*Publ. RIMS, Kyoto Univ.*
**22** (1989), 347–354

# A Remark on the Decomposition of Abelian Fiber Spaces over Curves

By

Yoshio FUJIMOTO*,**

## § 0. Introduction

By an Abelian fiber space $f: X \to Y$, we mean that $f$ is a proper surjective morphism from a complex manifold $X$ to a complex manifold $Y$ whose general fibers are Abelian varieties. If $g: X \dashrightarrow Y$ is a surjective meromorphic map of complex varieties with irreducible general fibers, $g$ is called a meromorphic fiber space. Using his Hodge theory with degenerating coefficients, Zucker [3] proved the theorem of the fixed part of Abelian varieties. ([3] Corollary 10.2) Inspired by his result, we consider the decomposition of Abelian fiber spaces over curves.

The purpose of this note is to prove the following Theorem A.

**Theorem (A).**

*Let $f: X \to C$ be a surjective morphism of a projective manifold $X$ to a non-singular curve $C$ such that the general fibers of $f$ are Abelian varieties. Then we have the following diagram*

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h} & \\ & C & \end{array}$$

*which enjoys the following properties.*

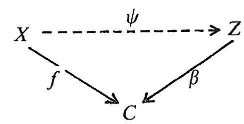
(1) *$g: X \dashrightarrow Y$ (resp. $h: Y \to C$) is a meromorphic fiber space (resp. a fiber space) whose general fibers are Abelian varieties and the period map associated to $h$ is constant.*

*(For the definition of a fiber space, a meromorphic fiber space and a period map, see notations below.)*

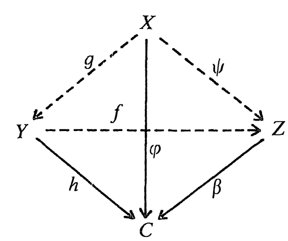
(2) *Suppose that there exists a commutative diagram*

---

Communicated by K. Saito, July 1, 1988. Revised December 1, 1988.
* Department of Mathematics, Kyoto University, Kyoto 606, Japan.
Current address: Department of Mathematics, Shizuoka University, Ohya, Shizuoka, Japan.
** Supported by the JSPS Fellowships for Japanese Junior Scientists




*where* $\psi$: $X \dashrightarrow Z$ (*resp.* $\beta: Z \to C$) *is a meromorphic fiber space* (*resp. a fiber space*) *whose general fibers are Abelian varieties and the period map associated to* $\beta$ *is constant. Then there exists a unique meromorphic map* $\varphi: Y \dashrightarrow Z$ *such that we obtain the following commutative diagram. Moreover we have* $\dim H^1(X, \mathcal{O}_X) = \dim H^1(Y, \mathcal{O}_Y)$.




The author wishes to express his thanks to Professor K. Ueno for useful suggestions.

**Notations.** We use the following convention and terminology.

$$q(X) := \dim H^1(X, \mathcal{O}_X): \text{ the irregularity of } X.$$

A fiber space is a proper surjective morphism of complex spaces with irreducible general fibers.

Let $f$: $X \to Y$ be a meromorphic map of complex varieties, $\Gamma \subset X \times Y$ be the graph of $f$, $p'$: $\Gamma \to Y$ be the natural projection. If $p'$ is a fiber space, we say that $g$ is *a meromorphic fiber space*.

Let $f$: $X \to C$ be an Abelian fiber space over the curve $C$ and let $f^{-1}(p) = \sum_{i=1}^{\lambda} m_i D_i$ be the irreducible decomposition of the singular fiber over a point $p \in C$. If the greatest common divisor $m$ of $(m_1, \cdots, m_\lambda)$ is greater than one, we say that $X$ has *multiple fibers* of multiplicity $m$ at $p \in C$.

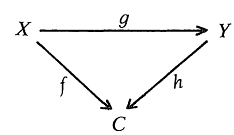
Let $\varphi$: $X \to Y$ be a projective family of $g$-dimensional Abelian varieties and $U \subset Y$ be a Zariski open subset over which $\varphi$ is smooth. Then by integrating a relative holomorphic one-form of $X$ along a continuous family of homology basis of $H_1(\varphi^{-1}(u), \mathbb{Z})$, $u \in U$, we can define a multi-valued holomorphic mapping (called the multi-valued *period mapping*) $T$: $U \to \mathfrak{S}_g$ into the Siegel upper half-plane of degree $g$. For the precise definition of a period mapping, see Ueno [2], p.4-8.

## §1. Preliminaries

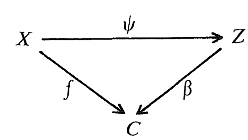
To prove Theorem A, we need the following propositions.

