Identification of Memory Kernels Depending on Time and on an Angular Variable

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Abstract. We deal with the problem of recovering a memory kernel $k(t, \eta)$, depending on time t and on an angular variable η, in a parabolic integrodifferential equation related to a toric domain. We show that the problem can be uniquely solved locally in time if the kernel k is not assumed to be necessarily periodic with respect to η. On the contrary, under a periodicity condition for $k(t, \cdot)$, we show uniqueness assuming existence.

Keywords: *Identification problems, parabolic integrodifferential equations, time and space dependent memory kernels, existence and uniqueness results*

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1. Introduction

Thought the problem of recovering time dependent memory kernels has been largely considered in literature, the corresponding one for memory kernels depending on both time and space is quite recent, at least as far as integrodifferential equations are concerned. We refer, in particular, to [1], [2], [5]–[7] where this problem was first attacked, in an essentially one-dimensional approach. In [1] and [2] the kernel depends on the time and on one space variable although the state function depends on a vector in $\mathbb{R}^n$, $n \geq 2$, whereas in [5]–[7] the kernel is assumed to be *degenerate*, i.e., of the form $k(t, x) = \sum_{j=1}^{N} m_j(t)\mu_j(x)$, but with the space-dependent functions μ_j, $j = 1, \ldots, N$, assumed to be *known*, too. As a consequence, the identification problem reduces to the vector-identification problem consisting of recovering the N unknown time-dependent functions m_j, $j = 1, \ldots, N$.

Starting from the abstract results of [2], two attempts of extending the theory to more general kernels have been worked out in [3] and [4]. There, the assumption of degenerateness has been skipped and the interest shifted to

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kernels depending on both time and space, but with the spatial dependence occurring through scalar functions of *all* the variables at disposal. However, a common feature in [3] and [4] is the radial character of the admissible unknown kernels and of the domains underlying the basic equations.

Here, instead, we consider kernels depending on time and on the angular variable η which, in dimension three, represents the angle between the x_1-axis and the projection of $x = (x_1, x_2, x_3)$ on the (x_1, x_2)-plane. For this purpose, as for the radial case was quite natural to deal with spherical coronas or balls, here the geometric domain Ω in the fundamental equations is a torus, i.e. a domain for which the sections with any plane containing the x_3-axis do not depend on η. Moreover, the fact of being able to solve our problem in a torus gives a further support to an idea already suggested by [3] and [4] and that we are led to believe to be of a general character. That is, that there must be a very strictly relationship between the geometry of the basic domain and the kind of kernels we can hope to recover.

We want to stress that, in contrast to one's expectations, in the first part of our treatment we do not require the kernel $k(t, \cdot)$ to have any periodicity with respect to η. Indeed, having in mind a torus, it could seem reasonable that, for instance, a relationship of type $k(t, -\frac{\pi}{2}) = k(t, 3\frac{\pi}{2})$ should hold for any t in the solvability interval. This is not the case: for, in general, an additional periodicity condition makes fail the equivalence result of Section 5. Therefore, unless we are not particularly lucky to get the periodicity free, the kernel k to be recovered may have a jump along one of the planes cutting the torus and containing the x_3-axis.

The case of periodic kernels is investigated too, but we are only able to show the uniqueness of a solution, assuming its existence. However, this is not quite unusual, since, dealing with inverse problems, we are often concerned only with the problem of the uniqueness of solutions, the existence being a priori justified physically.

Finally, we refer to [4, Remark 1.2] for the physical model which our identification problem refers to and for the physical motivations for investigating it. Of course, with respect to [4] the second dependence variable of the kernel k must be replaced by η and some changes are needed, but the essential sense remains the same.

2. Statement of the problem

Let $\delta > R > 0$ and let Ω be the torus $\{x = (x', x_3) \in \mathbb{R}^3 : (|x'| - \delta)^2 + x_3^2 < R^2\}$, where x' and $|x'|$ denote the pair (x_1, x_2) and the scalar $(x_1^2 + x_2^2)^{\frac{1}{2}}$, respectively. As usual, given a Banach space Y and two functions v, $w : [0, T] \to Y$, the symbol "$*$" stands for the convolution operator $(v * w)(t) = \int_0^t v(t - s)w(s)\, ds$.

We investigate the problem of recovering the unknown kernel k, depending on two scalar variables t, η, appearing in the following integrodifferential equation of parabolic type, where $(t, x) \in [0, T] \times \Omega$:

$$
\begin{aligned}
D_t u(t, x) = {} & \mathcal{A}u(t, x) + [k(\cdot, \rho(x')) * \mathcal{B}u(\cdot, x)](t) \\
& + [D_\eta k(\cdot, \rho(x')) * \mathcal{C}u(\cdot, x)](t) + f(t, x) \,.
\end{aligned}
\tag{2.1}
$$

Here $\mathcal{A}$ and $\mathcal{B}$ are two second-order linear differential operators, $\mathcal{C}$ is a first-order linear differential operator and $\rho(x')$ denotes the continuation of $\arctan\left(\frac{x_2}{x_1}\right)$ according to

$$
\rho(x') =
\begin{cases}
\arctan\left(\frac{x_2}{x_1}\right), & x_1 > 0 \,, \; x_2 \in \mathbb{R} \\
\frac{\pi}{2} \,, & x_1 = 0 \,, \; x_2 > 0 \\
\pi + \arctan\left(\frac{x_2}{x_1}\right), & x_1 < 0 \,, \; x_2 \in \mathbb{R} \,.
\end{cases}
\tag{2.2}
$$

Moreover, we assume that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ have, respectively, the following forms:

$$
\mathcal{A} = \sum_{j,k=1}^{3} D_{x_j}(a_{j,k}(x)D_{x_k}), \quad \mathcal{B} = \sum_{j,k=1}^{3} D_{x_j}(b_{j,k}(x)D_{x_k}), \quad \mathcal{C} = \sum_{j=1}^{3} c_j(x)D_{x_j}, \tag{2.3}
$$

with coefficients satisfying the properties

$$
a_{i,j} \in W^{2,+\infty}(\Omega) \,, \quad a_{i,j} = a_{j,i} \,, \quad b_{i,j}, c_i \in W^{1,+\infty}(\Omega) \,, \quad i, j = 1, 2, 3 \,. \tag{2.4}
$$

In particular, the $a_{i,j}$'s have to be such that $\mathcal{A}$ is uniformly elliptic.

Remark 2.1. Denoting by Γ the half-plane $\{x \in \mathbb{R}^3 : x_1 = 0, \; x_2 < 0\}$, the function ρ defined by (2.2) satisfies

$$
\begin{cases}
\displaystyle \liminf_{y \to x, \, y \in \Omega \backslash \Gamma, \, x \in \Omega \cap \Gamma} \rho(y') = -\frac{\pi}{2} \,, \\[2ex]
\displaystyle \limsup_{y \to x, \, y \in \Omega \backslash \Gamma, \, x \in \Omega \cap \Gamma} \rho(y') = \frac{3\pi}{2} \,.
\end{cases}
$$

Hence, for $x \in \Omega \cap \Gamma$, it might be not clear how to intend (2.1), for instance if $\mathcal{A}u(t, \cdot)$ is not continuous. This suggests to require that (2.1) is satisfied almost everywhere in space rather than everywhere and, consequently, that with respect to space the suitable function setting is that related to L^p-spaces rather than to spaces of continuous functions. To this purpose, in the sequel, for brevity, we will always write "$\forall (t, x) \in [0, T] \times \Omega$", but having well clear in mind that this notation stands for "$\forall t \in [0, T]$ and for a.e. $x \in \Omega$".

To establish our results the uniform ellipticity of $\mathcal{A}$ is not enough. Indeed, we have to restrict our attention to the class of differential operators $\mathcal{A}$ whose coefficients satisfy, in addition to (2.4), also the following further conditions:

$$|x'|^{-4}[a_{1,1}(x)x_2^2 + a_{2,2}(x)x_1^2 - 2a_{1,2}(x)x_1x_2] = \lambda\left(\tfrac{x_2}{x_1}\right) \qquad \forall\, x \in \overline{\Omega} \qquad (2.5)$$

$$[a_{1,1}(x) - a_{2,2}(x)]x_1x_2 + a_{1,2}(x)[x_2^2 - x_1^2] = 0 \qquad \forall\, x \in \partial\Omega \qquad (2.6)$$

$$a_{1,3}(x)x_2 = a_{2,3}(x)x_1 \quad \forall\, x \in \partial\Omega\,, \qquad (2.7)$$

where $\lambda \in \widetilde{C}_b^1(\mathbb{R})$, the set $\widetilde{C}_b^1(\mathbb{R})$ being defined by

$$\widetilde{C}_b^1(\mathbb{R}) := \left\{ g \in C^1(\mathbb{R}) : \lim_{y \to \pm\infty} g(y) = r_1 \in \mathbb{R},\; \lim_{y \to \pm\infty} y^2 g'(y) = 0 \right\}. \qquad (2.8)$$

We stress that, contrarly to [3] and [4] where all the coefficients $a_{i,j}$, $i, j = 1, 2, 3$, take part in a condition similar to (2.5) (cf., for instance, formula (1.20) in [3]), in this case only coefficients $a_{j,k}$, $j, k = 1, 2$, appear. This allows us to consider a largest class of admissible operators $\mathcal{A}$, since we can choose the coefficients $a_{i,3}$, $i = 1, 2, 3$, quite freely, provided that the ellipticity condition and the symmetry of the matrix $(a_{i,j})_{i,j=1}^3$ hold. Anyway, such a largest freedom is balanced by the boundary requirements (2.7) and (2.8) which were unnecessary in [3] and [4]. The technical reasons forcing us to impose them will be clarified later, in Lemma 4.1 and Remark 4.4.

Now, we introduce the co-ordinates $(r, \varphi, \theta) \in (0, +\infty) \times \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right) \times \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)$ related to Cartesian ones via the formula

$$(x_1, x_2, x_3) = \delta\widetilde{x}\left(\varphi, \tfrac{\pi}{2}\right) + r\widetilde{x}(\varphi, \theta)\,, \qquad (2.9)$$

where we have set $\widetilde{x}(\varphi, \theta) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$. Therefore, denoting respectively by $n(x) = (n_1(x), n_2(x), n_3(x))$ and $\mathrm{d}x$ the unit outer normal vector at $x \in \partial\Omega$ and the volume element of Ω, standard arguments of elementary calculus show that $n(x)$ is the vector $\widetilde{x}(\varphi, \theta)$ and

$$\mathrm{d}x = r(\delta + r\sin\theta)\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta\,, \qquad (2.10)$$

where r is allowed to vary in the interval $(0, R)$, only.

Remark 2.2. The set of co-ordinates defined by the right-hand side of (2.9) is called the set of toric co-ordinates (r, φ, θ). It differs from the set of spherical ones, since here r and θ represent the polar co-ordinates centered in $x_\varphi = \delta\widetilde{x}(\varphi, \tfrac{\pi}{2})$ and parameterizing the plane $P_\varphi = \mathrm{span}\{e_\varphi, e_3\}$, $\varphi \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}]$, where $e_\varphi = \widetilde{x}(\varphi, \tfrac{\pi}{2})$ and $e_3 = \widetilde{x}(\varphi, 0)$.

We observe that conditions (2.5)–(2.7) rewritten in terms of toric co-ordinates are equivalent to require, for any $(r, \varphi, \theta) \in [0, R] \times [-\frac{\pi}{2}, \frac{3\pi}{2}] \times [-\frac{\pi}{2}, \frac{3\pi}{2}]$:

$$\frac{\widetilde{a}_{1,1}(r, \varphi, \theta)\sin^2\varphi + \widetilde{a}_{2,2}(r, \varphi, \theta)\cos^2\varphi - \widetilde{a}_{1,2}(r, \varphi, \theta)\sin(2\varphi)}{(\delta + r\sin\theta)^2} = \lambda(\tan\varphi) \quad (2.11)$$

$$[\widetilde{a}_{2,2}(R, \varphi, \theta) - \widetilde{a}_{1,1}(R, \varphi, \theta)]\frac{\sin(2\varphi)}{2} + \widetilde{a}_{1,2}(R, \varphi, \theta)\cos(2\varphi) = 0 \qquad (2.12)$$

$$\widetilde{a}_{1,3}(R, \varphi, \theta)\sin\varphi - \widetilde{a}_{2,3}(R, \varphi, \theta)\cos\varphi = 0, \qquad (2.13)$$

where

$$\widetilde{g}(r, \varphi, \theta) = g(\delta\widetilde{x}(\varphi, \tfrac{\pi}{2}) + r\widetilde{x}(\varphi, \theta)) \quad \forall\, g \in L^1(\Omega). \qquad (2.14)$$

Coming back to our problem, $u_0 : \overline{\Omega} \to \mathbb{R}$ and $u_1 : [0, T] \times \overline{\Omega} \to \mathbb{R}$ being two prescribed smooth functions, we supplement equation (2.1) with the *initial condition*

$$u(0, x) = u_0(x) \quad \forall\, x \in \Omega, \qquad (2.15)$$

and with the *conormal boundary value condition*

$$D_\nu u(t, x) = D_\nu u_1(t, x) \quad \forall (t, x) \in [0, T] \times \partial\Omega, \qquad (2.16)$$

where $\nu(x) = (\nu_1(x), \nu_2(x), \nu_3(x))$ is defined by $\nu_j(x) = \sum_{k=1}^{3} a_{j,k}(x)n_k(x)$, $j = 1, 2, 3$.

