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Monogenic Wavelets over the Unit Ball

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Abstract. In this article we define a monogenic wavelet transform for quaternion valued functions on the unit ball B in $\mathbb{R}^3$ based on representations of the group of Möbius transformations which maps the unit ball onto itself.

Keywords: *Möbius transformations, square integrable group re presentations, monogenic functions, wavelet transform*

MSC 2000: 30G35

1. Introduction

Wavelets have undergone a rapid growth in the last fifteen years both in research and applications, mainly because they are based on a powerful mathematical theory. In its abstract definition we have a transitive action of a group of linear automorphisms of the domain and a representation of its group on a Hilbert space. If this representation is unitary, irreducible, and square-integrable we obtain a continuous wavelet transform, a reproducing kernel, etc. [9, 17]. More recently there have been developed continuous wavelet transforms on the 2D-sphere via group representations on the tangent bundle (see e.g. [8]) or based on coherent states associated to square integrable group representations of subgroups of the Euclidean group [9] and of SO(3,1) [1], which is the conformal group of the sphere. If we now take a look at the theory of monogenic functions, the so-called Clifford analysis, we can observe strong connections between both

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theories, which can be easily seen if we define monogenic functions via irreducible representations of the Spin-group [11]. These connections were already considered by M. Mitrea, V. Kisil and J. Cnops for the classical case of $\mathbb{R}^n$ and Hardy spaces, see [5, 7, 15, 18]. Moreover, it gives rise to the question of what happens in the particular case of the unit ball which has its own group of automorphisms.

The case of the complex unit disk can be found in [14] and [16]. In the last one we can find the discrete series of the group $SL(2,\mathbb{R})$, isomorphic to the group $SU(1,1)$:

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\} \cong SL(2,\mathbb{R}).$$

The group $SU(1,1)$ acts on $B = \{z \in \mathbb{C} : |z| < 1\}$ by $g(z) = \frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$, transitively. For f analytic on B and $n \geq 2$, we have the representations

$$D_n \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} f(z) = (-\overline{\beta}z + \alpha)^{-n} f\left(\frac{\overline{\alpha}z - \beta}{-\overline{\beta}z + \alpha} \right)$$

connected to the norm

$$\|f\|^2 = \int_B |f(z)|^2 \big(1 - |z|^2\big)^{n-2}\, dx\, dy,$$

which gives rise to the discrete series. In [14] it is proved that these representations are unitary, irreducible and square integrable representations.

If we want to consider the same theory in higher dimensions, in particular in the case of the unit ball in $\mathbb{R}^3$, we immediately encounter a lot of difficulties. These representations cannot be considered in higher dimensions using Clifford algebras due to the non-commutativity and the fact that the product of two monogenic functions is not necessary a monogenic function. If we modify the representation to preserve monogenicity we will lose the unitary property. Also, problems about the square-integrability of the representation arises. The purpose of this paper is to show that it is possible to overcome this difficulties about the representation in higher dimensions and finally to define monogenic wavelets over the unit ball and the corresponding continuous wavelet transform. Moreover, the approach presented in this paper works also in spaces which allow the functions to have a certain growth to infinity at the boundary, e.g., weighted Bergman spaces or $Q_{p,0}$-spaces.

2. Preliminaries

We will work in $\mathbb{H}$, the skew field of quaternions. Each element $z \in \mathbb{H}$ can be written in the form $z = x_0 + x_1 i + x_2 j + x_3 k$, $x_n \in \mathbb{R}$, where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. The conjugate element $\overline{z}$ is given by $\overline{z} = x_0 - x_1 i - x_2 j - x_3 k$ and we have the properties $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{w}\,\overline{z}$ and $z\overline{z} = \overline{z}z = x_0^2 + x_1^2 + x_2^2 + x_3^2 := |z|^2$, which is the Euclidean norm of z. Each non-zero quaternion has a inverse given by $z^{-1} = \frac{\overline{z}}{|z|^2}$. Since multiplication is not commutative the symbol $\frac{w}{z}$ is ambiguous. However, we will define the symbol by $\frac{w}{z} := wz^{-1}$.