**Proposition 1.1.** *Let $f: X \to C$ be an Abelian fiber space over a smooth curve $C$ and define $q(f) := q(X) - q(C)$. Then we have the inequality $q(f) \leq \dim(X) - 1$.*

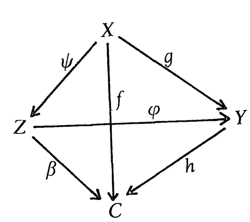
**Proposition 1.2.** *Let $f: X \to C$ be a proper surjective morphism of a projective manifold $X$ to a smooth curve $C$ such that the general fibers are Abelian varieties. Then we have the following diagram*



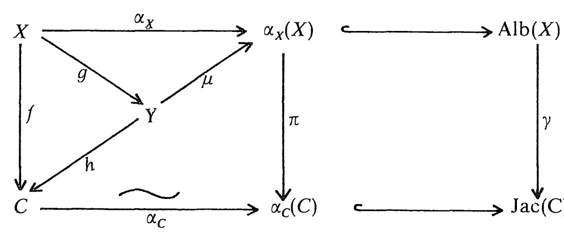

*which satisfies the following properties.*
*(1) $g: X \to Y$ and $h: Y \to C$ are Abelian fiber spaces. Moreover we have $q(X) = q(Y)$ and $h: Y \to C$ is a Seifert fiber space, that is, a fiber space with constant moduli which has only multiple singular fibers.*
*(2) Suppose that there exists a commutative diagram*




*such that $\psi: X \to Z$ and $\beta: Z \to C$ are Abelian fiber spaces with $q(X) = q(Z)$. Then there exists a unique morphism $\varphi: Z \to Y$ with the following commutative diagram.*




*Proof of Proposition* 1.1 *and Proposition* 1.2.
Let $\alpha_X$ be the Albanese map of $X$ and consider the following commutative diagram:



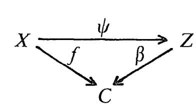
,

where $\mu \circ g$ is the Stein factorization of $\alpha_X$.

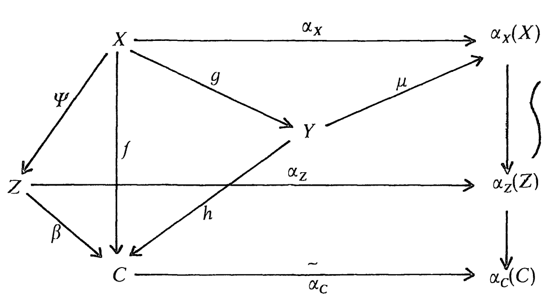
The Albanese map $\alpha_X$: $X \rightarrow Alb(X)$, when restricted to the general fiber of $f$, is a Lie group homomorphism, hence each fiber of $\pi$ and the general fiber of $g$ are Abelian varieties. And it is easy to see that the general fibers of $h$ are Abelian varieties.

As $\alpha_X(X)$ generates $Alb(X)$, for a general point $p \in C$, $\alpha_X(f^{-1}(p))$ is equal to the connected component of $\ker\{\gamma: Alb(X) \rightarrow Jac(C)\}$. Hence we have $\dim(Y) = \dim(\alpha_X(X)) = 1 + q(X) - q(C)$. This proves Proposition 1.1. Moreover we have $q(X) = q(Y)$, since $q(X) \geq q(Y) \geq q(\alpha(X)) = q(X)$. Each fiber of $h$ contains no rational curve and the ramification locus of $\mu$ are at most multiple fibers of $h$. Hence $h$: $Y \rightarrow C$ is a Seifert fiber space.

*Universality.* Suppose that there exists a commutative diagram



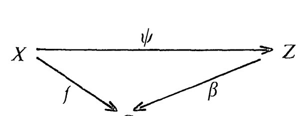

with $q(X) = q(Z)$. Then we have the following commutative diagram:



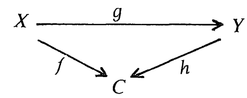

Since $q(X) = q(Y) = q(Z)$, we have $\alpha_X(X) \simeq \alpha_Y(Y) \simeq \alpha_Z(Z)$. Hence for each fiber $X_p := \psi^{-1}(p)$ of $\psi$, $\mu \circ g(X_p) = \alpha_X(X_p)$ consists of one point. Since $\mu$ is finite and $X_p$ is connected, $g(X_p)$ is also a point. Hence there exists a holomorphic map $\varphi$: $Z \rightarrow Y$ such that $g$ is isomorphic to $\varphi \circ \psi$.

q.e.d.

