have solved the auxiliary Problem (5), by passing to the limit as $\lambda \downarrow 0$, we shall obtain a solution for the original Problem (1).

Theorem 3.6. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(A)}_1$ *and* $\mathrm{H(F)}_1$ *hold, then Problem* (1) *has a solution* $x \in C^1_{per}(T, \mathbb{R}^N)$ *with* $a(x'(\cdot)) \in W^{1,q}_{per}(T, \mathbb{R}^N)$.

Proof. Let $\lambda_n \downarrow 0$ and let $x_n \in \widehat{D} \cap \overline{W}$ be solutions of the corresponding auxiliary problems (5). Evidently $\{x_n\}_{n \geq 1} \subseteq W^{1,p}_{per}(T, \mathbb{R}^N)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T, \mathbb{R}^N)$. For every $n \geq 1$, we have

$$\big(U(x_n), \widehat{A}_{\lambda_n}(x_n)\big)_{qp} + \|\widehat{A}_{\lambda_n}(x_n)\|_2^2 = -\big(f_n, \widehat{A}_{\lambda_n}(x_n)\big)_{qp} \tag{10}$$

From integration by parts and since $x_n(0) = x_n(b)$, $x_n'(0) = x_n'(b)$, we have

$$\big(U(x_n), \widehat{A}_{\lambda_n}(x_n)\big)_{qp} = \int_0^b \big(-(a(x_n'(t)))', A_{\lambda_n}(x_n(t))\big)_{\mathbb{R}^N} dt$$
$$= \int_0^b \Big(a(x_n'(t)), \frac{d}{dt} A_{\lambda_n}(x_n(t))\Big)_{\mathbb{R}^N} dt.$$

From the chain rule of Marcus-Mizel [17], we have that $\frac{d}{dt} A_{\lambda_n}(x_n(t)) = A_{\lambda_n}'(x_n(t)) x_n'(t)$ a.e. on T. So (see $\mathrm{H(A)}_1$)

$$\big(U(x_n), \widehat{A}_{\lambda_n}(x_n)\big)_{qp} = \int_0^b c(x_n'(t)) \big(x_n'(t), A_{\lambda_n}(x_n(t)) x_n'(t)\big)_{\mathbb{R}^N} dt \geq 0\,.$$

Using this inequality in (10), we obtain that $\{\widehat{A}_{\lambda_n}(x_n)\}_{n \geq 1} \subseteq L^2(T, \mathbb{R}^N)$ is bounded. So we may assume that $\widehat{A}_{\lambda_n}(x_n) \xrightarrow{w} u$ in $L^2(T, \mathbb{R}^N)$. If $\widehat{J}_{\lambda_n}(x_n)(\cdot) = J_{\lambda_n}(x_n(\cdot)) \in C(T, \mathbb{R}^N)$, we have $\widehat{J}_{\lambda_n}(x_n) \to x$ in $L^2(T, \mathbb{R}^N)$. Because $A_{\lambda_n}(x_n(t)) \in A(J_{\lambda_n}(x_n(t)))$ for all $n \geq 1$ and all $t \in T$, we have $\widehat{A}_{\lambda_n}(x_n) \in \widehat{A}(\widehat{J}_{\lambda_n}(x_n))$. Because $\widehat{A}_{\lambda_n}(x_n) \in \widehat{A}(\widehat{J}_{\lambda_n}(x_n))$, $\widehat{J}_{\lambda_n}(x_n) \to x$ in $L^2(T, \mathbb{R}^N)$ and $\widehat{A}_{\lambda_n}(x_n) \xrightarrow{w} u$ in $L^2(T, \mathbb{R}^N)$, we infer that $u \in \widehat{A}(x)$, i.e., $u(t) \in A(x(t))$ a.e. on T. Moreover, we may assume that $f_n \xrightarrow{w} f$ in $L^q(T, \mathbb{R}^N)$. Arguing as in the proof of Proposition 3.5 (see Claim 1), we obtain $x_n \to x$ in $C^1(T, \mathbb{R}^N)$. Then in the limit as $n \to \infty$, we have $f \in N_1(x)$ and $(a(x'(t)))' = u(t) + f(t) \in A(x(t)) + F(t, x(t), x'(t))$ a.e on T, $x(0) = x(b)$, $x'(0) = x'(b)$. ∎

4. Problems with the p-Laplacian and nonlinear boundary conditions

In this section we deal with Problem (2). Now, in contrast to the situation of Section 3, we assume that $D(A) = \mathbb{R}^N$. This permits the improvement of the growth condition on F and so we can have multivalued nonlinearities of the Nagumo-Hartman type (see also Mawhin-Urena [20]). More precisely our hypotheses on the data of (2) are the following:

$\mathbf{H(A)_2}$: $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map with $D(A) = \mathbb{R}^N$ and $0 \in A(0)$.