Let $x_0 =: \mathrm{Sc}(z)$ be the scalar part and $\underline{z} = x_1 i + x_2 j + x_3 k := \mathrm{Vec}(z)$ the vector part of the quaternion z. Then for w, z quaternions it follows

$$wz = w_0 z_0 - \underline{w} \cdot \underline{z} + w_0 \underline{z} + z_0 \underline{w} + \underline{w} \times \underline{z}\,.$$

Here $\underline{w} \cdot \underline{z}$ denotes the scalar product in $\mathbb{R}^3$ and $\underline{w} \times \underline{z}$ denotes Gibbs' cross product in $\mathbb{R}^3$. Also, we will identify each element $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with the pure quaternion $x = \underline{z} = x_1 i + x_2 j + x_3 k$.

For all what follows we will work over B, the unit ball in $\mathbb{R}^3$, and consider functions $f : B \mapsto \mathbb{H}$. Then any function f has a representation $f = f_0(x) + if_1(x) + jf_2(x) + kf_3(x)$ with real-valued components $f_i, i = 0, \ldots, 3$. Thus, notations $f \in C^k(B, \mathbb{H}), k \in \mathbb{N} \cup \{0\}$, and $f \in L_p(B, \mathbb{H}), 1 < p < \infty$, might be understood both coordinatewisely and directly. For instance, $f \in L_p(B, \mathbb{H}), 1 < p < \infty$, means that $\{f_i\} \subset L_p(B)$ or, equivalently, that $\int_B |f(x)|^p dB_x < \infty$. All these spaces are $\mathbb{H}$-bi-modules. $L_2(B, \mathbb{H})$ can be converted into a Hilbert $\mathbb{H}$-module, namely an inner product can be defined as $\langle f, g \rangle := \int_B \overline{f(x)} g(x)\, dB_x$ and thus $L_2(B, \mathbb{H})$ becomes a right $\mathbb{H}$-module. In the following we will use the short notation $C^k(B), L_p(B)$ etc., instead of $C^k(B, \mathbb{H}), L_p(B, \mathbb{H})$.

We now introduce the Dirac operator by

$$Df = i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3}$$

This operator is a hypercomplex analogue to the complex Cauchy-Riemann operator. In particular, we have $D^2 = -\Delta$, where Δ is the Laplacian in $\mathbb{R}^3$. All functions f which belong to $\ker D = \{f : Df = 0\}$ are called *left-monogenic*. Obviously, monogenic functions are also harmonic functions.

Of particular interest for us is the Bergman space $L_2(B) \cap \ker D$. For this space there exists a basis $\{H_\nu^k : \nu = 1, ..., K(3,k)\}_{k=0}^\infty$ of homogeneous monogenic polynomials, the so-called *inner spherical monogenics*, hereby k denotes the degree of homogeneity. These inner spherical monogenics can be made into an orthonomal basis satisfying the orthogonality condition $\langle H_k^\nu, H_l^\tau \rangle_{L_2(B)} = \delta_{k,l}\delta_{\nu\tau}$ (see [4] and [3]). For more details about monogenic functions see [2], [10], and [13].

3. Group theoretical background

Let us give in this section a short resume of the group theoretical background, which in its abstract context, represents the common group-theoretical denominator of the wavelet and windowed Fourier transform (see [17] and [9] for more details). Let H be a Hilbert space and let G be a separable Lie group with (right) Haar measure μ. A representation π of G in H is defined as a mapping $\pi : G \to L(H)$ of G into the space $L(H)$ of unitary operators on H, such that $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$ and $\pi(e) = Id$. The representation is continuous if for any $\phi, \psi \in H$, the map $G \ni g \mapsto \langle \phi, \pi(g)\psi \rangle_H$ is continuous and it is irreducible if the only invariant subspaces are the trivial spaces $\{0\}$ and H. A continuous, unitary representation π is said to be *square-integrable* if it is irreducible and there exists a nonzero function $\psi \in H$ such that

$$\int_G |\langle \pi(g)\psi, \psi \rangle_H|^2 d\mu(g) < \infty.$$

Such a function ψ if it exists is called *admissible.* The left or right-invariant Haar measure exists on locally compact groups and it is defined up to a normalization factor by its property of preserving the measure.