**Lemma 1.3.** *Let*



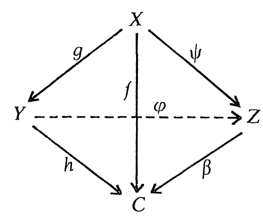

*be a commutative diagram of Abelian fiber spaces over a smooth curve, where the Albanese map of $Z$ is generically finite to its image. And let*



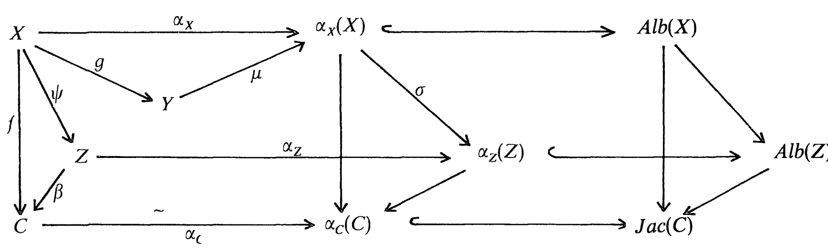

*be a commutative diagram of Abelian fiber spaces as in Proposition* 1.2.

*Then there exists a surjective meromorphic map* $\varphi$: $Y \dashrightarrow Z$ *with the following commutative diagram.*

(*)




*Proof.* There is a commutative diagram.




For a general fiber $X_p := g^{-1}(p)$ $(p \in Y)$ of $g$, we have

$$\alpha_Z \circ \psi(X_p) = \sigma \circ \alpha_X(X_p) = \sigma \circ \mu(p) = \{\text{one point}\}.$$

Since $\alpha_Z$ is generically finite and $X_p$ is connected, $\psi(X_p)$ is a point for general $p \in Y$. Hence there exists a meromorphic map $\varphi$: $Y \dashrightarrow Z$ such that $\varphi \circ g$ is bimeromorphically equivalent to $\psi$ and (*) holds.

**q.e.d.**

## §2. Proof of Theorem A

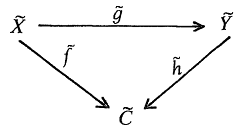
Now we shall prove Theorem A.

*Construction.* There exists a finite Galois covering $\pi$: $\tilde{C} \to C$ and an Abelian fiber space $\tilde{f}$: $\tilde{X} \to \tilde{C}$ over $\tilde{C}$ which satisfies the following conditions.

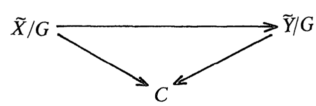
1) The fiber space $\tilde{f}$: $\tilde{X} \to \tilde{C}$ is bimeromorphically equivalent to $X \times_C \tilde{C} \to \tilde{C}$.
2) $\tilde{X}$ is a smooth projective manifold and $\tilde{f}$ is projective.
3) $\tilde{f}$: $\tilde{X} \to \tilde{C}$ has a section. (So it is free from multile fibers.)

4) $q(\tilde{f})$ is maximal among all finite Galois coverings $\tilde{C} \to C$ of $C$. (In general we have $q(\tilde{f}) \leq \dim(X)$.)

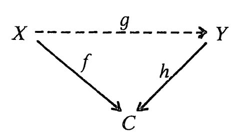
Let




be the decomposition of Abelian fiber spaces as in Proposition 1.2.. From the construction of $\tilde{Y}$, $\mathrm{Aut}(\tilde{X})$ induces $\mathrm{Aut}(\tilde{Y})$. Then if we put $G = \mathrm{Gal}(\tilde{X}/X)$, we have the following commutative diagram:

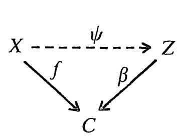



Taking a non-singular model of them, we obtain the commutative diagram

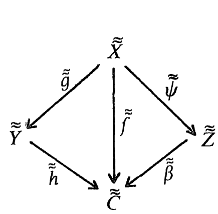


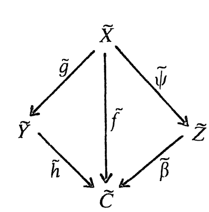

such that $g$: $X$ ----› $Y$ and $h$: $Y \to C$ are Abelian fiber spaces and the period map associated to $h$ is constant.

*Universality.* Suppose that there exists a commutative diagram such that the period map associated to $\beta$ is constant.




Then by Fujiki [1], Proposition 13.3, there exists a finite Galois covering $\tilde{\tilde{C}} \to \tilde{C}$ of $\tilde{C}$ and by pulling-back $X \to Z \to C$ and taking a suitable bimeromorphic model, we obtain





where $\tilde{\tilde{Z}} \to \tilde{\tilde{C}}$ is bimeromorphic to $\tilde{\tilde{C}} \times B$ ($B$ is an Abelian variety) and $\tilde{\tilde{\psi}}$, $\tilde{\psi}$ are morphisms. Then from the maximality of $q(\tilde{f})$, we have $q(\tilde{f}) = q(\tilde{\tilde{f}})$. And clearly we have $q(\tilde{\tilde{f}}) \geq q(\tilde{\tilde{h}}) \geq q(\tilde{h}) = q(\tilde{f})$, where the last equality holds from Proposition 1.2. Therefore we have $q(\tilde{\tilde{f}}) = q(\tilde{\tilde{h}})$ and $q(\tilde{\tilde{X}}) = q(\tilde{\tilde{Y}})$.
Next we show that