Since we are concerned with an identification problem, we will assume also that the following two additional pieces of information are available

$$\Phi[u(t, \cdot)](\varphi) := g_1(t, \varphi) \quad \forall\, (t, \varphi) \in [0, T] \times [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}], \qquad (2.17)$$

$$\Psi[u(t, \cdot)] := g_2(t) \qquad \forall\, t \in [0, T], \qquad (2.18)$$

where Φ is a linear operator acting on variables r and θ only, while Ψ is a linear operator acting on all the space variables r, φ, θ.

By using the shortening $\Phi[w(\cdot)] = \Phi[w]$, $\Psi[w(\cdot)] = \Psi[w]$ for any $w : \Omega \to \mathbb{R}$, from (2.15)–(2.18) we (formally) deduce that our data have to satisfy the following consistency conditions, where $\varphi \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}]$:

$$\Phi[u_0](\varphi) = g_1(0, \varphi), \quad \Psi[u_0] = g_2(0), \quad D_\nu u_0 = D_\nu u_1(0, \cdot). \qquad (2.19)$$

Convention: from now on we will always denote by P(C) the identification problem consisting of (2.1) and (2.15)–(2.18).

In order to give a concrete example of admissible linear operators Φ and Ψ, first, recalling the definitions of P_φ and x_φ given in Remark 2.2, we denote by

$\Sigma(\varphi)$, $\varphi \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$, the subset of P_φ consisting in the two-dimensional ball of centre x_φ and radius R, i.e.,

$$\Sigma(\varphi) = \{x \in \Omega : \rho(x') = \varphi, \; |x - x_\varphi| < R\} \quad \forall \varphi \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}]. \tag{2.20}$$

Then, when $\phi : [\delta - R, \delta + R] \to \mathbb{R}$ and $\psi : \overline{\Omega} \to \mathbb{R}$ are two smooth assigned functions, taking into account (2.9), (2.10) and (2.14) we set

$$\begin{aligned}
\Phi[v](\varphi) &= \frac{1}{m_2(\Sigma(\varphi))} \int_{\Sigma(\varphi)} \phi(|x'|)v(x) \, d\sigma(x) \\
&= \frac{1}{\pi R^2} \int_0^R r \, dr \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \overline{\phi}(r, \theta)\widetilde{v}(r, \varphi, \theta) \, d\theta
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\Psi[v] &= \int_\Omega \psi(x)v(x) \, dx \\
&= \int_0^R r \, dr \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (\delta + r \sin\theta) \, d\theta \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \widetilde{\psi}(r, \varphi, \theta)\widetilde{v}(r, \varphi, \theta) \, d\varphi.
\end{aligned} \tag{2.22}$$

Here $m_2(\Sigma(\varphi))$ and $d\sigma(x)$ denote, respectively, the two-dimensional Lebesgue measure and the two-dimensional volume element of $\Sigma(\varphi)$, whereas

$$\overline{\varkappa}(r, \theta) = \varkappa(\delta + r \sin\theta) \quad \forall \varkappa \in L^1(\delta - R, \delta + R). \tag{2.23}$$

For the brevity's sake, we introduce the following definition.

Definition 2.3. An operator triplet $(\mathcal{A}, \Phi, \Psi)$ is said to be *admissible* for the problem P(C) if

(i) the operator $\mathcal{A}$ defined by (2.3) is uniformly elliptic and its coefficients $a_{i,j}$, $i, j = 1, 2, 3$, satisfy also the further conditions (2.5)–(2.7);

(ii) the operators Φ and Ψ are defined, respectively, by (2.21) and (2.22), where functions ϕ and ψ are chosen so that $\phi \in C^1([\delta - R, \delta + R])$ and $\psi \in C^1(\overline{\Omega})$.

Remark 2.4. In order to show that conditions (2.5)–(2.7) are meaningful we exhibit a class of coefficients $a_{i,j}$, $i, j = 1, 2, 3$, satisfying such properties. Suppose there exist $a, b \in C_b(\mathbb{R}) \cap W^{2,+\infty}(\mathbb{R})$, $d \in W^{2,+\infty}(\delta - R, \delta + R)$, $c_j \in W^{2,+\infty}(\Omega)$, $j = 1, 2, 3$, with a and c_1, respectively, *positive* and *non-positive* and $c_2|_{\partial\Omega} = c_3|_{\partial\Omega}$, such that

$$\begin{aligned}
a_{j,j}(x) &= |x'|^2 a\big(\tfrac{x_2}{x_1}\big) + x_{3-j}^2 b\big(\tfrac{x_2}{x_1}\big) - x_j^2[c_1(x) + d(|x'|)], \quad j = 1, 2 \\
a_{1,2}(x) &= a_{2,1}(x) = -x_1 x_2 \big[b\big(\tfrac{x_2}{x_1}\big) + c_1(x) + d(|x'|)\big] \\
a_{j,3}(x) &= a_{3,j}(x) = c_{j+1}(x)x_j, \quad\quad\quad\quad\quad\quad\quad\quad j = 1, 2.
\end{aligned}$$

Easy computations show that in this case (2.5)–(2.7) are satisfied with $\lambda = a+b$. In particular, (2.6) holds true in all of $\overline{\Omega}$ and not only on its boundary $\partial\Omega$. Moreover, denoting by f^+, f^-, respectively, the positive and the negative parts of a function f, for every $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^3$ it easily follows

$$
\sum_{j,k=1}^{3} a_{j,k}(x)\xi_j\xi_k \geq \left\{ a\left(\tfrac{x_2}{x_1}\right) - b^-\left(\tfrac{x_2}{x_1}\right) - d^+(|x'|) \right\} |x'|^2 |\xi'|^2
$$

$$
+ a_{3,3}(x)\xi_3^2 - 2 \max_{j=2,3} \| c_j^- \|_{C(\overline{\Omega})} |x'||\xi'|\xi_3
$$

$$
\geq \left\{ a\left(\tfrac{x_2}{x_1}\right) - b^-\left(\tfrac{x_2}{x_1}\right) - d^+(|x'|) - \max_{j=2,3} \| c_j^- \|_{C(\overline{\Omega})} \right\} |x'|^2 |\xi'|^2
$$

$$
+ \left\{ a_{3,3}(x) - \max_{j=2,3} \| c_j^- \|_{C(\overline{\Omega})} \right\} \xi_3^2 .
$$

Hence, assuming

$$
D := \min_{t\in[-\infty,+\infty]} [a(t) - b^-(t)] - \max_{s\in[-R,R]} d^+(\delta - s) - \max_{j=2,3} \| c_j^- \|_{C(\overline{\Omega})} > 0,
$$

which amounts to requiring that a is large enough with respect to b^-, d^+, c_j^-, $j = 2, 3$, and choosing, for instance, $a_{3,3}$ greater than $(\delta - R)^2 D + \max_{j=2,3} \| c_j^- \|_{C(\overline{\Omega})}$, the uniform ellipticity of $\mathcal{A}$ is guaranteed for a very large class of functions c_{j+1}, $j = 1, 2$. In this sense, it must be interpreted the freedom in the choice of the coefficients $a_{i,3}$, $i = 1, 2, 3$, previously remarked. Of course, (2.4) must be satisfied, too. To this purpose, an example of function a, b and d could be $a(t) = C + e(t)$, $b(t) = e(t)$ and $d(t) = t$, where C is a great enough positive constant and $e(t) = t^2/(t^2 + 1)$.

3. Basic assumptions and main results

Let the triplet $(\mathcal{A}, \Phi, \Psi)$ be admissible according to Definition 2.3. As in [3], in order to find out the right hypotheses on the linear operators Φ and Ψ it is convenient to rewrite the differential operator $\mathcal{A}$ in the co-ordinates (r, φ, θ) introduced in (2.9). First of all, we observe that performing easy computations the gradient $\nabla_x = (D_{x_1}, D_{x_2}, D_{x_3})$ can be expressed in terms of $\nabla_{(r,\varphi,\theta)} = (D_r, D_\varphi, D_\theta)$ by the following formulae:

$$
D_{x_j} = \tau_{j,1}(\varphi, \theta)D_r + \frac{\tau_{j,2}(\varphi, \theta)}{\delta + r\sin\theta}D_\varphi + \frac{\tau_{j,3}(\varphi, \theta)}{r}D_\theta, \quad j = 1, 2, 3, \qquad (3.1)
$$

where

$$
\begin{cases}
\tau_{1,1}(\varphi, \theta) = \cos\varphi\sin\theta, & \tau_{1,2}(\varphi, \theta) = -\sin\varphi, & \tau_{1,3}(\varphi, \theta) = \cos\varphi\cos\theta \\
\tau_{2,1}(\varphi, \theta) = \sin\varphi\sin\theta, & \tau_{2,2}(\varphi, \theta) = \cos\varphi, & \tau_{2,3}(\varphi, \theta) = \sin\varphi\cos\theta \\
\tau_{3,1}(\varphi, \theta) = \cos\theta, & \tau_{3,2}(\varphi, \theta) = 0, & \tau_{3,3}(\varphi, \theta) = -\sin\theta.
\end{cases} \qquad (3.2)
$$

As a consequence, for any $j = 1, 2, 3$, we obtain

$$\sum_{k=1}^{3} a_{j,k}(x)D_{x_k} = f_{j,1}(r, \varphi, \theta)D_r + \frac{f_{j,2}(r, \varphi, \theta)}{\delta + r \sin \theta} D_\varphi + \frac{f_{j,3}(r, \varphi, \theta)}{r} D_\theta, \quad (3.3)$$

where the functions $f_{j,k}$, $j, k = 1, 2, 3$, are defined by

$$f_{j,l}(r, \varphi, \theta) := \sum_{k=1}^{3} \tilde{a}_{j,k}(r, \varphi, \theta)\tau_{k,l}(\varphi, \theta), \quad j, l = 1, 2, 3. \quad (3.4)$$

Hence, using (3.1) and applying it to relations (3.3), easy computations lead to

$$\mathcal{A}w\big(\delta\tilde{x}\big(\varphi, \tfrac{\pi}{2}\big) + r\tilde{x}(\varphi, \theta)\big)$$
$$= \left[\sum_{j=1}^{2} \frac{\tau_{j,2}(\varphi, \theta)}{\delta + r \sin \theta} D_\varphi \Big(\frac{f_{j,2}(r, \varphi, \theta)}{\delta + r \sin \theta} D_\varphi\Big) + \sum_{j=1}^{3} \mathcal{P}_j\right]\tilde{w}(r, \varphi, \theta), \quad (3.5)$$

where we have used $\tau_{3,2} = 0$ and where $\tilde{w}$ is related to $w \in W^{2,p}(\Omega)$ via formula (2.14). The second-order linear differential operators $\mathcal{P}_j$, $j = 1, 2, 3$, are defined, respectively, by

$$\mathcal{P}_1 = D_r\left(k_1(r, \varphi, \theta)D_r + \frac{k_2(r, \varphi, \theta)}{\delta + r \sin \theta} D_\varphi + \frac{k_3(r, \varphi, \theta)}{r} D_\theta\right) \quad (3.6)$$

$$\mathcal{P}_2 = \sum_{j=1}^{2} \frac{\tau_{j,2}(\varphi, \theta)}{\delta + r \sin \theta} D_\varphi\left[f_{j,1}(r, \varphi, \theta)D_r + \frac{f_{j,3}(r, \varphi, \theta)}{r} D_\theta\right] \quad (3.7)$$

$$\mathcal{P}_3 = \sum_{j=1}^{3} \frac{\tau_{j,3}(\varphi, \theta)}{r} D_\theta\left[f_{j,1}(r, \varphi, \theta)D_r + \frac{f_{j,2}(r, \varphi, \theta)}{\delta + r \sin \theta} D_\varphi + \frac{f_{j,3}(r, \varphi, \theta)}{r} D_\theta\right], \quad (3.8)$$

where the functions k_j, $j = 1, 2, 3$, appearing in (3.6) are defined by

$$k_l(r, \varphi, \theta) = \sum_{j=1}^{3} \tau_{j,1}(\varphi, \theta) f_{j,l}(r, \varphi, \theta), \quad l = 1, 2, 3. \quad (3.9)$$