Any admissible function $\psi \in H$, with $\psi \neq 0$, gives rise to a wavelet transform $V_\psi f$ as an operator on H defined by

$$V_\psi f(g) := \langle \pi(g)\psi, f \rangle_H, \quad g \in G,$$

which is an isometry from H onto the reproducing Hilbert space $M_2 = \{F \in L_2(G) : \langle F, R(g, \cdot) \rangle = F(g)\}$ with the reproducing kernel

$$R(g, l) := \langle \pi(g)\psi, \pi(l)\psi \rangle_H = V_\psi(\pi(l)\psi)(g).$$

Thus V_ψ can be inverted on its range M_2 by its adjoint V_ψ^* given by

$$V_\psi^* F(s) = \int_G \big(\pi(g)\psi\big)(s) F(g)\, d\mu(g)$$

For $f \in H$ this provides us with the reconstruction formula

$$f = V_\psi^* V_\psi f = \int_G \big(\pi(g)\psi\big)(s) \langle \pi(g)\psi, f \rangle_H \, d\mu(g).$$

Moreover, the spaces H and M_2 are isometrically isomorphic.

4. Monogenic wavelets over the unit ball in $\mathbb{R}^3$

Let us now consider our case of the unit ball in $\mathbb{R}^3$ and investigate how the general theory translates into our case.

The group of automorphisms of the unit ball consists of the group of Möbius transformations, which map the unit ball onto itself and the rotations. In what follows we consider, without loss of generality, only the group of Möbius transformations (up to rotations) G consisting of the mappings

$$\varphi_a(x) = (x-a)(1-\overline{a}x)^{-1}, \quad a \in \mathbb{R}^3,\ |a| < 1\,,$$

which maps the unit ball onto itself. Due to the fact that monogenic functions are Spin(m)-invariant (Spin(m) is a double covering of SO(m)) and the Spin-representation is unitary the results are also valid in the more general setting, i.e., the problems come only from the above group of Möbius transformations.

The composition of two Möbius transformations is another Möbius transformation up to a rotation

$$\varphi_a \circ \varphi_b(x) = \frac{1+a\overline{b}}{|1+a\overline{b}|}\,\frac{x-(1+a\overline{b})^{-1}(a+b)}{1-\overline{(1+a\overline{b})^{-1}(a+b)}x}\,\frac{\overline{1+a\overline{b}}}{|1+a\overline{b}|} = q\,\varphi_{(1+a\overline{b})^{-1}(a+b)}(x)\,\overline{q}$$

with $q = \frac{1+a\overline{b}}{|1+a\overline{b}|}$. We denote by $b \times a = (1+a\overline{b})^{-1}(a+b)$ the symbol of the new Möbius transformation. It's important to observe that we have the property

$$\begin{aligned}(1+a\overline{b})^{-1}(a+b) &= \frac{\overline{1+a\overline{b}}}{|1+a\overline{b}|^2}\,(a+b)\\ &= \frac{a+b+b\overline{a}a+b\overline{a}b}{|1+a\overline{b}|^2}\\ &= \frac{a+b+a\overline{a}b+b\overline{a}b}{|1+a\overline{b}|^2}\\ &= \frac{(a+b)(1+\overline{a}b)}{|1+a\overline{b}|^2}\\ &= (a+b)(1+\overline{b}a)^{-1}.\end{aligned} \tag{1}$$

With the neutral element $\varphi_0(x)$ and the inversion $(\varphi_a(x))^{-1} = \varphi_{-a}(x)$ we have that G is a (non-abelian) locally compact group. We must remark that we have a natural isomorphism between this group of M öbius transformations and the group of points of the unit ball G^* identifying each $\varphi_a(x)$ with the element $a \in B$ and the operation $\varphi_a \circ \varphi_b$ with $b \times a$. So the integration over the group can be seen as an integration over the unit ball.