$\mathbf{H(F)_2}$: $F : T \times \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that $\mathrm{H(F)}_1$(i), (ii) hold and

(iii) for almost all $t \in T$, all $\|x\| \leq M$ and all $\|y\|^{p-1} \geq M_1 > 0$ we have

$$\sup\big[\|v\| : v \in F(t, x, y)\big] \leq \eta(\|y\|^{p-1})$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ is a locally bounded Borel measurable function such that $\int_{M_1}^{\infty} \frac{s ds}{\eta(s)} = +\infty$;

(iv) if $\|x_0\| = M$ (with $M > 0$ as in (iii)), hypothesis $\mathrm{H(F)}_1$(iv) holds;

(v) for all $r > 0$, there exists $\gamma_r \in L^q(T)_+$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that for almost all $t \in T$, all $\|x\|, \|y\| \le r$ and all $v \in F(t, x, y)$, we have $\|v\| \le \gamma_r(t)$;

(vi) is the same as $\mathrm{H(F)}_1$(iii).

Recall that if $A : D(A) \subseteq X \to 2^{X^*}$ is a maximal monotone operator, we define $\widehat{A} : D(\widehat{A}) \subseteq L^p(T, X) \to 2^{L^q(T, X^*)}$ by $\widehat{A}(x) = \{h \in L^q(T, X^*) : h(t) \in A(x(t))$ a.e. on $T\}$ for all $x \in D(\widehat{A}) = \{x \in L^p(T, X) : x(t) \in D(A)$ a.e. on T and $S^q_{A(x(\cdot))} \neq \varnothing\}$.

Proposition 4.1. *If X is a separable reflexive Banach space and $A : D(A) \subseteq X \to 2^{X^*}$ is a maximal monotone operator with $0 \in A(0)$, then $\widehat{A} : D(\widehat{A}) \subseteq L^p(T, X) \to 2^{L^q(T, X^*)}$ is maximal monotone too.*

Proof. By Troyanski's renorming theorem (see Hu-Papageorgiou [13, p. 316]), without any loss of generality we may assume that both X and X^* are locally uniformly convex spaces. Let $\mathcal{F} : X \to X^*$ be the duality map of X (i.e., $\mathcal{F}(x) = \partial\varphi(x)$ with $\varphi(x) = \frac{1}{2}\|x\|^2$, see Hu-Papageorgiou [13, p. 30] and Zeidler [22, p. 860]). We know that $\mathcal{F}$ is a homeomorphism (see Zeidler [22, p. 861]). We introduce the operator $J_0 : L^p(T, X) \to L^q(T, X^*)$ defined by $J_0(x)(\cdot) = \|\mathcal{F}(x(\cdot))\|^{p-2}\mathcal{F}(x(\cdot))$. It is easy to see that J_0 is continuous, strictly monotone, thus maximal monotone. Clearly $\widehat{A}$ is monotone. We show that $R(\widehat{A} + J_0) = L^q(T, X^*)$ (i.e., surjectivity of $\widehat{A} + J_0$). For this purpose let $h \in L^q(T, X^*)$ and consider the multifunction $\Gamma : T \to 2^{X^*}$ defined by $\Gamma(t) = \{x \in X : A(x) + \varphi(x) \ni h(t)\}$, where $\varphi : X \to X^*$ is the monotone continuous map defined by $\varphi(x) = \|\mathcal{F}(x)\|^{p-2}\mathcal{F}(x)$. Note that $A + \varphi : D(A) \subseteq X \to 2^{X^*}$ is maximal monotone. Moreover, because $0 \in A(0)$, we have that $A + \varphi$ is coercive. Therefore $R(A+\varphi) = X^*$ and so we infer that for all $t \in T$, $\Gamma(t) \neq \varnothing$. Remark that $Gr\Gamma = \{(t, x) \in T \times X : (x, \varphi(x) - h(t)) \in GrA\}$. Let $\xi : T \times X \to X \times X^*$ be defined by $\xi(t, x) = (x, \varphi(x) - h(t))$. Evidently ξ is a Caratheodory function, thus jointly measurable. Note that $Gr\Gamma = \xi^{-1}(GrA)$ and since GrA is sequentially closed in $X \times X^*_w$, we have $GrA \in B(X \times X^*_w)$ (the Borel σ-field). But X^*_w is a Souslin space and so $B(X \times X^*_w) = B(X) \times B(X^*_w)$ (see Hu-Papageorgiou [13, p. 153]). Also $B(X^*_w) = B(X^*)$. Therefore $GrA \in B(X) \times B(X^*) = B(X \times X^*)$ and so $Gr\Gamma = \xi^{-1}(GrA) \in \mathcal{L} \times B(X)$ with $\mathcal{L}$ being the Lebesgue σ-field of T. We can apply the Yankon-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [13, p. 158]) to obtain a measurable map $x : T \to X$ such that $x(t) \in \Gamma(t)$ a.e. on T. We have $h(t) \in A(x(t)) + \varphi(x(t))$ a.e. on T. Taking duality brackets with $x(t)$, we obtain $\|x(t)\|^p \le \langle h(t), x(t)\rangle_{X^*,X}$ and so $\|x(t)\|^{p-1} \le \|h(t)\|$ a.e. on T, i.e., $x \in L^p(T, X)$. So we have proved that $R(\widehat{A} + J_0) = L^q(T, X^*)$. Then arguing as in the proof of Proposition 3.3 and exploiting the strict monotonicity of J_0, we obtain the maximality of $\widehat{A}$. ∎