Now, recalling the definitions of $\tau_{j,2}$, $j = 1, 2$, in (3.2) and using Leibniz's formula for the derivative of the product, we easily deduce

$$\sum_{j=1}^{2} \frac{\tau_{j,2}(\varphi, \theta)}{\delta + r \sin \theta} D_\varphi\left[\frac{f_{j,2}(r, \varphi, \theta)}{\delta + r \sin \theta} D_\varphi\right]$$
$$= D_\varphi\left[\frac{l_1(r, \varphi, \theta)}{(\delta + r \sin \theta)^2} D_\varphi\right] + \frac{l_2(r, \varphi, \theta)D_\varphi}{(\delta + r \sin \theta)^2}, \quad (3.10)$$

where we have set

$$l_j(r, \varphi, \theta) = f_{3-j,2}(r, \varphi, \theta) \cos \varphi + (-1)^j f_{j,2}(r, \varphi, \theta) \sin \varphi, \quad j = 1, 2. \quad (3.11)$$

Taking into account (3.4), we see that the functions $(\delta + r\sin\theta)^{-2}l_1(r, \varphi, \theta)$ and $l_2(r, \varphi, \theta)$ coincide, respectively, with the left-hand side of (2.11) and (2.12) (with R being replaced by r). Therefore, by virtue of assumption (2.5), from (3.10) it follows

$$\sum_{j=1}^{2} \frac{\tau_{j,2}(\varphi, \theta)}{\delta + r\sin\theta} D_\varphi \left[\frac{f_{j,2}(r, \varphi, \theta)}{\delta + r\sin\theta} D_\varphi \right] = \mathcal{A}_1 + \mathcal{P}_4, \qquad (3.12)$$

where we have set

$$\mathcal{A}_1 = D_\varphi[\lambda(\tan\varphi)D_\varphi], \qquad (3.13)$$

$$\mathcal{P}_4 = (\delta + r\sin\theta)^{-2}l_2(r, \varphi, \theta)D_\varphi. \qquad (3.14)$$

Replacing (3.12) in (3.5) we finally obtain the basic decomposition formula

$$\mathcal{A}w\left(\delta\widetilde{x}\left(\varphi, \tfrac{\pi}{2}\right) + r\widetilde{x}(\varphi, \theta)\right) = \mathcal{A}_1\widetilde{w}(r, \varphi, \theta) + \sum_{j=1}^{4} \mathcal{P}_j\widetilde{w}(r, \varphi, \theta). \qquad (3.15)$$

We can now list the main properties of operators Φ and Ψ appearing in (2.14) and (2.15) in the framework of Sobolev spaces related to $L^p(\Omega)$ with

$$p \in (3, +\infty). \qquad (3.16)$$

As usual, Z_j, $j = 1, 2$, being Banach spaces, $\mathcal{L}(Z_1; Z_2)$ denotes the Banach space of all bounded linear operators from Z_1 to Z_2 equipped with the uniform operatorial norm. In particular, given a Banach space X, $\mathcal{L}(X) = \mathcal{L}(X; X)$ and $X^* = \mathcal{L}(X; \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We assume:

$$\Phi \in \mathcal{L}\left(L^p(\Omega); L^p\left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)\right), \quad \Psi \in L^p(\Omega)^* \qquad (3.17)$$

$$\Phi[wu] = w\Phi[u] \quad \forall\, (w, u) \in L^p\left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right) \times L^p(\Omega) \qquad (3.18)$$

$$D_\varphi\Phi[u](\varphi) = \Phi[D_\varphi u](\varphi) \quad \forall\, u \in W^{1,p}(\Omega),\ \forall\, \varphi \in \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right) \qquad (3.19)$$

$$\Phi\mathcal{A} = \mathcal{A}_1\Phi + \Phi_1 \text{ on } W_{\mathrm{C}}^{2,p}(\Omega), \quad \Phi_1 \in \mathcal{L}\left(W^{1,p}(\Omega); L^p\left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)\right) \qquad (3.20)$$

$$\Psi\mathcal{A} = \Psi^1 \text{ on } W_{\mathrm{C}}^{2,p}(\Omega), \quad \Psi_1 \in W^{1,p}(\Omega)^*. \qquad (3.21)$$

Here $\mathcal{A}_1$ is the second-order differential operator defined by (3.13), whereas $W_{\mathrm{C}}^{2,p}(\Omega)$ is the function space defined by

$$W_{\mathrm{C}}^{2,p}(\Omega) = \{\omega \in W^{2,p}(\Omega): \ D_\nu\omega = 0\}.$$

To state our result concerning the identification problem P(C) we need (see, e.g., [3]) also the following assumptions on the data f, u_0, u_1, g_1, g_2:

$$f \in C^{1+\beta}([0,T]; L^p(\Omega)), \quad f(0,\cdot) \in W^{2,p}(\Omega) \tag{3.22}$$

$$u_0 \in W^{4,p}(\Omega), \quad \mathcal{B}u_0 \in B_{\mathrm{C}}^{2\alpha,p,+\infty}(\Omega) \tag{3.23}$$

$$u_1 \in C^{2+\beta}([0,T]; L^p(\Omega)) \cap C^{1+\beta}([0,T]; W^{2,p}(\Omega)) \tag{3.24}$$

$$v_0 := \mathcal{A}u_0 + f(0,\cdot) - D_t u_1(0,\cdot) \in W_{\mathrm{C}}^{2,p}(\Omega) \tag{3.25}$$

$$[\mathcal{A}^2 + k_0\mathcal{B} + k_0'\mathcal{C}]u_0 + [D_t + \mathcal{A}]f(0,\cdot) - D_t^2 u_1(0,\cdot) \in B_{\mathrm{C}}^{2\beta,p,+\infty}(\Omega) \tag{3.26}$$

$$g_1 \in C^{2+\beta}([0,T]; L^p(-\tfrac{\pi}{2}, \tfrac{3\pi}{2})) \cap C^{1+\beta}([0,T]; W^{2,p}(-\tfrac{\pi}{2}, \tfrac{3\pi}{2})) \tag{3.27}$$

$$\mathcal{A}_1 D_t g_1(t,\cdot) \in L^p(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}) \quad \forall t \in [0,T], \quad j = 1,2 \tag{3.28}$$

$$g_2 \in C^{2+\beta}([0,T]; \mathbb{R}), \tag{3.29}$$

where $0 < \beta < \delta < \frac{1}{2}$ and function k_0 in (3.26) is defined by formula (5.17). Here, for $\gamma \in (0,1)$, $p_1 \in (1,+\infty)$, $p_2 \in [1,+\infty]$, $2\gamma - 1/p_1 \neq 1$, the Besov spaces $B_{\mathrm{C}}^{2\gamma,p_1,p_2}(\Omega)$ represent the interpolation spaces $(L^{p_1}(\Omega), W_{\mathrm{C}}^{2,p_1}(\Omega))_{\gamma,p_2}$ (cf. [10, Subsection 4.3.3]).

Remark 3.1. We can now make it clear why we have assumed $\lambda \in \widetilde{C}_b^1(\mathbb{R})$ (cf. (2.8)). Essentially, such belonging is for assumption (3.28) to make sense when (3.27) holds. Indeed, from (3.13) we get

$$\mathcal{A}_1 D_t g_1(t,\varphi) = \frac{\lambda'(\tan\varphi)}{\cos^2\varphi} D_\varphi D_t g_1(t,\varphi) + \lambda(\tan\varphi)D_\varphi^2 D_t g_1(t,\varphi),$$

and hence, when φ goes to $(-\tfrac{\pi}{2})^+$, $(\tfrac{\pi}{2})^{\pm}$ and $(3\tfrac{\pi}{2})^-$, $\mathcal{A}_1 D_t g_1(t,\varphi)$ might happen to blow up, seriously compromising its belonging to $L^p(-\tfrac{\pi}{2}, 3\tfrac{\pi}{2})$. Now, from (2.9) and (3.2) we deduce the formulae

$$D_\varphi = (\delta + r\sin\theta)\sum_{j=1}^{2}\tau_{j,2}(\varphi,\theta)D_{x_j},$$

$$D_\varphi^2 = (\delta + r\sin\theta)^2\sum_{j,k=1}^{2}\tau_{j,2}(\varphi,\theta)\tau_{k,2}(\varphi,\theta)D_{x_j x_k}$$

$$+ (\delta + r\sin\theta)\sum_{\substack{j,k=1 \\ j \neq k}}^{2}(-1)^k\tau_{j,2}(\varphi,\theta)D_{x_k}.$$

Therefore it is easily seen that both $D_\varphi D_t g_1(t,\cdot)$ and $D_\varphi^2 D_t g_1(t,\cdot)$ belong to $L^p(-\tfrac{\pi}{2}, \tfrac{3\pi}{2})$ and hence, to ensure $\mathcal{A}_1 D_t g_1(t,\cdot) \in L^p(-\tfrac{\pi}{2}, \tfrac{3\pi}{2})$, it is quite natural to search for a function λ such that both $\lambda'(\tan\varphi)/\cos^2\varphi$ and $\lambda(\tan\varphi)$ belong

to $L^\infty(-\frac{\pi}{2}, \frac{3\pi}{2})$. This is true, for instance, if λ belongs to $\widetilde{C}_b^1(\mathbb{R})$. Indeed, in such a case, the following holds for $\varphi_0 = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$:

$$\lim_{\varphi \to \varphi_0^\pm} \frac{\lambda'(\tan \varphi)}{\cos^2 \varphi} = \lim_{y \to \pm\infty} (y^2 + 1)\lambda'(y) = 0, \quad \lim_{\varphi \to \varphi_0^\pm} \lambda(\tan \varphi) = \lim_{y \to \pm\infty} \lambda(y) = r_1.$$

Coming back to our basic assumptions we assume that u_0 satisfies also the following conditions for some positive constant m_0:

$$J_0(u_0)(\varphi) := \left|\Phi[\mathcal{C}u_0](\varphi)\right| \geq m_0 \quad \forall \varphi \in (-\tfrac{\pi}{2}, \tfrac{3\pi}{2}) \tag{3.30}$$

$$J_1(u_0) := \Psi[J(u_0)] \neq 0, \tag{3.31}$$

where, for any $x \in \Omega$, we have set

$$J(u_0)(x) := \left(\mathcal{B}u_0(x) - \frac{\Phi[\mathcal{B}u_0](\rho(x'))}{\Phi[\mathcal{C}u_0](\rho(x'))}\mathcal{C}u_0(x)\right) \exp\left[\int_{\rho(x')}^{-\frac{\pi}{2}} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} \, d\xi\right].$$

Finally, we list the consistency conditions for the function v_0 defined by (3.26):

$$\Phi[v_0](\varphi) = D_t g_1(0, \varphi) - \Phi[D_t u_1(0, \cdot)](\varphi) \quad \forall \varphi \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \tag{3.32}$$

$$\Psi[v_0] = D_t g_2(0) - \Psi[D_t u_1(0, \cdot)], \tag{3.33}$$

and we introduce the following Banach spaces, where $s \in \mathbb{N} \setminus \{0\}$:

$$\mathcal{U}^{s,\beta,p}(T) = C^{s+\beta}\big([0,T]; L^p(\Omega)\big) \cap C^{s-1+\beta}\big([0,T]; W^{2,p}(\Omega)\big). \tag{3.34}$$

Similarly, replacing $W^{2,p}(\Omega)$ with $W_C^{2,p}(\Omega)$ in (3.34), we define spaces $\mathcal{U}_C^{s,\beta,p}(T)$.

In this context we can now state our first result.