Lemma 4.1. *The right-invariant Haar measure μ_R, on G or G^* is given by*

$$\mu_R(a) = \left(\frac{1}{1-|a|^2}\right)^3 dB_a\,.$$

Proof. Let us consider the group G^*. The right-invariant measure for G^* is

$$\mu_R(G^*) = \int_{G^*} 1\, d\mu_R(a) = \int_{G^*} \left(\frac{1}{1-|a|^2}\right)^3 dB_a\,.$$

We must prove that for $b \in B$ (and, therefore, $b \in G^*$) $\mu_R(G^* \circ b) = \mu_R(G^*)$. We have

$$\mu_R(G^* \circ b) = \int_{G^* \circ b} 1\, d\mu_R(a) = \int_{\Phi(B)} \left(\frac{1}{1-|a|^2}\right)^3 dB_a\,,$$

where the transformation $\Phi : B \to B$ is given by

$$\Phi(a) = b \times a = (1 + a\bar{b})^{-1}(a+b) = (a+b)(1+\bar{b}a)^{-1} = \varphi_{-b}(a)\,.$$

The Jacobian of this transformation is $\left(\frac{1-|b|^2}{|1+\bar{b}a|^2}\right)^3$. Using the relation

$$\frac{1-|\varphi_a(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{|1-\bar{a}x|^2} \tag{2}$$

and the transformation theorem yields

$$\begin{aligned}
\mu_R(G^* \circ b) &= \int_B \left(\frac{1}{1-|b \times a|^2}\right)^3 |\det \mathrm{J}_\Phi(a)|\, dB_a \\
&= \int_B \left(\frac{1}{1-|\varphi_{-b}(a)|^2}\right)^3 \left(\frac{1-|b|^2}{|1+\bar{b}a|^2}\right)^3 dB_a \\
&= \int_B \left(\frac{|1+\bar{b}a|^2}{(1-|a|^2)(1-|b|^2)} \frac{1-|b|^2}{|1+\bar{b}a|^2}\right)^3 dB_a \\
&= \int_B \left(\frac{1}{(1-|a|^2)}\right)^3 dB_a \\
&= \mu_R(G^*).
\end{aligned}$$

∎

The next step is to find a square integrable representation of G in $L_2(B) \cap \ker D$. Initially we can think in the trivial representation $\pi(a)f(x) = f\big(\varphi_a(x)\big)$ for functions $f : L_2(B) \cap \ker D \to \mathbb{H}$. It is a homomorphism. The main problem is that if the function f is monogenic, the function $f\big(\varphi_a(x)\big)$ is not monogenic for dimensions higher or equal to 3, so the representation is not in the space $L_2(B) \cap \ker D$. But we know that $\frac{1-\bar{x}a}{|1-\bar{a}x|^3} f\big(\varphi_a(x)\big)$ is again monogenic. We refer to [19] for the general case and to [20] who studied this problem for the four-dimensional case already in 1979. Therefore, we get

Theorem 4.2. *Let* $\pi : G \to L(L_2(B) \cap \ker D)$ *defined by*

$$\pi(a)f(x) = \frac{(1-|a|^2)(1-\overline{x}a)}{|1-\overline{a}x|^3} f\big(\varphi_a(x)\big).$$

Then π *defines a representation of the group of Möbius transformations into the linear automorphisms over* $L_2(B) \cap \ker D$*.*

Proof. We have to prove that $\pi(a)f(x)$ is a homomorphism, i.e., $\pi(b)(\pi(a)f(x)) = \pi(b \times a)f(x)$ and $\pi_0 f(x) = f(x)$. On one hand,