$$\begin{array}{ccc} \tilde{\tilde{X}} & \xrightarrow{\tilde{\tilde{g}}} & \tilde{\tilde{Y}} \\ & {\scriptstyle \tilde{\tilde{f}}}\searrow \quad \swarrow{\scriptstyle \tilde{\tilde{h}}} & \\ & \tilde{\tilde{C}} & \end{array}$$

is obtained from the Stein factorization of the Albanese map of $\tilde{\tilde{X}}$. Let

$$\begin{array}{ccc} \tilde{\tilde{X}} & \longrightarrow & W \\ & {\scriptstyle \tilde{\tilde{f}}}\searrow \quad \swarrow & \\ & \tilde{\tilde{C}} & \end{array}$$

be the factorization as in Proposition 1.2. Then from the universality of $W$, there exists a unique morphism $\tilde{\tilde{Y}} \to W$ such that the following diagram is commutative.

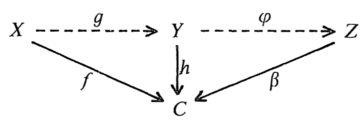
$$\begin{array}{ccccc} \tilde{\tilde{X}} & \xrightarrow{\tilde{\tilde{g}}} & \tilde{\tilde{Y}} & \longrightarrow & W \\ & {\scriptstyle \tilde{\tilde{f}}}\searrow & \downarrow{\scriptstyle \tilde{\tilde{h}}} & \swarrow & \\ & & \tilde{\tilde{C}} & & \end{array}$$

We have $\dim(\tilde{\tilde{Y}}) = \dim(\tilde{Y}) = 1 + q(\tilde{f}) = 1 + q(\tilde{\tilde{f}}) = \dim(W)$ from Proposition 1.2. Hence $\tilde{\tilde{Y}} \to W$ is a generically finite morphism. Since both $\tilde{\tilde{X}} \to \tilde{\tilde{Y}}$ and $\tilde{\tilde{X}} \to W$ have connected fibers, $\tilde{\tilde{Y}}$ is bimeromorphic to $W$ and the claim has been proved.

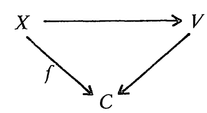
By our construction, the Albanese map of $\tilde{\tilde{Z}}$ is birational to its image. So it follows from Lemma 1.3 that there exists a meromorphic map $\tilde{\tilde{\varphi}}$: $\tilde{\tilde{Y}} \dashrightarrow \tilde{\tilde{Z}}$ with the following commutative diagram.

$$\begin{array}{ccccc} \tilde{\tilde{X}} & \xrightarrow{\tilde{\tilde{g}}} & \tilde{\tilde{Y}} & \overset{\tilde{\tilde{\varphi}}}{\dashrightarrow} & \tilde{\tilde{Z}} \\ & {\scriptstyle \tilde{\tilde{f}}}\searrow & \downarrow{\scriptstyle \tilde{\tilde{h}}} & \swarrow{\scriptstyle \tilde{\tilde{\beta}}} & \\ & & \tilde{\tilde{C}} & & \end{array}$$

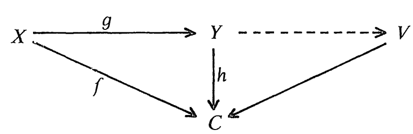
By the universality, the meromorphic map is compatible with the action of $G := \mathrm{Gal}(\tilde{X}/X)$. Hence $\tilde{\varphi}$ descends to a meromorphic map $\varphi$: $Y \dashrightarrow Z$ with the following commutative diagram.




and the universality has been proved. Let




be a commutative diagam of Abelian fiber spaces as in Proposition 1.2. Then we have $q(V)=q(X)$ and $V\to C$ is a Seifert fiber space. Hence from the universality of $Y$, there exists a meromorphic map $Y \dashrightarrow V$ with the commutative diagram.




Clearly we have $q(X)\geq q(Y)\geq q(V)$. Since $q(X)=q(V)$, we have $q(X)=q(Y)$.
q.e.d.

## References.


[1] Fujiki, A., On the structure of compact complex manifolds in $\mathscr{C}$. *Advanced studies in Pure Math.* 1, Algebraic varieties and analytic varieties, 231–302, Kinokuniya, Tokyo (1983).

[2] Ueno, K., On algebraic fiber spaces of Abelian varieties. *Math. Ann.* **237** (1978), 1-22

[3] Zucker, S., Hodge theory with degenerating coefficients: $L^2$-cohomology in the Poincare metric. *Annals of Math.* **109** (1979), 415–476.