Theorem 3.2. *Let assumptions (2.4) and (3.16)–(3.21) be fulfilled together with the uniform ellipticity of $\mathcal{A}$. Moreover, assume that the vector (u_0, u_1, g_1, g_2, f) enjoy properties (3.22)–(3.29) and satisfy (3.30), (3.31) as well as the consistency conditions (2.19), (3.32) and (3.33). Then, there exists a $T^* \in (0, T]$ such that the identification problem P(C) admits a unique solution $(u, k) \in \mathcal{U}^{2,\beta,p}(T^*) \times C^\beta([0, T^*]; W^{1,p}(-\frac{\pi}{2}, \frac{3\pi}{2}))$ depending continuously on the data. The result is true even in the case of an operator triplet $(\mathcal{A}, \Phi, \Psi)$ admissible in the sense of Definition 2.3.*

The techniques developed for proving Theorem 3.2 do not guarantee that k satisfies

$$k(t, -\tfrac{\pi}{2}) = k(t, \tfrac{3\pi}{2}) \quad \forall t \in [0, T^*], \tag{3.35}$$

and, consequently, we should expect $k(t, \cdot)$ to have a jump along the section $\Sigma(-\frac{\pi}{2}) = \Sigma(\frac{3\pi}{2})$ (cf. (2.20)). We stress that the choice of $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$ is made

only for convenience and for agreeing with the definition (2.2) of $\rho(x')$. Actually, any other interval of amplitude 2π could have been chosen, the only effect of a different choice being that of making the discontinuity section of $k(t, \cdot)$ rotate.

The point is that (3.35) makes, in general, the equivalence Theorem 5.2 below fail. In Section 7 we will give a detailed explanation of this fact. Here, instead, we want to emphasize that for periodic kernels we are still able to show uniqueness, assuming existence. To this purpose, we first need to define some function spaces suitable for working with periodic functions on an interval $[a, b] \subset \mathbb{R}$, $a < b$. The spaces $\widetilde{W}^{j,p}(a, b)$, $j \in \mathbb{N} \cup \{0\}$, are defined by

$$\widetilde{W}^{j,p}(a, b) = \left\{ \omega \in W^{j,p}(a, b) : \ \omega(a) = \omega(b) \right\}. \tag{3.36}$$

In particular, in accordance with the definition of L^p-spaces as Sobolev spaces of order zero, when $j = 0$ in (3.36) we use the shortening:

$$\widetilde{L}^p(a, b) = \left\{ \omega \in L^p(a, b) : \ \omega(a) = \omega(b) \right\}. \tag{3.37}$$

Note that, due to the Sobolev imbedding $W^{k,p}(a, b) \hookrightarrow C^{k-1/p}([a, b])$ for any $k \geq 1$ and $p > 1$, definition (3.36) is meaningful (for $j \geq 1$) whereas (3.37) is forced.

We then get the following result.

Theorem 3.3. *Let all the assumptions of Theorem* 3.3 *be satisfied, except for replacing any space* $W^{j,p}(-\frac{\pi}{2}, \frac{3\pi}{2})$, $j \in \mathbb{N} \cup \{0\}$, *with the correspondent* $\widetilde{W}^{j,p}(-\frac{\pi}{2}, \frac{3\pi}{2})$. *There exists* $T^* \in (0, T]$ *such that, if the identification problem* P(C) *admits a solution* $(u, k) \in \mathcal{U}^{2,\beta,p}(T^*) \times C^\beta([0, T^*]; \widetilde{W}^{1,p}(-\frac{\pi}{2}, \frac{3\pi}{2}))$, *then the solution is unique. The result is true even when the triplet* $(\mathcal{A}, \Phi, \Psi)$ *is admissible in the sense of Definition* 2.3.

Observe that the replacement of $L^p(-\frac{\pi}{2}, \frac{3\pi}{2})$ with $\widetilde{L}^p(-\frac{\pi}{2}, \frac{3\pi}{2})$ does make sense, since (2.21) implies $\Phi[v](-\frac{\pi}{2}) = \Phi[v](\frac{3\pi}{2})$ for every $v \in L^p(\Omega)$.

4. Preliminary lemmata

Here we show that if the triplet $(\mathcal{A}, \Phi, \Psi)$ is admissible in the sense of Definition 2.3, then, for $p \in (3, +\infty)$, the pair (Φ, Ψ) satisfies the abstract assumption (3.17)–(3.21). As a consequence, the last part of Theorems 3.2 and 3.3 will simply derive from what we will prove later in the general abstract situation. FFrom now on, by $\langle \cdot, \cdot \rangle$ and $I(r, \varphi, \theta)$ we denote, respectively, the canonical inner product in $\mathbb{R}^3$ and the Cartesian product $(0, R) \times (-\frac{\pi}{2}, \frac{3\pi}{2}) \times (-\frac{\pi}{2}, \frac{3\pi}{2})$.

Lemma 4.1. *Let the triplet* $(\mathcal{A}, \Phi, \Psi)$ *be admissible according to Definition* 2.3. *Then, if* $p \in (3, +\infty)$, *the linear operator* Φ *satisfies the decomposition* (3.20).

Proof. Let $w \in W_\mathrm{C}^{2,p}(\Omega)$ and let apply Φ to $\mathcal{A}w$ taking (3.15) into account. First, $\widetilde{w} \in W^{2,p}(I(r, \varphi, \theta))$ being related to $w \in W_\mathrm{C}^{2,p}(\Omega)$ via (2.14), from (2.21) and (3.13) we easily get $\Phi[\mathcal{A}_1\widetilde{w}](\varphi) = \mathcal{A}_1\Phi[w](\varphi)$. Therefore, from (3.15) it follows

$$\Phi[\mathcal{A}w] = \mathcal{A}_1\Phi[w] + \sum_{j=1}^{4} \Phi[\mathcal{P}_j\widetilde{w}] \quad \forall\, w \in W_\mathrm{C}^{2,p}(\Omega). \tag{4.1}$$

Now, from the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C^{1-3/p}(\overline{\Omega})$, $p \in (3, +\infty)$, and from expressing $\nabla_{(r,\varphi,\theta)}$ in terms of ∇_x (cf. (2.9) and (3.2))

$$\nabla_{(r,\varphi,\theta)} = \sum_{k=1}^{3} \left(\tau_{k,1}(\varphi, \theta)D_{x_k}, (\delta + r\sin\theta)\tau_{k,2}(\varphi,\theta)D_{x_k}, r\tau_{k,3}(\varphi,\theta)D_{x_k}\right), \tag{4.2}$$

we find that $D_r\widetilde{w}$, $(D_\varphi\widetilde{w})/(\delta + r\sin\theta)$ and $(D_\theta\widetilde{w})/r$ belong to $C^{1-3/p}\left(\overline{I(r, \varphi, \theta)}\right)$. Hence, after observing that (cf. (3.14))

$$\Phi[\mathcal{P}_4 w](\varphi) = \frac{1}{\pi R^2} \int_0^R r\,\mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\overline{\phi}(r,\theta)l_2(r,\varphi,\theta)}{(\delta + r\sin\theta)^2} D_\varphi\widetilde{w}(r,\varphi,\theta)\,\mathrm{d}\theta, \tag{4.3}$$

recalling (3.6)–(3.8) we can proceed to transform the expression of $\sum_{j=1}^{3} \Phi[\mathcal{P}_j\widetilde{w}]$ in (4.1) by integration by parts. To this purpose we first introduce the following vector functions:

$$F(r, \varphi, \theta) = \left(k_1(r,\varphi,\theta), \frac{k_2(r,\varphi,\theta)}{\delta + r\sin\theta}, \frac{k_3(r,\varphi,\theta)}{r}\right) \tag{4.4}$$

$$F_j(r, \varphi, \theta) = \left(f_{j,1}(r,\varphi,\theta), \frac{f_{j,2}(r,\varphi,\theta)}{\delta + r\sin\theta}, \frac{f_{j,3}(r,\varphi,\theta)}{r}\right), \quad j = 1, 2, 3. \tag{4.5}$$

Then, using the definition of the conormal vector ν given after formula (2.16) and the fact that $\widetilde{x}(\varphi, \theta)$ is the unit outer normal vector at $x \in \partial\Omega$, from formulae (3.2), (3.4), (3.9) and (4.2) through lengthy computations we get

$$\langle F(R,\varphi,\theta), \nabla_{(r,\varphi,\theta)}\widetilde{w}(R,\varphi,\theta)\rangle = \sum_{j=1}^{3}\langle\widetilde{A}_j(R,\varphi,\theta), \widetilde{x}(\varphi,\theta)\rangle\widetilde{(D_{x_j}w)}(R,\varphi,\theta)$$

$$= \widetilde{(D_\nu w)}(R,\varphi,\theta). \tag{4.6}$$

where $\widetilde{A}_j(R,\varphi,\theta) = (\widetilde{a}_{j,1}(R,\varphi,\theta), \widetilde{a}_{j,2}(R,\varphi,\theta), \widetilde{a}_{j,3}(R,\varphi,\theta))$, $j = 1, 2, 3$. Now, observe that (2.23), (3.4) and (4.2) imply, respectively, $\overline{\lambda}(r, -\frac{\pi}{2}) = \overline{\lambda}(r, \frac{3\pi}{2})$, $f_{j,k}(r, \varphi, -\frac{\pi}{2}) = f_{j,k}(r, \varphi, \frac{3\pi}{2})$, $j, k = 1, 2, 3$, and $D_\theta\widetilde{w}(0, \varphi, \theta) = 0$. Therefore,

taking into account formula (4.6) with $\widetilde{(D_\nu w)}(R, \varphi, \theta) = 0$ (recall that $w \in W_C^{2,p}(\Omega)$), an integration by parts lead us to

$$\Phi[\mathcal{P}_1 \widetilde{w}](\varphi) = -\frac{1}{\pi R^2} \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \langle F(r, \varphi, \theta), \nabla_{(r,\varphi,\theta)} \widetilde{w}(r, \varphi, \theta) \rangle \tag{4.7}$$
$$\times D_r[r\overline{\phi}(r, \theta)] \, \mathrm{d}\theta$$

$$\Phi[\mathcal{P}_3 \widetilde{w}](\varphi) = -\frac{1}{\pi R^2} \sum_{j=1}^3 \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \langle F_j(r, \varphi, \theta), \nabla_{(r,\varphi,\theta)} \widetilde{w}(r, \varphi, \theta) \rangle \tag{4.8}$$
$$\times D_\theta[\tau_{j,3}(\varphi, \theta) \overline{\phi}(r, \theta)] \, \mathrm{d}\theta \,.$$

Moreover, since the $\tau_{j,2}$'s, $j = 1, 2, 3$, depend on φ, only, we obtain

$$\Phi[\mathcal{P}_2 \widetilde{w}](\varphi)$$
$$= \frac{1}{\pi R} \sum_{k=0}^1 \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\overline{\phi}(R, \theta) l_{4-k}(R, \varphi, \theta) D_\varphi^k \widetilde{w}(R, \varphi, \theta)}{\delta + R \sin \theta} \, \mathrm{d}\theta$$
$$- \frac{1}{\pi R^2} \sum_{j,k=1}^2 \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \tau_{j,2}(\varphi) D_\varphi^{k-1} \widetilde{w}(r, \varphi, \theta) \tag{4.9}$$
$$\times D_\varphi^{2-k} \left\{ D_r \left[\frac{r\overline{\phi}(r, \theta) f_{j,1}(r, \varphi, \theta)}{\delta + r \sin \theta} \right] + D_\theta \left[\frac{\overline{\phi}(r, \theta) f_{j,3}(r, \varphi, \theta)}{\delta + r \sin \theta} \right] \right\} \mathrm{d}\theta \,,$$

where we have set

$$l_j(r, \varphi, \theta) = [D_\varphi^{j-3} f_{2,1}(r, \varphi, \theta)] \cos \varphi - [D_\varphi^{j-3} f_{1,1}(r, \varphi, \theta)] \sin \varphi, \quad j = 3, 4. \tag{4.10}$$

Easy computations, taking advantage of definitions (3.4), shows that

$$l_3(r, \varphi, \theta) = \left\{ [\widetilde{a}_{2,2}(r, \varphi, \theta) - \widetilde{a}_{1,1}(r, \varphi, \theta)] \frac{\sin(2\varphi)}{2} + \widetilde{a}_{1,2}(r, \varphi, \theta) \cos(2\varphi) \right\} \sin \theta$$
$$+ [\widetilde{a}_{1,3}(r, \varphi, \theta) \sin \varphi - \widetilde{a}_{2,3}(r, \varphi, \theta) \cos \varphi] \cos \theta \,.$$