The second auxiliary result concerns the periodic problem

$$\begin{cases} -(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}x(t) = g(t) \ \text{ a.e. on } T = [0,b] \\ (\varphi_p(x'(0)), -\varphi_p(x'(b))) \in \xi(x(0), x(b)),\ 1 < p < \infty. \end{cases} \tag{11}$$

From Gasinski-Papageorgiou [10] we have the following result:

Proposition 4.2. *If* $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ *is a maximal monotone map with* $(0,0) \in \xi(0,0)$ *and* $g \in L^q(T, \mathbb{R}^N)$ $(\frac{1}{p} + \frac{1}{q} = 1)$, *then Problem* (11) *has a unique solution* $x \in C^1(T, \mathbb{R}^N)$ *with* $\|x'\|^{p-2}x' \in W^{1,q}(T, \mathbb{R}^N)$.

Let $D_0 = \{x \in C^1(T, \mathbb{R}^N) : \|x'\|^{p-2}x' \in W^{1,q}(T, \mathbb{R}^N), (\varphi_p(x'(0)), -\varphi_p(x'(b))) \in \xi(x(0), x(b))\}$ and let $V : D_0 \subseteq L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be defined by $V(x) = -(\|x'\|^{p-2}x')'$, $x \in D_0$. Arguing as in the proof of Proposition 3.3, using this time Proposition 4.2, we obtain

Proposition 4.3. *If* $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ *is a maximal monotone map with* $(0,0) \in \xi(0,0)$, *then* $V : D_0 \subseteq L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ *is maximal monotone.*

For the existence theorem for Problem (2) we will use the following hypotheses on ξ:

H(ξ) : $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map with $(0,0) \in \xi(0,0)$ and one of the following holds:

(i) for every $(a', d') \in \xi(a, d)$, we have $(a', a)_{\mathbb{R}^N} \geq 0$ and $(d', d)_{\mathbb{R}^N} \geq 0$; or

(ii) $D(\xi) = \{(a, d) \in \mathbb{R}^N \times \mathbb{R}^N : a = d\}$.

Proposition 4.4. *If the hypotheses* $\mathrm{H(A)}_2$, $\mathrm{H(F)}_2$, $\mathrm{H}(\xi)$ *and* H_0 *hold, then Problem* (2) *has a solution* $x \in C^1(T, \mathbb{R}^N)$ *with* $\|x'\|^{p-2}x' \in W^{1,q}(T, \mathbb{R}^N)$.