$$\begin{aligned}
&\pi(b)\big(\pi(a)f(x)\big)\\
&\quad= \pi(b)\left(\frac{(1-|a|^2)(1-\overline{x}a)}{|1-\overline{a}x|^3} f\big(\varphi_a(x)\big)\right)\\
&\quad= \frac{(1-|b|^2)(1-\overline{x}b)}{|1-\overline{b}x|^3}\,\frac{(1-|a|^2)(1-\overline{\varphi_b(x)}a)}{|1-\overline{a}\varphi_b(x)|^3} f\big(\varphi_a(\varphi_b(x))\big)\\
&\quad= \frac{(1-|b|^2)(1-\overline{x}b)}{|1-\overline{b}x|^3}\,\frac{(1-|a|^2)(1-\overline{(x-b)(1-\overline{b}x)^{-1}})a}{|1-\overline{a}(x-b)(1-\overline{b}x)^{-1}|^3} f\big(\varphi_{b\times a}(x)\big)\\
&\quad= \frac{(1-|a|^2)(1-|b|^2)(1-\overline{x}b)}{|1-\overline{b}x|^3}\,\frac{1-(1-\overline{x}b)^{-1}(\overline{x}-\overline{b})a}{|1-\overline{a}(x-b)(1-\overline{b}x)^{-1}|^3} f\big(\varphi_{b\times a}(x)\big)\\
&\quad= \frac{(1-|a|^2)(1-|b|^2)(1-\overline{x}b)(1-\overline{x}b)^{-1}[(1-\overline{x}b)-(\overline{x}-\overline{b})a]}{|1-\overline{b}x|^3|1-\overline{b}x-\overline{a}(x-b)|^3|(1-\overline{b}x)^{-1}|^3} f\big(\varphi_{b\times a}(x)\big)\\
&\quad= \frac{(1-|a|^2)(1-|b|^2)(1+\overline{b}a-\overline{x}(a+b))}{|1+\overline{a}b-(\overline{a}+\overline{b})x|^3} f\big(\varphi_{b\times a}(x)\big)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\pi(b\times a)f(x)\\
&\quad= \frac{(1-|b\times a|^2)(1-\overline{x}(1+a\overline{b})^{-1}(a+b))}{|1-\overline{(1+a\overline{b})^{-1}(a+b)}x|^3} f\big(\varphi_{b\times a}(x)\big)\\
&\quad= \frac{(1-|a|^2)(1-|b|^2)(1-\overline{x}(a+b)(1+\overline{b}a)^{-1})}{|1+b\overline{a}|^2|1-\overline{(a+b)(1+\overline{b}a)^{-1}}x|^3} f\big(\varphi_{b\times a}(x)\big),\\
&\quad= \frac{(1-|a|^2)(1-|b|^2)(1+\overline{b}a-\overline{x}(a+b))}{|1+b\overline{a}|^2|1+\overline{a}b-(\overline{a}+\overline{b})x|^3|(1+\overline{a}b)^{-1}|^3}(1+\overline{b}a)^{-1} f\big(\varphi_{b\times a}(x)\big)\\
&\quad= \frac{(1-|a|^2)(1-|b|^2)(1+\overline{b}a-\overline{x}(a+b))}{|1+\overline{a}b-(\overline{a}+\overline{b})x|^3}\,\frac{1+\overline{a}b}{|1+\overline{a}b|^2}|1+\overline{a}b| f\big(\varphi_{b\times a}(x)\big)\\
&\quad= \frac{(1-|a|^2)(1-|b|^2)(1+\overline{b}a-\overline{x}(a+b))}{|1+\overline{a}b-(\overline{a}+\overline{b})x|^3}\,\frac{1+\overline{a}b}{|1+\overline{a}b|} f\big(\varphi_{b\times a}(x)\big)\,.
\end{aligned}$$

As we can see π is a homomorphism up to a constant factor with modulus 1, hence being a rotation in H. The condition $\pi(0)f(x) = f(x)$ is trivially satisfied. So π is a representation. ∎

Unfortunately, this representation is not unitary in the space $L_2(B) \cap \ker D$ but in the space $L_2(B, \frac{1}{1-|x|^2} dB_x) \cap \ker D$. Here we have another problem. We can't work in the space $L_2(B, \frac{1}{1-|x|^2} dB_x) \cap \ker D$ because the functions in this space are zero almost everywhere. If we want the unitary property in the space $L_2(B) \cap \ker D$ we must introduce the "multiplier" $\frac{(1-|a|^2)^{1/2}}{|1-\bar{a}x|}$ but in that case we loose the monogenicity property of the representation. Therefore, we will drop the unitary property and continue to work with our representation π. However, we remark that the representation π is irreducible since the subgroup $SO(3)$ induces already the irreducibility of this representation into our space (of monogenic functions) [11].