Hence, from (2.12) and (2.13) we deduce $l_3(R, \varphi, \theta) = 0$. Therefore, rearranging (4.3) and (4.7)–(4.9) we finally deduce

$$\Phi[\mathcal{A}w] = \mathcal{A}_1 \Phi[w] + \Phi_1[w] \quad \forall \, w \in W_C^{2,p}(\Omega) \,,$$

where we have set $\Phi_1[w](\varphi) := \sum_{j=1}^{4} \Phi[\mathcal{P}_j \widetilde{w}](\varphi)$ with

$$
\begin{aligned}
\sum_{j=1}^{4} & \Phi[\mathcal{P}_j \widetilde{w}](\varphi) \\
&= \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{3\frac{\pi}{2}} \frac{\overline{\phi}(R,\theta) l_4(R,\varphi,\theta) \widetilde{w}(R,\varphi,\theta)}{\delta + R \sin\theta} \, \mathrm{d}\theta \\
&\quad + \frac{1}{\pi R^2} \int_0^R r \, \mathrm{d}r \int_{-\frac{\pi}{2}}^{3\frac{\pi}{2}} \frac{\overline{\phi}(r,\theta) l_2(r,\varphi,\theta)}{(\delta + r \sin\theta)^2} \, D_\varphi \widetilde{w}(r,\varphi,\theta) \, \mathrm{d}\theta \\
&\quad - \frac{1}{\pi R^2} \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{3\frac{\pi}{2}} \langle F(r,\varphi,\theta), \nabla_{(r,\varphi,\theta)} \widetilde{w}(r,\varphi,\theta) \rangle D_r[r\overline{\phi}(r,\theta)] \, \mathrm{d}\theta \\
&\quad - \frac{1}{\pi R^2} \sum_{j=1}^{3} \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{3\frac{\pi}{2}} \langle F_j(r,\varphi,\theta), \nabla_{(r,\varphi,\theta)} \widetilde{w}(r,\varphi,\theta) \rangle \\
&\quad \times D_\theta[\tau_{j,3}(\varphi,\theta) \overline{\phi}(r,\theta)] \, \mathrm{d}\theta \\
&\quad - \frac{1}{\pi R^2} \sum_{j,k=1}^{2} \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{3\frac{\pi}{2}} \tau_{j,2}(\varphi) D_\varphi^{k-1} \widetilde{w}(r,\varphi,\theta) \\
&\quad \times D_\varphi^{2-k} \left\{ D_r \left[\frac{r\overline{\phi}(r,\theta) f_{j,1}(r,\varphi,\theta)}{\delta + r \sin\theta} \right] + D_\theta \left[\frac{\overline{\phi}(r,\theta) f_{j,3}(r,\varphi,\theta)}{\delta + r \sin\theta} \right] \right\} \mathrm{d}\theta.
\end{aligned}
\tag{4.11}
$$

We now prove $\Phi_1 \in \mathcal{L}(W^{1,p}(\Omega); L^p(-\frac{\pi}{2}, \frac{3\pi}{2}))$. Using assumption (2.4) for the $a_{i,j}$'s, it is easily shown that functions $f_{i,j}$, k_j and l_m, $i,j = 1,2,3$, $m = 2,4$, defined by (3.4), (3.9), (3.11) and (4.10) they all belong to $W^{2,+\infty}(I(r,\varphi,\theta))$. Therefore, since $0 < \delta - R < \delta + r \sin\theta < \delta + R$, $(r,\theta) \in (0,R) \times (-\frac{\pi}{2}, \frac{3\pi}{2})$, from (4.11) it follows

$$
\begin{aligned}
|\Phi_1[w](\varphi)| \leq C_1 \Bigg\{ & \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |\widetilde{w}(R,\varphi,\theta)| \, \mathrm{d}\theta + \sum_{k=0}^{1} \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |D_r^k \widetilde{w}(r,\varphi,\theta)| \, \mathrm{d}\theta \\
& + \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\left| \frac{D_\varphi \widetilde{w}(r,\varphi,\theta)}{\delta + r \sin\theta} \right| + \left| \frac{D_\theta \widetilde{w}(r,\varphi,\theta)}{r} \right| \right] \mathrm{d}\theta \Bigg\},
\end{aligned}
\tag{4.12}
$$

the constant C_1 being positive and depending on δ, R, $\max_{i,j=1,2,3} \|a_{i,j}\|_{W^{2,+\infty}(\Omega)}$ and $\|\phi\|_{C^1([\delta-R,\delta+R])}$, only. Now, from the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C^{1-3/p}(\overline{\Omega})$, $p \in (3,+\infty)$, the trace of a function $w \in W^{1,p}(\Omega)$ is well defined. Consequently, $\widetilde{w}$ being related to w via (2.14), for any $\varphi \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ we have

$$
\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |\widetilde{w}(R,\varphi,\theta)| \, \mathrm{d}\theta \leq 2\pi \|w\|_{C^{1-3/p}(\partial\Omega)} \leq C_2 \|w\|_{W^{1,p}(\Omega)},
\tag{4.13}
$$

where the positive constant C_2 is independent of w. In addition, denoting by p' the conjugate exponent of p, for any $v \in L^p(I(r, \varphi, \theta))$ we have

$$
\int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} |v(r, \varphi, \theta)| \,\mathrm{d}\theta
$$
$$
\leq \left[2\pi \int_0^R r^{-\frac{p'}{p}} \,\mathrm{d}r \right]^{\frac{1}{p'}} \tag{4.14}
$$
$$
\times \left[\frac{1}{(\delta - R)} \int_0^R r \,\mathrm{d}r \int_{-\frac{\pi}{2}}^{3\frac{\pi}{2}} (\delta + r \sin \theta) |v(r, \varphi, \theta)|^p \,\mathrm{d}\theta \right]^{\frac{1}{p}},
$$

and the right-hand side of (4.14) is in $L^p(-\frac{\pi}{2}, \frac{3\pi}{2})$ when $p \in (3, +\infty)$. Formula (4.2) imply that $D_r \widetilde{w}$, $(D_\varphi \widetilde{w})/(\delta + r \sin \theta)$ and $(D_\theta \widetilde{w})/r$ belong to $L^p(I(r, \varphi, \theta))$ when $w \in W^{1,p}(\Omega)$. Hence, recalling (2.10), from (4.12)–(4.14) we easily obtain

$$
\|\Phi_1[w]\|_{L^p(-\frac{\pi}{2},3\frac{\pi}{2})} \leq C_3 \left[\|w\|_{W^{1,p}(\Omega)} + \|w\|_{L^p(\Omega)} + \sum_{j=1}^3 \|D_{x_j} w\|_{L^p(\Omega)} \right] \tag{4.15}
$$
$$
\leq 2C_3 \|w\|_{W^{1,p}(\Omega)},
$$

and this completes the proof. ∎

Lemma 4.2. *Let the triplet $(\mathcal{A}, \Phi, \Psi)$ be admissible according to Definition 2.3. Then, if $p \in (3, +\infty)$, the linear operator Ψ satisfies decomposition (3.21).*

Proof. We only sketch the proof since the procedure is very close to that of Lemma 4.1. First, taking (3.15) into account, let us apply Ψ to $\mathcal{A}w$, where $w \in W_C^{2,p}(\Omega)$. Then, performing integration by parts similar to those in Lemma 4.1 it is easy to check
$$
\Psi[\mathcal{A}w] = \Psi_1[w] \quad \forall w \in W_C^{2,p}(\Omega),
$$
where, recalling $\tau_{3,2}(\varphi) = 0$ and denoting by $\widetilde{F}_j(r, \varphi, \theta)$, $j = 1, 2, 3$, the vector function $F_j(r, \varphi, \theta)$ defined by (4.5), but with zero second component, Ψ_1 is defined by $\Psi_1[w] = I_1 + I_2 + I_3 + I_4$ with

$$
I_1 := -\int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \langle F(r, \varphi, \theta), \nabla_{(r,\varphi,\theta)} \widetilde{w}(r, \varphi, \theta) \rangle
$$
$$
\times D_r[r(\delta + r \sin \theta) \widetilde{\psi}(r, \varphi, \theta)] \,\mathrm{d}\varphi
$$
$$
I_2 := -\sum_{j=1}^2 \int_0^R r \,\mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \langle \widetilde{F}_j(r, \varphi, \theta), \nabla_{(r,\varphi,\theta)} \widetilde{w}(r, \varphi, \theta) \rangle
$$
$$
\times D_\varphi[\tau_{j,2}(\varphi) \widetilde{\psi}(r, \varphi, \theta)] \,\mathrm{d}\varphi
$$

$$I_3 := -\sum_{j=1}^{3} \int_0^R \mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \langle F_j(r,\varphi,\theta), \nabla_{(r,\varphi,\theta)} \widetilde{w}(r,\varphi,\theta) \rangle$$

$$\times D_\theta[(\delta + r\sin\theta)\tau_{j,3}(\varphi,\theta)\widetilde{\psi}(r,\varphi,\theta)]\,\mathrm{d}\varphi$$

$$I_4 := -\int_0^R r\,\mathrm{d}r \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (\delta + r\sin\theta)\,\mathrm{d}\theta \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} D_\varphi \widetilde{w}(r,\varphi,\theta)$$

$$\times \left\{ \lambda(\tan\varphi)D_\varphi \widetilde{\psi}(r,\varphi,\theta) - \frac{\widetilde{\psi}(r,\varphi,\theta)l_2(r,\varphi,\theta)}{(\delta + r\sin\theta)^2} \right\} \mathrm{d}\varphi\,.$$

As said before, due to (2.4), all the functions $f_{i,j}$, k_j, $i,j = 1,2,3$, and l_2 belong to $W^{2,+\infty}(I(r,\varphi,\theta))$. Moreover, due to (2.4) and (2.5) (cf. also (2.11)), the function $\lambda \circ \tan$ belongs to $W^{2,+\infty}(-\frac{\pi}{2}, \frac{3\pi}{2})$. Therefore, since $\psi \in C^1(\overline{\Omega})$, using formula (4.2) and an estimate similar to (4.15) we easily deduce $\Psi_1 \in W^{1,p}(\Omega)^*$. This completes the proof. ∎

Corollary 4.3. *Let the triplet* $(\mathcal{A}, \Phi, \Psi)$ *be admissible according to Definition* 2.3. *Then, if* $p \in (3, +\infty)$, *the linear operators* Φ *and* Ψ *satisfy assumptions* (3.17)–(3.21).

Proof. From definitions (2.22) and Hölder's inequality it immediately follows

$$|\Psi[v]| \leq \|\psi\|_{C(\overline{\Omega})}\|v\|_{L^1(\Omega)} \leq C_4\|\psi\|_{C(\overline{\Omega})}\|v\|_{L^p(\Omega)} \quad \forall v \in L^p(\Omega)\,,$$

where the positive constant C_4 is independent of v. Hence, $\Psi \in L^p(\Omega)^*$. Concerning operator Φ defined by (2.21), instead, using an estimate similar to (4.14), it can be easily checked that Φ belongs to $\mathcal{L}(L^p(\Omega); L^p(-\frac{\pi}{2}, \frac{3\pi}{2}))$. Therefore (3.17) is satisfied. Since definition (2.21) easily implies (3.18) and (3.19) and since decompositions (3.20) and (3.21) are established, respectively, by Lemma 4.1 and Lemma 4.2 the proof is complete. ∎

Remark 4.4. From the first integral on the right-hand side of (4.9) and the fact that the trace of the derivatives of $w \in W^{1,p}(\Omega)$ are not well-defined, we see that without the boundary assumptions (2.12) and (2.13) no estimate of type (4.15) is allowed, unless of requiring the tangential derivative of w to be zero on $\partial\Omega$, too. Moreover, contrarily to [3] and [4] where also the Dirichlet case was treated, here we have limited ourselves only to the conormal boundary condition. The reasons for this choice lie in formulae (4.6), (4.7) and (4.9). Indeed, from such formulae we deduce that in order to obtain estimate (4.15) in the Dirichlet case, we are however forced to require the conormal derivative to be zero on $\partial\Omega$ and, in addition, either assumptions (2.12) and (2.13) to be satisfied or the tangential derivative to be zero on $\partial\Omega$, too.