Proof. Because A is maximal monotone with $D(A) = \mathbb{R}^N$, we have that $\theta = \sup[\,\|u\| : u \in A(x), \|x\| \leq M] < +\infty$ (see Hu-Papageorgiou [13, p. 308]). Without any loss of generality we may assume that for almost all $r \geq 0$, $0 < \beta \leq \eta(r)$. Set $\eta_1(r) = \theta + \eta(r)$. If $\widehat{\gamma} \geq \frac{\theta}{\beta} + 1$, then we have $\eta_1(r) \leq \widehat{\gamma}\eta(r)$ for all $r \geq 0$ and so $\int_{M_1}^{\infty} \frac{sds}{\eta_1(s)} = +\infty$.

As we did with Problem (1) (see Section 3), first we assume that the multivalued nonlinearity F satisfies (6) (with $c_0 = 1$) instead of $\mathrm{H(F)}_2$(iv). Let

$$M_1' > \max\left\{ b^{\frac{1}{p}} \left(\frac{p}{rc_0} \left[c_1 M^p b + \frac{rc_2^{\frac{p}{r}} M^p b^{\frac{p}{r}}}{c_0 p} + \|c_3\|_1 M^s \right] \right)^{\frac{p-1}{p}}, M_1 \right\}$$

and then take $M_2 > 0$ such that $M_2^{p-1} > M_1'$ and $\int_{M_1}^{M_2^{p-1}} \frac{sds}{\eta_1(s)} = M_1'$. Also let $W \subseteq C^1(T, \mathbb{R}^N)$ be defined by

$$W = \{x \in C^1(T, \mathbb{R}^N) : \|x(t)\| < M, \|x'(t)\| < M_2 \text{ for all } t \in T\}.$$

The set W is open, bounded in $C^1(T, \mathbb{R}^N)$ and $0 \in W$. Moreover, we have

$$\partial W = \{x \in C^1(T, \mathbb{R}^N) : \|x\|_\infty = M, \|x'\|_\infty = M_2\}.$$

Let $N : \overline{W} \to P_{wkc}(L^q(T, \mathbb{R}^N))$ be defined by $N(x) = S^q_{F(\cdot,x(\cdot),x'(\cdot))}$. We know that N is usc from $\overline{W}$ with the $C^1(T, \mathbb{R}^N)$-norm topology into $L^q(T, \mathbb{R}^N)$ with the weak topology. For each $g \in L^q(T, \mathbb{R}^N)$, we consider Problem (11). By Proposition 4.2 we know that this problem has a unique solution $x = K(g) \in C^1(T, \mathbb{R}^N)$. So we can define the map $K : L^q(T, \mathbb{R}^N) \to C^1(T, \mathbb{R}^N)$ which to each $g \in L^q(T, \mathbb{R}^N)$ assigns the unique solution of (11). It is easy to check that K is completely continuous.

Let $J : C^1(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be the bounded continuous map defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$. Also let $\widehat{A} : C^1(T, \mathbb{R}^N) \to 2^{L^q(T,\mathbb{R}^N)}$ be defined by $\widehat{A}(x) = S^q_{A(x(\cdot))}$. We have that $\widehat{A}$ is usc from $C^1(T, \mathbb{R}^N)$ into $L^q(T, \mathbb{R}^N)_w$. Set $N_1(x) = -N(x) + J(x) - \widehat{A}(x)$. Evidently $N_1 : \overline{W} \to P_{wkc}(L^q(T, \mathbb{R}^N))$ is usc from $\overline{W}$ with the $C^1(T, \mathbb{R}^N)$-norm topology into $L^q(T, \mathbb{R}^N)_w$. Problem (2) is equivalent to the fixed point problem

$$x \in (K \circ N_1)(x). \tag{12}$$

`Claim:` For every $x \in \partial W$ and every $\xi \in (0, 1)$, we have $x \notin \xi(K \circ N_1)(x)$.