Now we study the square integrable property of our representation. To this end we will consider the function $\psi(x) \equiv 1$. We have to prove that $\int_G |\langle \pi(a)1, 1\rangle|^2 d\mu(a) < \infty$. Using the fact that $(\pi(a)1)(x) = \frac{(1-|a|^2)(1-\bar{x}a)}{|1-\bar{a}x|^3}$ is a left monogenic function we can expand it in a generalized power series with respect to the inner spherical monogenics $H_\nu^k(x)$ already mentioned in the end of the preliminaries and use their orthogonality property. Thus,

$$
\begin{aligned}
\langle \pi(a)1, 1\rangle &= \int_B \overline{\frac{(1-|a|^2)(1-\bar{x}a)}{|1-\bar{a}x|^3}}\, dB_x \\
&= \int_B (1-|a|^2) dB_x + \sum_{k=1}^{\infty} \sum_{\nu=1}^{K(3,k)} \int_B \overline{H_\nu^k(x)\, c_\nu}\, dB_x \\
&= \int_B (1-|a|^2) dB_x + \sum_{k=1}^{\infty} \sum_{\nu=1}^{K(3,k)} \overline{c_\nu} \int_B \overline{H_\nu^k(x)}\, dB_x \\
&= \frac{4}{3}\pi(1-|a|^2)\,.
\end{aligned}
$$

with $c_\nu \in \mathbb{H}$. Finally, using the Haar measure we obtain

$$
\int_G |\langle \pi(a)1, 1\rangle|^2 d\mu(a) = \left(\frac{4}{3}\pi\right)^2 \int_B (1-|a|^2)^2 \frac{1}{(1-|a|^2)^3}\, dB_a.
$$

Here appears a problem with the Haar measure which does not exist in the complex case. Because the exponent of the term $1-|a|^2$ in the Haar measure is larger than in the representation the above integral is equal to infinity. We cannot simply change the weight in the representation or we lose the property of being an homomorphism. The way out we suggest here is to use a different measure, e.g., $d\mu * (a) = \left(\frac{1}{1-|a|^2}\right)^2 dB_a$, such that $\int_B (1-|a|^2)^2 d\mu(a) < \infty$. Let us remark that although this measure is not invariant, we can easily get the transformation formulae from the proof of the Haar measure:

$$\int_G |\langle \pi(a)1, 1\rangle|^2 d\mu^*(a) = \left(\frac{4}{3}\pi\right)^2 \int_B (1-|a|^2)^2 \frac{1}{(1-|a|^2)^2}\, dB_a = \left(\frac{4}{3}\pi\right)^3 < \infty\,.$$

This leads to the following theorem.

Theorem 4.3. *The representation π is square integrable with respect to the measure $d\mu^*(a)$.*

If we consider a more general Hilbert space H with measure that satisfies $\int_B d\mu(a) < \infty$ we can prove that the representation is square integrable, too. For instance, we can prove it for the following Hilbert spaces ($p \geq 0$):

$$L_2(B, (1-|x|^2)^p dB_x) \cap \ker D$$

and

$$Q_{p,0} = \left\{ f \in \ker D : |f(0)| + \lim_{a\to 1} \int_B |\nabla f|^2 (1-|\varphi_a(x)|^2)^p dB_x < \infty \right\}.$$

Using the orthogonality property of our (orthonormalized) inner spherical monogenics the above considerations lead to the following proposition.

Proposition 4.4. *A function f defined in the unit ball B is admissible if the quaternionic series $\sum_{k=0}^{\infty}\sum_\nu \overline{c_\nu} d_\nu$ converges where c_ν and d_ν are the coefficients of the generalized power series*

$$\pi(a)\psi(x) = (1-|a|^2)\sum_{k=0}^{\infty}\left(\sum_\nu H_\nu^k c_\nu\right) \quad \text{and} \quad \psi(x) = \sum_{l=0}^{\infty}\left(\sum_\tau H_\tau^k d_\tau\right)$$

with $c_\nu, d_\tau \in \mathbb{H}$.