5. The equivalence result and the basic system

Similarly to what is done in [3], we reduce our problem P(C) to an abstract fixed-point system having the same functional form of that detailly studied in [2]. First, let us suppose that $(u, k) \in \mathcal{U}^{2,\beta,p}(T) \times C^\beta([0, T]; W^{1,p}(-\frac{\pi}{2}, \frac{3\pi}{2}))$ is a solution to problem P(C), and introduce the new triplet of unknown functions (v, h, q) defined by

$$v = D_t u - D_t u, \quad h(t) = k\left(t, -\tfrac{\pi}{2}\right), \quad q(t, \varphi) = D_\varphi k(t, \varphi), \tag{5.1}$$

and related to u and k via the following formulae

$$u(t, x) = u_1(t, x) - u_1(0, x) + u_0(x) + \int_0^t v(s, x)\, \mathrm{d}s \tag{5.2}$$

$$k(t, \varphi) = h(t) + \int_{-\frac{\pi}{2}}^{\varphi} q(t, \xi)\, \mathrm{d}\xi =: h(t) + Eq(t, \varphi). \tag{5.3}$$

Now, for any $(t, \varphi, x) \in [0, T] \times \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \times \Omega$ we set

$$
\begin{aligned}
N_1(v, h, q)&(t, \rho(x')) \\
&= -\Big(\big[h(\cdot) + Eq(\cdot, \rho(x'))\big] * \big[\mathcal{B}v(\cdot, x) + \mathcal{B}D_t u_1(\cdot, x)\big]\Big)(t) \\
&\quad - \Big(q(\cdot, \rho(x')) * \big[\mathcal{C}v(\cdot, x) + \mathcal{C}D_t u_1(\cdot, x)\big]\Big)(t)
\end{aligned}
\tag{5.4}
$$

and

$$z(t, x) = D_t f(t, x) - (D_t - \mathcal{A})D_t u_1(t, x), \tag{5.5}$$

$$
\begin{aligned}
N_1^0(u_1, g_1, f)(t, \varphi) &= (D_t - \mathcal{A}_1)D_t g_1(t, \varphi) \\
&\quad - \Phi_1[D_t u_1(t, \cdot)](\varphi) - \Phi[D_t f(t, \cdot)](\varphi),
\end{aligned}
\tag{5.6}
$$

$$N_2^0(u_1, g_2, f)(t) = D_t^2 g_2(t) - \Psi_1[D_t u_1(t, \cdot)] - \Psi[D_t f(t, \cdot)]. \tag{5.7}$$

Hence, differentiating (2.1) with respect to t and using (2.15)–(2.18) and (5.4)–(5.7), we deduce that the triplet (v, h, q) solves the following identification problem, for any $(t, \varphi, x) \in [0, T] \times (-\frac{\pi}{2}, \frac{3\pi}{2}) \times \Omega$:

$$
\begin{aligned}
D_t v(t, x) &= \mathcal{A}v(t, x) - N_1(v, h, q)(t, \rho(x')) + q(t, \rho(x'))\mathcal{C}u_0(t, x) \\
&\quad + [h(t) + Eq(t, \rho(x'))]\mathcal{B}u_0(t, x) + z(t, x)
\end{aligned}
\tag{5.8}
$$

$$v(0, x) = v_0(x) \tag{5.9}$$

v satisfies the homogeneous conormal boundary condition $\qquad$ (5.10)

$$\Phi[v(t,\cdot)](\varphi) = D_t g_1(t,\varphi) - \Phi[D_t u_1(t,\cdot)](\varphi) \tag{5.11}$$

$$\Psi[v(t,\cdot)] = D_t g_2(t) - \Psi[D_t u_1(t,\cdot)] \tag{5.12}$$

$$\begin{aligned} q(t,\varphi)\Phi[\mathcal{C}u_0](\varphi) &+ Eq(t,\varphi)\Phi[\mathcal{B}u_0](\varphi) \\ &= N_1^0(u_1, g_1, f)(t,\varphi) - \Phi_1[v(t,\cdot)](\varphi) \\ &+ \Phi[N_1(v,h,q)(t,\cdot)](\varphi) - h(t)\Phi[\mathcal{B}u_0](\varphi) \end{aligned} \tag{5.13}$$

$$\begin{aligned} \Psi[q(t,\cdot)\mathcal{C}u_0 + Eq(t,\cdot)]\mathcal{B}u_0] &= N_2^0(u_1, g_2, f)(t) - \Psi_1[v(t,\cdot)] \\ &+ \Psi[N_1(v,h,q)(t,\cdot)] - h(t)\Psi[\mathcal{B}u_0]. \end{aligned} \tag{5.14}$$

Here the latter two equations are easily obtained by applying Φ and Ψ to (5.8) and taking advantage from (3.18) and decomposition (3.20), (3.21).

Remark 5.1. Assume now that the triplet (v,h,q) belongs to $\mathcal{U}_{\mathrm{C}}^{1,\beta,p}(T) \times C^\beta([0,T];\mathbb{R}) \times C^\beta([0,T];L^p(-\frac{\pi}{2},\frac{3\pi}{2}))$ and solves (5.8)–(5.14). Then, if we define u and k according to (5.2) and (5.3) it clearly follows that the pair (u,k) belongs to $\mathcal{U}^{2,\beta,p}(T) \times C^\beta([0,T];W^{1,p}(-\frac{\pi}{2},3\frac{\pi}{2}))$. In addition, if the function u_0 appearing in (5.2) satisfies the consistency condition (2.19), then, performing integrations with respect to time, we easily deduce that the pair (u,k) solves problem P(C). Hence, problem P(C) and problem (5.8)–(5.14) are *equivalent*.

Since definition (5.4) implies $N_1(v,h,q)(0,x) = 0$ for every $x \in \Omega$, from (5.13) and (5.14) we easily find the initial value $k(0,\cdot)$ of k. Indeed, letting

$$\widetilde{v}_1(\varphi) := N_1^0(u_1, g_1, f)(0,\varphi) - \Phi_1[v_0](\varphi) \quad \forall\, \varphi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

and recalling (5.3), from (5.13) and (5.14) we deduce, for any $\varphi \in (-\frac{\pi}{2}, \frac{3\pi}{2})$

$$D_\varphi k(0,\varphi)\Phi[\mathcal{C}u_0](\varphi) + k(0,\varphi)\Phi[\mathcal{B}u_0](\varphi) = \widetilde{v}_1(\varphi), \tag{5.15}$$

$$\Psi[D_\varphi k(0,\cdot)\mathcal{C}u_0 + k(0,\cdot)\mathcal{B}u_0] = N_2^0(u_1, g_2, f)(0) - \Psi_1[v_0]. \tag{5.16}$$

Integrating the first-order linear differential equation (5.15) we obtain the following general integral depending on an arbitrary constant C:

$$\begin{aligned} k(0,\varphi) &= C\exp\left[\int_\varphi^{-\frac{\pi}{2}} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)}\,\mathrm{d}\xi\right] \\ &+ \int_{-\frac{\pi}{2}}^{\varphi} \exp\left[\int_\varphi^{\sigma} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)}\,\mathrm{d}\xi\right]\frac{\widetilde{v}_1(\sigma)}{\Phi[\mathcal{C}u_0](\sigma)}\,\mathrm{d}\sigma := k_0(\varphi). \end{aligned} \tag{5.17}$$

Then, substituting this representation of $k(0,\cdot)$ into (5.16), we find that the constant C is equal to $[J_1(u_0)]^{-1}\{N_2^0(u_1, g_2, f)(0) - \Psi[\widetilde{v}_2] - \Psi_1[v_0]\}$ where $J_1(u_0)$ and $\widetilde{v}_2$ are defined, respectively, by (3.31) and formula (3.17) in [3] (with the vector $(\widetilde{l}_1, \widetilde{l}_2, |x|, R_2)$ being replaced by $(\widetilde{v}_1, \widetilde{v}_2, \rho(x'), -\frac{\pi}{2})$).

Of course, the previous argument works only for $t = 0$ since when $t = 0$, the data (u_0, u_1, g_1, g_2, f) occur, only. Therefore, due to (5.3), to determine $k(t, \cdot)$ for any $t \in [0, T]$ we have to solve system (5.13) and (5.14) for h and q. To this purpose, for any $(t, \varphi) \in [0, T] \times (-\frac{\pi}{2}, \frac{3\pi}{2})$, we first consider the following integral equation for q, where $g \in L^1((0, T) \times (-\frac{\pi}{2}, \frac{3\pi}{2}))$ is an arbitrary given function:

$$q(t, \varphi)\Phi[\mathcal{C}u_0](\varphi) + Eq(t, \varphi)\Phi[\mathcal{B}u_0](\varphi) = g(t, \varphi). \tag{5.18}$$

Since (5.3) implies $D_\varphi Eq(t, \varphi) = q(t, \varphi)$ the solution to the differential equation (5.18) for $Eq(t, \cdot)$ satisfying $Eq(t, -\frac{\pi}{2}) = 0$ is given by

$$Eq(t, \varphi) = \int_{-\frac{\pi}{2}}^{\varphi} \exp\left[\int_\varphi^\sigma \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} \,d\xi\right] \frac{g(t, \sigma)}{\Phi[\mathcal{C}u_0](\sigma)} \,d\sigma := Lg(t, \varphi). \tag{5.19}$$

Therefore, differentiating (5.19) with respect to φ we obtain the following representation formula for q:

$$q(t, \varphi) = \frac{1}{\Phi[\mathcal{C}u_0](\varphi)}\left[I - \Phi[\mathcal{B}u_0](\varphi)L\right]g(t, \varphi). \tag{5.20}$$

Replacing g with the right-hand side of (5.13) and denoting by $J_3(u_0)$ the operator

$$J_3(u_0)\widetilde{g}(\varphi) := \frac{1}{\Phi[\mathcal{C}u_0](\varphi)}\left[I - \Phi[\mathcal{B}u_0](\varphi)L\right]\widetilde{g}(\varphi), \quad \widetilde{g} \in L^1(-\frac{\pi}{2}, \frac{3\pi}{2}),$$

from (5.20) we easily find, for any $(t, \varphi) \in [0, T] \times (-\frac{\pi}{2}, \frac{3\pi}{2})$:

$$q(t, \varphi) = h(t)\frac{\Phi[\mathcal{B}u_0](\varphi)}{\Phi[\mathcal{C}u_0](\varphi)}\left[L\Phi[\mathcal{B}u_0](\varphi) - 1\right] + N_2(v, h, q)(t, \varphi)$$
$$+ N_3^0(u_0, u_1, g_1, f)(t, \varphi), \tag{5.21}$$

where we have set

$$N_2(v, h, q)(t, \varphi) = J_3(u_0)\{\Phi[N_1(v, h, q)(t, \cdot)](\varphi) - \Phi_1[v(t, \cdot)](\varphi)\} \tag{5.22}$$

$$N_3^0(u_0, u_1, g_1, f)(t, \varphi) = J_3(u_0)N_1^0(u_1, g_1, f)(t, \varphi).$$

Now, from (5.19) we derive

$$L\Phi[\mathcal{B}u_0](\varphi) - 1 = -\exp\left[\int_\varphi^{-\frac{\pi}{2}} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)} \,d\xi\right], \tag{5.23}$$

and we substitute this expression into (5.21). Then from (5.14) it is easy to check that h solves the following equation, for any $t \in [0, T]$:

$$h(t)J_1(u_0) = N_0(u_0, u_1, g_1, g_2, f)(t) - \Psi_1[v(t, \cdot)] + \Psi[N_1(v, h, q)(t, \cdot)]$$
$$- \Psi[N_2(v, h, q)(t, \cdot)\mathcal{C}u_0] + \Psi[E(N_2(v, h, q)(t, \cdot))\mathcal{B}u_0], \tag{5.24}$$

where $J_1(u_0)$ and $N_0(u_0, u_1, g_1, g_2, f)$ are defined, respectively, by (3.31) and

$$N_0(u_0, u_1, g_1, g_2, f)(t) := N_2^0(u_1, g_2, f)(t) - \Psi[N_3^0(u_0, u_1, g_1, f)(t, \cdot)\mathcal{C}u_0]$$
$$+ \Psi[E(N_3^0(u_0, u_1, g_1, f)(t, \cdot))\mathcal{B}u_0].$$

Hence, from (5.24) we conclude that h solves the fixed-point equation

$$h(t) = h_0(t) + N_3(v, h, q)(t) \quad \forall\, t \in [0, T], \tag{5.25}$$

where we have set

$$h_0(t) = [J_1(u_0)]^{-1} N_0(u_0, u_1, g_1, g_2, f)(t), \tag{5.26}$$

$$N_3(v, h, q)(t) = [J_1(u_0)]^{-1} \Big\{ \Psi[N_1(v, h, q)(t, \cdot)] - \Psi[N_2(v, h, q)(t, \cdot)\mathcal{C}u_0]$$
$$+ \Psi[E(N_2(v, h, q)(t, \cdot))\mathcal{B}u_0] - \Psi_1[v(t, \cdot)] \Big\}. \tag{5.27}$$

So, using again (5.23) and replacing the right-hand side of (5.25) into (5.21), we conclude that q satisfies the fixed-point equation

$$q(t, \varphi) = q_0(t, \varphi) + J_2(u_0)(\varphi)N_3(v, h, q)(t) + N_2(v, h, q)(t, \varphi), \tag{5.28}$$

$J_2(u_0)$ and q_0 being defined, respectively, by

$$J_2(u_0)(\varphi) = -\frac{\Phi[\mathcal{B}u_0](\varphi)}{\Phi[\mathcal{C}u_0](\varphi)} \exp\left[\int_\varphi^{-\frac{\pi}{2}} \frac{\Phi[\mathcal{B}u_0](\xi)}{\Phi[\mathcal{C}u_0](\xi)}\, \mathrm{d}\xi \right], \tag{5.29}$$

$$q_0(t, \eta) = J_2(u_0)(\varphi)h_0(t) + N_3^0(u_0, u_1, g_1, f)(t, \varphi). \tag{5.30}$$

Thus, the pair (h, q) solves the fixed-point system (5.25) and (5.28) and the following equivalence theorem holds true.