Let $x \in \overline{W}$ and suppose that for some $\xi \in (0, 1)$ we have $x \in \xi(K \circ N_1)(x)$. Arguing as in the proof of Proposition 3.5 (claim 2), we obtain

$$\|x'\|_p^p \le \frac{p}{rc_0}\left[c_1 M^p b + \frac{rc_2^{\frac{p}{r}} M^p b^{\frac{p}{r}}}{c_0 p} + \|c_3\|_1 M^s\right],$$

and hence $\|x'\|_p^{p-1} < \frac{1}{b^{\frac{1}{p}}} M_1'$. The function $\vartheta(u) = u^{\frac{p-1}{p}}$, $u \ge 0$, is concave. So using Jensen's inequality, we have

$$\frac{1}{b^{\frac{1}{p}}}\|x'\|_p^{p-1} \ge \frac{1}{b}\int_0^b \|x'(t)\|^{p-1} dt.$$

Therefore it follows (since $\frac{1}{p} + \frac{1}{q} = 1$) that

$$\int_0^b \|x'(t)\|^{p-1} dt < M_1'. \tag{13}$$

We claim that $\|x'(t)\| < M_2$ for all $t \in T$. Suppose that this is not the case. Then we can find $t_0 \in T$ such that $\|x'(t_0)\| = M_2$, hence $\|x'(t_0)\|^{p-1} > M_1'$. So from (13) we infer that there exists a $t_1 \in T$ such that $\|x(t_1)\|^{p-1} = M_1'$ (take the $t_1 \in T$ which is closest to t_0). Let $\chi : [M_1', +\infty) \to \mathbb{R}_+$ be the function defined by $\chi(r) = \int_{M_1'}^{r} \frac{s}{\eta_1(s)} ds$. Clearly χ is continuous, strictly increasing, $\chi(M_1') = 0$ and $\chi(M_2^{p-1}) = M_1'$. We have

$$\begin{aligned} M_1' &= \chi(M_2^{p-1}) \\ &= |\chi(\|x'(t_0)\|^{p-1})| \\ &= \left| \int_{M_1}^{\|x'(t_0)\|^{p-1}} \frac{s}{\eta_1(s)}\, ds \right| \\ &= \left| \int_{\|x'(t_0)\|^{p-1}}^{\|x'(t_1)\|^{p-1}} \frac{s}{\eta_1(s)}\, ds \right| \\ &\le \left| \int_{t_0}^{t_1} \frac{\|(\|x'(t)\|^{p-2}x'(t))'\|}{\eta_1(\|(\|x'(t)\|^{p-2}x'(t))\|)} \|x'(t)\|^{p-1} dt \right|. \end{aligned} \tag{14}$$

We also have

$$\begin{aligned} \big\|(\|x'(t)\|^{p-2}x'(t))'\big\| &\le \theta + \eta(\|x'(t)\|^{p-1}) \\ &= \eta_1(\|x'(t)\|^{p-1}) \\ &= \eta_1\big(\, \big\|(\|(x'(t))\|^{p-2}x'(t))\big\|\, \big). \end{aligned}$$

Using this in (14), we obtain (see (13))

$$M_1' \le \left| \int_{t_0}^{t_1} \|x'(t)\|^{p-1} dt \right| = \int_{\min\{t_0,t_1\}}^{\max\{t_0,t_1\}} \|x'(t)\|^{p-1} dt < M_1',$$

a contradiction. Therefore $\|x'(t)\| < M_2$ for all $t \in T$. Moreover, following the argument in the proof of Proposition 3.5 and using hypotheses H(ξ), we can show that $\|x(t)\| < M$ for all $t \in T$. Therefore $x \in W$ and we have proved the claim.

Apply Proposition 2.1 to obtain $x \in D_0 \cap \overline{W}$ which solves (12). Evidently this is a solution of (2) when (6) (with $c_0 = 1$) is in effect. As in the proof of Proposition 3.5 we remove this extra restriction. ∎

Remark 4.5. It will be interesting to have this existence result when $D(A) \neq \mathbb{R}^N$.

5. Special cases and examples

We show that our general formulation of Problem (2) unifies the classical Dirichlet, Neumann and periodic problems and goes beyond them:

(a) Let $K_1,\ K_2 \in P_{fc}(\mathbb{R}^N)$ with $0 \in K_1 \cap K_2$. By $\delta_{K_1 \times K_2}$ we denote the indicator function of the set $K_1 \times K_2$, i.e.,

$$\delta_{K_1\times K_2}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in K_1 \times K_2 \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently $\delta_{K_1\times K_2}$ is proper, lower semicontinuous and convex, i.e., $\delta_{K_1\times K_2} \in \Gamma_0(\mathbb{R}^N \times \mathbb{R}^N)$. Set $\xi = \partial\delta_{K_1\times K_2} = N_{K_1\times K_2} = N_{K_1} \times N_{K_2}$ (given $C \in P_{fc}(\mathbb{R}^N)$ by $N_C(x)$ we denote the normal cone to the set C at $x \in C$, see Hu-Papageorgiou [13, p. 624]). Then Problem (2) becomes

$$\begin{cases} (\|x'(t)\|^{p-2}x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T \\ x(0) \in K_1,\ x(b) \in K_2 \\ (x'(0), x(0))_{\mathbb{R}^N} = \sigma(x'(0), K_1),\ (-x'(b), x(b))_{\mathbb{R}^N} = \sigma(-x'(b), K_2). \end{cases} \tag{15}$$

