

On some Weighted Inequalities for Fractional Integrals on Nonhomogeneous Spaces

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Abstract. Necessary and sufficient conditions on a measure governing two-weight inequality with the weights of power type for fractional integrals on nonhomogeneous spaces are established. Various applications are given, in particular to potentials with Radon and Hausdorff measures.

Keywords: *Potential operators, nonhomogeneous spaces, two-weight inequality, boundedness of operators, non-doubling measures*

MSC 2000: Primary 26A33, secondary 42B20, 47B38, 47G10

1. Introduction

The main goal of the present paper is to give a complete description of those measure spaces for which the two-weight estimate for potentials with measure holds, where the weights are of power type. This enables us to generalize the well-known classical theorem of E. M. Stein and G. Weiss [15] concerning the two-weight inequality

$$\left(\int_{R^n} |T_\gamma f(x)|^q |x|^{\lambda_2} dx \right)^{\frac{1}{q}} \leq A \left(\int_{R^n} |f(x)|^p |x|^{\lambda_1} dx \right)^{\frac{1}{p}}, \quad 1 < p \leq q < \infty,$$

for the operator

$$T_\gamma f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy, \quad 0 < \gamma < n,$$

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in the case of non-doubling measures. The boundedness of the potential operator

$$\bar{T}_\gamma f(x) = \int_0^\infty \frac{f(t)}{|x-t|^{1-\gamma}} dt$$

from $L^p_{x^{\lambda_1}}(0, \infty)$ to $L^q_{x^{\lambda_2}}(0, \infty)$, $1 < p \leq q < \infty$, was obtained by G. H. Hardy and J. E. Littlewood [9] (see also [14, p. 495], and [13] for two-weight estimates with power weights for the operator T_γ).

For the first time weighted estimates for integral transforms with positive kernel defined on nonhomogeneous spaces (X, ρ, μ) were obtained in [7] (see also [8, Chapter 2]), where the authors showed that the weak-type inequality

$$\nu\{x \in X : Kf(x) > \lambda\} \leq \frac{c}{\lambda^q} \left(\int_X |f(x)|^p w(x) d\mu(x) \right)^{\frac{q}{p}}, \quad 1 < p < q < \infty,$$

for the operator $Kf(x) = \int_X k(x, y)f(y) d\mu(y)$, $k \geq 0$, holds if

$$\sup_{x \in X, r > 0} (\nu B(x, 2N_0r))^{\frac{1}{q}} \left(\int_{X \setminus B(x,r)} k^{p'}(x, y) w^{1-p'}(y) d\mu(y) \right)^{\frac{1}{p'}} < \infty,$$

where N_0 is a positive constant depending on the quasimetric ρ ; ν is another non-doubling measure on X , and w is a weight function defined on X . Using this result they have established necessary and sufficient conditions governing two-weight strong-type inequality for the operator K defined on measure spaces with quasimetric and doubling measure, i.e., spaces of homogeneous type (SHT) (see, e.g., [2] and [8] for the definition and some examples of SHT).

In [12] (see also [4, Chapter 6]) a complete description of non-doubling measure μ guaranteeing the boundedness of the potential operator

$$I_\alpha f(x) = \int_X \frac{f(y)}{\rho(x, y)^{1-\alpha}} d\mu(y)$$

from $L^p(\mu, X)$ to $L^q(\mu, X)$, $1 < p < q < \infty$, has been obtained. In the same paper theorems of Sobolev and Adams type for fractional integrals defined on nonhomogeneous spaces have been established. Analogous problems in the case of Euclidean spaces and curves were considered in [10, 11]. Some two-weight norm inequalities for fractional maximal functions and potentials on R^n with non-doubling measure were studied in [6].

The paper is organized as follows: In Section 2 we give a definition of nonhomogeneous spaces and some well-known results concerning fractional integrals on nonhomogeneous spaces. In Section 3 we formulate the main results of the paper, while in Section 4 we prove them. Constants (often different constants in the same series of inequalities) will generally be denoted by c .