Theorem 5.2. *The pair $(u, k) \in \mathcal{U}^{2,\beta,p}(T) \times C^\beta([0, T]; W^{1,p}(-\frac{\pi}{2}, \frac{3\pi}{2}))$ solves problem P(C) if and only if the triplet (v, h, q) defined by (5.1) belongs to $\mathcal{U}_{\mathrm{C}}^{1,\beta,p}(T) \times C^\beta([0, T]; \mathbb{R}) \times C^\beta([0, T]; L^p(-\frac{\pi}{2}, \frac{3\pi}{2}))$ and solves problem (5.8)–(5.12), (5.25), (5.28).*

6. The auxiliary abstract result

Taking advantage of the equivalence Theorem 5.2, here we sketch out how to reformulate problem P(C) in an abstract space framework. Then, for the reformulated problem we recall the main local existence and uniqueness result stated in [2] and to which we refer for the proof.

Let X be a complex Banach space with norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator, with a non–necessarily dense domain, satisfying the following assumption:

(H1) the resolvent $\rho(A)$ of A contains the half-plane $S_0 = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq 0\}$ and there exists $M_0 > 0$ such that $\|(zI - A)^{-1}\|_{\mathcal{L}(X)} \leq M_0|1 + z|^{-1}$ for every $z \in S_0$.

Since for any $z \in S_0$ we have $|z| < |1 + z|$, due to Proposition 2.1.11 in [8] assumption (H1) guarantees that A is sectorial, the constant ω, θ and M of Definition 2.0.1 in [8] being replaced, respectively, by 0, $\zeta = \pi - \arctan(2M_0)$ and $M_0' = 2M_0[1 + 1/(4M_0)^2]^{\frac{1}{2}}$. As a consequence, A generates an analytic semigroup $\{e^{tA}\}_{t \geq 0}$ of linear bounded operators from X to itself. Moreover, the non emptiness of $\rho(A)$ implies that A is a closed operator, so that $\mathcal{D}(A)$, endowed with the graph norm $\|x\|_{\mathcal{D}(A)} = \|x\|_X + \|Ax\|_X$, turns out to be a Banach space. Hence, as usual, we can define the family of spaces $\mathcal{D}_A(\gamma, q)$, $0 < \gamma < 1$, $1 \leq q \leq +\infty$, which are intermediate between $\mathcal{D}(A)$ and X (cf. [8, Subsection 2.2.1]). Actually, from the equivalence $\mathcal{D}_A(\gamma, q) = (X, \mathcal{D}(A))_{\gamma, q}$ (cf. [8, Proposition 2.2.2]) the spaces $\mathcal{D}_A(\gamma, q)$ turn out to be interpolation spaces between X and $\mathcal{D}(A)$. This implies (cf. [8, Corollary 1.2.7]) that there exists a constant $c(\gamma, q)$ such that

$$\|x\|_{\mathcal{D}_A(\gamma, q)} \leq c(\gamma, q)\|x\|_X^{1-\gamma}\|x\|_{\mathcal{D}(A)}^{\gamma} \quad \forall\, y \in \mathcal{D}(A). \tag{6.1}$$

Therefore, the sectorialness of A and (6.1) imply

$$\|(zI - A)^{-1}\|_{\mathcal{L}(X; \mathcal{D}_A(\frac{1}{2}, 1))} \leq M_1|z|^{-\frac{1}{2}} \quad \forall\, z \in \Sigma_\zeta, \tag{6.2}$$

where Σ_ζ denotes the open sector $\{\mu \in \mathbb{C} : |\arg \mu| < \zeta\} \cup \{0\}$. To see that (6.2) holds true, observe first that (H1) with $z = 0$ implies that A is an isomorphism from $\mathcal{D}(A)$ onto X. Consequently, for $y \in \mathcal{D}(A)$, we find

$$\|y\|_{\mathcal{D}(A)} = \|A^{-1}(Ay)\|_X + \|Ay\|_X \leq (M_0 + 1)\|Ay\|_X. \tag{6.3}$$

Now, for every $x \in X$ and $z \in \Sigma_\zeta$, from the sectorialness of A we get

$$\|(zI - A)^{-1}x\|_X \leq M_0'|z|^{-1}\|x\|_X, \tag{6.4}$$

whereas, using (6.3) with $y = (zI - A)^{-1}x$ and taking advantage of $A(zI - A)^{-1} = z(zI - A)^{-1} - I$ and sectorialness again, we easily obtain

$$\|(zI - A)^{-1}x\|_{\mathcal{D}(A)} \leq (M_0 + 1)\|A(zI - A)^{-1}x\|_X \leq M_0''\|x\|_X, \tag{6.5}$$

with $M_0'' = (M_0 + 1)(M_0' + 1)$. Replacing x with $(zI - A)^{-1}x$ in (6.1) and using (6.4) and (6.5), we deduce (6.2) with $M_1 = c(\frac{1}{2}, 1)(M_0 M_0'')^{\frac{1}{2}}$. Observe also that by the previous arguments, (H1) implies (H1)–(H3) in [3] and [4].

In order to reformulate in an abstract form problem (5.8)–(5.12), (5.25) and (5.28) we need the following list of assumptions involving spaces, operators and data, where $0 < \beta < \alpha < \frac{1}{2}$ and $\widetilde{q}_0$ is defined in the next Remark 6.2:

(H2) Y, Y_1, $\mathcal{D}(B)$, $\mathcal{D}(C)$ are Banach spaces such that $Y_1 \hookrightarrow Y$ and $\mathcal{D}(A) \hookrightarrow \mathcal{D}(B) \hookrightarrow \mathcal{D}(C) \hookrightarrow X$, $\mathcal{D}_A(\frac{1}{2}, 1) \hookrightarrow \mathcal{D}(C)$;

(H3) $B : \mathcal{D}(B) \to X$ and $C : \mathcal{D}(C) \to X$ are linear operators such that $BA^{-1} \in \mathcal{L}(X)$ and $CA^{-1} \in \mathcal{L}(X; \mathcal{D}(C))$;

(H4) $E \in \mathcal{L}(Y; Y_1)$, $\Phi \in \mathcal{L}(X; Y)$, $\Phi_1 \in \mathcal{L}(\mathcal{D}(C); Y)$, $\Psi \in X^*$, $\Psi_1 \in \mathcal{D}(C)^*$;

(H5) $\mathcal{M} \in \mathcal{B}(Y \times \mathcal{D}(C); X) \cap \mathcal{B}(Y_1 \times X; X)$;

(H6) $J_1 : \mathcal{D}(B) \to \mathbb{R}$, $J_2 : \mathcal{D}(B) \to Y$, $J_3 : \mathcal{D}(B) \to \mathcal{L}(Y)$, are three prescribed (nonlinear) operators;

(H7) $u_0 \in \mathcal{D}(B)$, $v_0 \in \mathcal{D}(A)$, $Cu_0 \in \mathcal{D}(C)$, $J_1(u_0) \neq 0$, $Bu_0 \in \mathcal{D}_A(\alpha, +\infty)$;

(H8) $h_0 \in C^\beta([0, T]; \mathbb{R})$, $q_0 \in C^\beta([0, T]; Y)$;

(H9) $z_0 \in C^\beta([0, T]; X)$; $z_1 \in C^\beta([0, T]; \mathcal{D}(C))$, $z_2 \in C^\beta([0, T]; X)$;

(H10) $Av_0 + \mathcal{M}(\widetilde{q}_0, Cu_0) - \mathcal{M}(E\widetilde{q}_0, Bu_0) + z_2(0, \cdot) \in \mathcal{D}_A(\beta, +\infty)$.

Now, denoting by $\mathcal{K}$ the convolution $\mathcal{K}(\chi, \kappa)(t) := \int_0^t \mathcal{M}\big(\chi(t - s), \kappa(s)\big)\, \mathrm{d}s$ our direct problem depending on the pair (h, q) is the following: to determine a function $v \in C^1([0, T]; X) \cap C([0, T]; \mathcal{D}(A))$ such that

$$
\begin{cases}
v'(t) = [\lambda_0 I + A]v(t) + [h * (Bv + z_0)](t) + \mathcal{K}(Eq, Bv + z_0)(t) \\
\qquad + \mathcal{K}(q, Cv + z_1)(t) + \mathcal{M}(q(t), Cu_0) \\
\qquad + h(t)Bu_0 + \mathcal{M}(Eq(t), Bu_0) + z_2(t) \quad \forall t \in [0, T], \\
v(0) = v_0,
\end{cases}
\tag{6.6}
$$

Remark 6.1. In the explicit case (5.8), we have $A = \mathcal{A} - \lambda_0 I$, with a large enough positive λ_0, and $z_0 = \mathcal{B}D_t u_1$, $z_1 = \mathcal{C}D_t u_1$, $z_2 = D_t f - (D_t - \mathcal{A})D_t u_1$. Functions v_0, h_0 and q_0 appearing in (H7) and (H8) are defined, respectively, by (3.26), (5.26) and (5.30), whereas the bilinear operator $\mathcal{M}$ is the multiplication operator $\mathcal{M}(\omega_1, \omega_2) = \omega_1 \omega_2$.

To rewrite the fixed-point system (5.25), (5.28) in an abstract form we need first to introduce the following operators:

$$
\widetilde{R}_1(v, h, q) = -\widetilde{J}_1(u_0)\Big\{ \Psi\big[\mathcal{M}(J_3(u_0)\Phi[N_1(v, h, q)], Cu_0) - N_1(v, h, q)\big]
$$
$$
\qquad\qquad - \Psi\big[\mathcal{M}(EJ_3(u_0)\Phi[N_1(v, h, q)], Bu_0)\big] \Big\}
$$
$$
\widetilde{R}_2(v, h, q) = J_2(u_0)\widetilde{R}_1(v, h, q) + J_3(u_0)\Phi[N_1(v, h, q)]
$$

and

$$
\widetilde{S}_1(v) = \widetilde{J}_1(u_0)\Big\{ \Psi\big[\mathcal{M}(J_3(u_0)\Phi_1[v], Cu_0) - \mathcal{M}(EJ_3(u_0)\Phi_1[v], Bu_0)\big] - \Psi_1[v] \Big\}
$$
$$
\widetilde{S}_2(v) = J_2(u_0)\widetilde{S}_1(v) - J_3(u_0)\Phi_1[v],
$$

where (cf. (H7) and (5.4)) we have set $\widetilde{J}_1(u_0) = [J_1(u_0)]^{-1}$ and

$$N_1(v, h, q) = -h * (Bv + z_0) - \mathcal{K}(Eq, Bv + z_0) - \mathcal{K}(q, Cv + z_1).\qquad(6.7)$$

Then, the fixed-point system for h and q can be rewritten more compactly:

$$h = h_0 + \widetilde{R}_1(v, h, q) + \widetilde{S}_1(v), \qquad q = q_0 + \widetilde{R}_2(v, h, q) + \widetilde{S}_2(v).\qquad(6.8)$$

h_0 and q_0 being the elements appearing in (H8).

Remark 6.2. Since $N_1(v, h, q)(0) = 0$, from (6.7) we can easily compute the initial values $\widetilde{h}_0$ and $\widetilde{q}_0$ of functions h and q:

$$\widetilde{h}_0 = h_0(0) + \widetilde{S}_2(v_0) = h(0), \qquad \widetilde{q}_0 = q_0(0) + \widetilde{S}_3(v_0) = q(0).$$

In particular, in the explicit case we get $\widetilde{h}_0 = k_0(\eta_0)$, $\widetilde{q}_0(\eta) = D_\eta k_0(\eta)$, the function k_0 being defined by (5.17).

We can now recall the following theorem (cf. [2]).

Theorem 6.3. *Under assumptions* (H1)–(H10) *there exists* $T^* \in (0, T)$ *such that for any* $\tau \in (0, T^*]$ *the problem* (6.6), (6.8) *admits a unique solution* $(v, h, q) \in [C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(A))] \times C^\beta([0, \tau]; \mathbb{R}) \times C^\beta([0, \tau]; Y)$.