Note that $\xi = \partial\delta_{K_1\times K_2}$ is maximal monotone, $(0,0) \in \xi(0,0)$ and hypothesis H(ξ) is valid (the first option).

(b) In the previous case, let $K_1 = K_2 = \{0\}$. Then Problem (15) becomes the usual Dirichlet problem.

(c) Again in the first example let $K_1 = K_2 = \mathbb{R}^N$. Then $\xi = N_{K_1} \times N_{K_2} = \{(0,0)\}$ and so we have Neumann problem. The Neumann problem was not examined before in the presence of Nagumo-Hartman nonlinearities (compare with Mawhin-Urena [20]).

(d) Let $K = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x = y\}$ and let $\xi = \partial\delta_K$. Then $\xi(x,y) = K^{\perp} = \{(v,w) \in \mathbb{R}^N \times \mathbb{R}^N : v = -w\}$. So Problem (2) becomes the usual periodic problem.

(e) Let $\xi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ be defined by

$$\xi(x,y) = \left(\frac{1}{\theta^{\frac{1}{q-1}}}\varphi_p(x), \frac{1}{\eta^{\frac{1}{q-1}}}\varphi_p(y) \right) \quad \text{with } \theta, \eta > 0.$$

Evidently, ξ is continuous, monotone (hence maximal monotone) and $\xi(0,0) = (0,0)$. With this choice of ξ, Problem (2) becomes a Sturm-Liouville type problem

$$\begin{cases} (\|x'(t)\|^{p-2}x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T \\ x(0) - \theta x'(0) = 0,\ x(b) + \eta x'(b) = 0. \end{cases} \tag{16}$$

Hypothesis H(ξ) is satisfied.

(f) Let $\xi_1, \xi_2 : \mathbb{R}^N \to \mathbb{R}^N$ be two monotone, continuous maps such that $\xi_1(0) = \xi_2(0) = 0$. Let $\xi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ be defined by $\xi(x,y) = (\xi_1(x), \xi_2(y))$. Evidently ξ satisfies hypothesis H(ξ). Then Problem (2) becomes

$$\begin{cases} (\|x'(t)\|^{p-2}x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T \\ x'(0) = \varphi_q(\xi_1(x(0))),\ -x'(b) = \varphi_q(\xi_2(x(b))). \end{cases} \tag{17}$$

Next let $\psi = \delta_{\mathbb{R}^N_+}$, $A = \partial\psi$, $K = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x = y\}$ and $\xi = \partial\delta_K = K^\perp$. We have

$$A(x) = \partial\psi(x) = N_{\mathbb{R}^N_+}(x) = \begin{cases} \{0\} & \text{if } x_k > 0 \text{ for all } k \in \{1,...,N\} \\ -\mathbb{R}^N_+ \cap \{x\}^\perp & \text{if } x_k = 0 \text{ for some } k \in \{1,...,N\}. \end{cases}$$

Then Problem (2) becomes the following differential variational inequality:

$$\begin{cases} \left(\|x'(t)\|^{p-2}x'(t)\right)' \in F(t,x(t),x'(t)) \\ \text{a.e. on } \{t \in T : x_k(t) > 0 \text{ for all } \ k = 1,...,N\} \\ \left(\|x'(t)\|^{p-2}x'(t)\right)' \in F(t,x(t),x'(t)) - u(t) \\ \text{a.e. on } \{t \in T : x_k(t) = 0 \text{ for some } k = 1,...,N\} \\ x(t) = \left(x_k(t)\right)_{k=1}^N \in \mathbb{R}^N_+ \text{ for all } t \in T,\ u \in L^q(T,\mathbb{R}^N_+) \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases} \tag{18}$$

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