Proof. By the Fourier expansion of $\pi(a)\psi$ and ψ we have

$$\begin{aligned}
\langle \pi(a)\psi, \psi\rangle &= \int_B \overline{\pi(a)\psi(x)}\psi(x) dB_x \\
&= \int_B (1-|a|^2)\overline{\left(\sum_{k=0}^{\infty}\sum_\nu H_\nu^k(x)\, c_\nu\right)}\left(\sum_{l=0}^{\infty}\sum_\tau H_\tau^l(x)\, d_\tau\right) dB_x \\
&= (1-|a|^2)\sum_{k=0}^{\infty}\sum_\nu\sum_{l=0}^{\infty}\sum_\tau \int_B \overline{c_\nu}\,\overline{H_\nu^k(x)}\, H_\tau^l(x)\, d_\tau\, dB_x \\
&= (1-|a|^2)\sum_{k=0}^{\infty}\sum_\nu\sum_{l=0}^{\infty}\sum_\tau \overline{c_\nu}\int_B \overline{H_\nu^k(x)}\, H_\tau^l(x)\, dB_x\, d_\tau \\
&= \frac{4}{3}\pi(1-|a|^2)\sum_{k=0}^{\infty}\sum_\nu \overline{c_\nu} d_\nu \\
&= \frac{4}{3}\pi(1-|a|^2)\alpha
\end{aligned}$$

with $\alpha \in \mathbb{H}$. Finally,

$$\int_G |\langle \pi(a)\psi(x), \psi(x)\rangle|^2 \, d\mu * (a) = \left(\frac{4}{3}\pi\right)^2 \int_B (1-|a|^2)^2 |\alpha|^2 \frac{1}{(1-|a|^2)^2} \, dB_a$$
$$= \left(\frac{4}{3}\pi\right)^3 |\alpha|^2 < \infty. \qquad \blacksquare$$

As an immediate consequence of this proposition we get the following corollary.

Corollary 4.5. *Each inner spherical monogenic $H_\nu^k(x)$ of degree k is an admissible function.*

Now, we are ready to define the continuous wavelet transform for an admissible function ψ. Let $\psi \neq 0$ be admissible and let $f \in L_2(B) \cap \ker D$, then the continuous left integrable wavelet transform is given by

$$V_\psi f(a) := \langle \pi(a)\psi, f\rangle_{L_2(B)} = (1-|a|^2) \int_B \overline{\frac{(1-\overline{x}a)}{|1-\overline{a}x|^3} \psi\big(\varphi_a(x)\big)} f(x) \, dB_x$$

which maps $L_2(B) \cap \ker D$ into $L_2(G)$. Let us remark that because our representation is neither unitary nor square integrable with respect to the Haar measure, we cannot state the immediate consequences from the general theory, directly. But, we can use the knowledge of the existence of the reproducing kernel in our Hilbert space, the Bergman kernel (see [2] and [6]), to establish nearly the same results.

While in the classic wavelet theory one uses the knowledge of the representation and the Wavelet transform to establish the reproducing kernel we can go the other way around and get from the existence of the Bergman projection the invertibility of our wavelet transform, i.e. $V_\psi^{-1} V_\psi f = Pf$, where P denotes the Bergman projection. Therefore, for an admissible function $\psi \neq 0$ our wavelets are given by

$$\pi(a)\psi(x) = \frac{(1-|a|^2)(1-\overline{x}a)}{|1-\overline{a}x|^3} \psi\big(\varphi_a(x)\big).$$

As an example, we obtain in the special case $\psi(x) \equiv 1$

$$\pi(a)\psi(x) = \frac{(1-|a|^2)(1-\overline{x}a)}{|1-\overline{a}x|^3} = \left(1 - \frac{1}{|y|^2}\right) \frac{\overline{y}-\overline{x}}{|y-x|^3} y|y|$$

with $y = \frac{a}{|a|^2} \in \mathbb{R}^3 \backslash B$. If we now take all the y from a dense set in $\mathbb{R}^3 \backslash B$ we obtain (up to constants) the complete function system from [12] which was used to prove the orthogonal decomposition of $L_2(B)$.

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