7. Proof of Theorems 3.2 and 3.3

To prove Theorem 3.2 we take advantage of the equivalence results of Section 5 and of the abstract ones of Section 6. Indeed, by virtue of Theorems 5.2 and 6.3, all we need to show is that the abstract assumptions (H1)–(H10) are satisfied when the Banach space framework of Section 6 is that related to the L^p and Sobolev spaces of Section 3. For saving space, here we limit ourselves only into sketching the basic ideas, since the proof of (H1)–(H10) is really close to that in [3]. Instead, we want to focus our attention on the difficulties arising in checking the periodic condition (3.35) and on the reasons for why such condition makes the "if" part of Theorem 5.2 fail. Due to this failure, our solvability procedure enables us to prove only the uniqueness of a periodic solution, assuming its existence.

Proof of Theorem 3.2. First, for any $p \in (3, +\infty)$, we choose the Banach spaces X, $\mathcal{D}(A)$, $\mathcal{D}_A(\frac{1}{2}, 1)$, $\mathcal{D}(B)$, $\mathcal{D}(C)$, Y and Y_1 according to the rule

$$\begin{cases} X = L^p(\Omega), & \mathcal{D}(A) = W_C^{2,p}(\Omega), & \mathcal{D}_A(\frac{1}{2}, 1) = B_C^{1,p,1}(\Omega), \\ \mathcal{D}(B) = W^{2,p}(\Omega), & \mathcal{D}(C) = W^{1,p}(\Omega), \\ Y = L^p(-\frac{\pi}{2}, \frac{3\pi}{2}), & Y_1 = W^{1,p}(-\frac{\pi}{2}, \frac{3\pi}{2}), \end{cases}\qquad(7.1)$$

where L^p and $W^{1,p}$ in the definition of Y and Y_1 are replaced, respectively, by $\widetilde{L}^p$ and $\widetilde{W}^{1,p}$ (cf. (3.36) and (3.37)) in the periodic case. Observe that, due to definition of spaces $B_C^{2\gamma,p_1,p_2}(\Omega)$ given after formula (3.29), the choice of $\mathcal{D}_A(\frac{1}{2},1)$ makes sense, since $\mathcal{D}_A(\frac{1}{2},1) = (X,\mathcal{D}(A))_{\frac{1}{2},1}$. Moreover, $B_C^{1,p,1}(\Omega)$ being a subset of the space $B^{1,p,1}(\Omega)$ defined in [10, Subsection 4.3.1], the choice (7.1) ensures that assumption (H2) is satisfied. Indeed, from $p \in (3,+\infty)$ and Theorem 4.6.1(a), (b) in [10] we have $B^{1,p,1}(\Omega) \hookrightarrow B^{1,p,2}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ and hence $\mathcal{D}_A(\frac{1}{2},1) \hookrightarrow \mathcal{D}(C)$, as required in (H2). The other inclusions in (H2) are trivial. Then, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ being defined by (2.3) and λ_0 being a large enough (fixed) positive constant, we define the operators A, B and C with domains $\mathcal{D}(A)$, $\mathcal{D}(B)$ and $\mathcal{D}(C)$ as follows:

$$Au = (\mathcal{A} - \lambda_0 I)u, \ u \in \mathcal{D}(A); \quad Bu = \mathcal{B}u, \ u \in \mathcal{D}(B); \quad Cu = \mathcal{C}u, \ u \in \mathcal{D}(C).$$

Now, the proof of (H1) is the same as that of (H1) in [3]. Due to definition (2.3) and assumption (2.4) we can easily show the estimates

$$\|BA^{-1}g\|_{L^p(\Omega)} \leq 2^{5\frac{p-1}{p}} \max_{i,j=1,2,3} \|b_{i,j}\|_{W^{1,\infty}(\Omega)} \|A^{-1}g\|_{W^{2,p}(\Omega)}, \qquad (7.2)$$

$$\|CA^{-1}g\|_{W^{1,p}(\Omega)} \leq 2^{3\frac{p-1}{p}} \max_{i=1,2,3} \|c_i\|_{W^{1,\infty}(\Omega)} \|A^{-1}g\|_{W^{2,p}(\Omega)}. \qquad (7.3)$$

In addition, endowing $\mathcal{D}(A)$ with the graph-norm, Theorem 3.1.1 in [8] and inequality (5.8) in [3] imply, for every $g \in L^p(\Omega)$

$$\|A^{-1}g\|_{W^{2,p}(\Omega)} \leq c_p \|A^{-1}g\|_{\mathcal{D}(A)} \leq c_p(M+1)\|g\|_{L^p(\Omega)}, \qquad (7.4)$$

where c_p and M are positive constants depending, respectively, on p and the uniform ellipticity constants of $\mathcal{A}$. Combining (7.2)–(7.4) and recalling (7.1) we get $BA^{-1} \in \mathcal{L}(X)$ and $CA^{-1} \in \mathcal{L}(X;\mathcal{D}(C))$. Hence (H3) is satisfied, too. Now, the proof of (H4)–(H10) follows exactly that of (H7)–(H13) in [3], but with r, R_1 and R_2 being replaced, respectively, by φ, $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$. ∎

Let us now turn to the periodic case. First, we briefly show where condition (3.35) makes the equivalence result of Section 5 fail. Recalling (5.1) and (5.3) the 2π-periodicity of $k(t,\cdot)$ must be reinterpreted as the requirement that q satisfies the additional property

$$Eq(t,\tfrac{3\pi}{2}) = 0 \quad \forall\, t \in [0,T]. \qquad (7.5)$$

Now, if the triplet (v,h,q), v being defined in (5.1), solves problem (5.8)–(5.14) then q satisfies the fixed-point equation (5.28). Hence, for (7.5) to be satisfied

we must have

$$I_1(t) := \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} q_0(t,\xi)\, \mathrm{d}\xi = 0 \qquad\qquad \forall\, t \in [0,T] \qquad\qquad (7.6)$$

$$I_2(t) := \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} N_2(v,h,q)(t,\xi)\, \mathrm{d}\xi = 0 \qquad\qquad \forall\, t \in [0,T] \qquad\qquad (7.7)$$

$$I_3(t) := N_3(v,h,q)(t) \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} J_2(u_0)(\xi)\, \mathrm{d}\xi = 0 \quad \forall\, t \in [0,T], \qquad\qquad (7.8)$$

where $J_2(u_0)$, q_0, $N_2(v,h,q)$ and $N_3(v,h,q)$ are defined, respectively, by (5.29), (5.30), (5.22) and (5.27). Now, since q_0 and $J_2(u_0)$ depend on the data, (7.6) and (7.8) can be easily satisfied, only by forcing the assumptions on the vector (u_0, u_1, g_1, g_2, f). In particular, due to definition of $J_2(u_0)$, (7.8) reduces to

$$1 = \exp\left[\int_{\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{\Phi[\mathcal{B}u_0](\sigma)}{\Phi[\mathcal{C}u_0](\sigma)}\, \mathrm{d}\sigma \right],$$

i.e., to the non-restrictive requirement

$$\int_{\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{\Phi[\mathcal{B}u_0](\sigma)}{\Phi[\mathcal{C}u_0](\sigma)}\, \mathrm{d}\sigma = 0.$$

More difficult , instead, is to satisfy (7.7). Indeed, from definition (5.22) we easily deduce that, for every $t \in [0,T]$, (7.7) is equivalent to

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\Phi[N_1(v,h,q)(t,\cdot)](\xi) - \Phi_1[v(t,\cdot)](\xi)}{\Phi[\mathcal{C}u_0](\xi)} \exp\left[\int_{\frac{3\pi}{2}}^{\xi} \frac{\Phi[\mathcal{B}u_0](\sigma)}{\Phi[\mathcal{C}u_0](\sigma)}\, \mathrm{d}\sigma \right] \mathrm{d}\xi = 0. \quad (7.9)$$

Unfortunately, $N_1(v,h,q)$ (cf. (5.4)) contains v, h and q in so an involved way, that there is a very little hope in (7.9) to hold true. This is the reason for why, in general, problem P(N) with the additional condition (3.35) and problem (5.8)–(5.14) are *not equivalent*.

Proof of Theorem 3.3. From Theorem 6.3 it follows that problem (5.8)–(5.12), (5.25) and (5.28) admits a unique solution (v,h,q) which must be that defined through (5.1). Therefore, the assert follows by a contradiction argument if we show that the existence of two different solutions (u_j, k_j), $j = 1, 2$, to problem P(N) with condition (3.35) implies the existence of two different solutions (v_j, h_j, q_j), $j = 1, 2$, to problem (5.8)–(5.12), (5.25) and (5.28). So, first, let $u_1 \neq u_2$ but $v_1 = v_2$. Then u_1 and u_2 differ one to each other for a function c_1 depending on the spatial variable x, only. Due to the initial condition $u_1(0,x) = u_2(0,x) = u_0(x)$, such function c_1 turns out to be identically zero, i.e.,

$u_1 = u_2$, and we get a contradiction. Similarly, if $k_1 \neq k_2$ but $(h_1, q_1) = (h_2, q_2)$, then k_1 and k_2 differ for a function c_2 depending on t, only. On the other hand, $h_1 = h_2$ implies $c_2 \equiv 0$. In fact, for any $t \in [0, T]$, we have

$$h_1(t) = k_1(t, -\tfrac{\pi}{2}) = k_2(t, -\tfrac{\pi}{2}) + c_2(t) = h_2(t) + c_2(t) = h_1(t) + c_2(t).$$

Then k_1 is equal to k_2, completing the contradiction. ∎

We conclude with some final remarks which justifies further the existence assumption of Theorem 3.3. Due to the solvability condition (3.30), equations (5.13) and (7.5) can be rewritten, for any $t \in [0, T]$, as the following first order scalar differential equation

$$\begin{cases} D_\varphi Eq(t, \varphi) = -F(t, \varphi, Eq(t, \varphi)), \\ Eq(t, -\tfrac{\pi}{2}) = Eq(t, \tfrac{3\pi}{2}) = 0, \end{cases} \tag{7.10}$$

where we have set

$$F(t, \varphi, Eq(t, \varphi)) = \frac{1}{\Phi[\mathcal{C}u_0](\varphi)} \Big\{ [h(t) + Eq(t, \varphi)]\Phi[\mathcal{B}u_0](\varphi) + \Phi_1[v(t, \cdot)](\varphi)$$

$$- \Phi[N_1(v, h, q)(t, \cdot)](\varphi) - N_1^0(u_1, g_1, f)(t, \varphi) \Big\}.$$

Of course, if we find a solution $Eq(t, \cdot)$ to (7.10), then problem P(N) with the additional periodicity condition (3.35) and problem (5.8)–(5.14) turn out to be equivalent, and we are done. Since (7.10) is a first order boundary value problem, it seems, at a first glance, that we can find a solution by applying the results in [9]. Indeed, in [9] it is shown that if $f \in C([0, 2\pi] \times \mathbb{R})$ satisfies $\lim_{|x| \to +\infty} f(\varphi, x) = +\infty$ uniformly on $[0, 2\pi]$, then there exists $s_1 \in \mathbb{R}$ with $s_1 \geq \min_{[0, 2\pi] \times \mathbb{R}} f$, such that the problem

$$x'(\varphi) + f(\varphi, x(\varphi)) = s \quad (s \in \mathbb{R}), \qquad x(0) = x(2\pi),$$

has zeros, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$. Therefore, we try to apply this result to (7.10), by showing that, $t \in [0, T]$ being fixed, $0 \geq s_1 \geq \min_{[-\frac{\pi}{2}, \frac{3\pi}{2}] \times \mathbb{R}} F(t, \cdot, \cdot)$. Unfortunately, here the situation is quite different, since, actually, $F(t, \cdot, \cdot)$ depends also on $v(t, \cdot)$, $h(t)$ and $q(t, \cdot)$ and, in fact, a better notation for F would be $F(t, v(t, \cdot), h(t), q(t, \cdot), \varphi, Eq(t, \varphi))$. As a consequence, the minimum of F on $[-\frac{\pi}{2}, \frac{3\pi}{2}] \times \mathbb{R}$ is not uniquely determined, but it depends on some a priori estimates for the unknowns v, h and q. This means that we need to know an additional constraint on the growth of the solution (u, k) to the original problem P(N). In general such constraint is not satisfied, so that the previous technique cannot be applied to our case.

Another possible approach suggested by (3.35) is that of rewriting $k(t, \cdot)$ in terms of its Fourier expansions, but in this case we encounter the following

problem: u has to be rewritten in terms of its Fourier expansion, too, not only k, and series product appear from the convolutions $k(\cdot, \rho(x')) * \mathcal{B}u(\cdot, x)$ and $D_\eta k(\cdot, \rho(x')) * \mathcal{C}u(\cdot, x)$ in (2.1). This implies that the Fourier method cannot be applied successfully because of the different dependence on the local variables of the named convolution terms.

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