2. Preliminaries

Throughout the paper we assume that (X, ρ, μ) is a topological space X , endowed with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a non-negative real-valued function (quasimetric) $\rho : X \times X \rightarrow R^1$ satisfying the conditions

- (i) $\rho(x, x) = 0$ for arbitrary $x \in X$;
- (ii) $\rho(x, y) > 0$ for arbitrary $x, y \in X, x \neq y$;
- (iii) there exists a positive constant a_0 such that for all $x, y \in X$ the inequality holds $\rho(x, y) \leq a_0\rho(y, x)$ holds;
- (iv) there exists a positive constant a_1 such that for arbitrary $x, y, z \in X$ the inequality $\rho(x, y) \leq a_1(\rho(x, z) + \rho(z, y))$ holds;
- (v) for every neighbourhood V of the point $x \in X$ there exists a positive number r such that the ball $B(x, r) = \{y \in X : \rho(x, y) < r\}$ with center in x and radius r is contained in V ;
- (vi) the balls $B(x, r)$ are measurable for all $x \in X, r > 0$ and, in addition, $0 < \mu B(x, r) < \infty$.

The spaces (X, ρ, μ) with the above mentioned properties are called *non-homogeneous spaces*. We shall also assume that $\mu(X) = \infty, \mu\{a\} = 0$ for all $a \in X$; and $B(x, r_2) \setminus B(x, r_1) \neq \emptyset$ for all x, r_1 and r_2 ($x \in X, 0 < r_1 < r_2 < \infty$).

Let w be μ -a.e. positive function on X . We denote by $L_w^p(X)$ ($1 \leq p < \infty$) the weighted Lebesgue space which is the class of all μ -measurable functions $f : X \rightarrow R^1$, for which

$$\|f\|_{L_w^p(X)} = \left(\int_X |f(x)|^p w(x) d\mu(x) \right)^{\frac{1}{p}} < \infty.$$

If $w \equiv 1$, then instead of $L_w^p(X)$ we use the symbol $L^p(X)$.

We consider the integral operator of the form

$$I_\alpha f(x) = \int_X \frac{f(y)}{\rho(x, y)^{1-\alpha}} d\mu(y), \quad 0 < \alpha < 1.$$

The next statement is from [12] (see also [4, Theorem 6.1.1]).

Theorem A. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1$. Then the operator I_α is bounded from $L^p(X)$ into $L^q(X)$ if and only if there exists a positive constant $c > 0$ such that*

$$\mu B(x, r) \leq cr^\beta, \quad \beta = \frac{pq(1 - \alpha)}{pq + p - q},$$

for arbitrary balls $B(x, r)$.

In particular, from Theorem A it follows (see Corollary 6.1.1 of [4])

Theorem B. *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. Then I_α acts boundedly from $L^p(X)$ into $L^q(X)$ if and only if $\mu B(x, r) \leq cr$, where the constant c is independent of x and r .*

The latter statement by the different proof was also derived in [5] for metric spaces. We shall need the following Hardy-type transforms defined on X :

$$H_{x_0}f(x) = \int_{\{y:\rho(x_0,y)\leq\rho(x_0,x)\}} f(y) d\mu(y)$$

$$H'_{x_0}f(x) = \int_{\{y:\rho(x_0,y)\geq\rho(x_0,x)\}} f(y) d\mu(y),$$

where x_0 is a fixed point of X . The next statement is from [3] (see also [4, Section 1.1]).

Theorem C. *Let $1 < p \leq q < \infty$. Suppose that v and w are μ -a.e. positive functions on X . Then:*

(a) *The operator H_{x_0} is bounded from $L^p_w(X)$ to $L^q_v(X)$ if and only if*

$$A_1 \equiv \sup_{t \geq 0} \left(\int_{\{y:\rho(x_0,y)\geq t\}} v(y) d\mu(y) \right)^{\frac{1}{q}} \left(\int_{\{y:\rho(x_0,y)\leq t\}} w^{1-p'}(y) d\mu(y) \right)^{\frac{1}{p'}} < \infty,$$

$p' = p/(p - 1);$

(b) *The operator H'_{x_0} is bounded from $L^p_w(X)$ to $L^q_v(X)$ if and only if*

$$A_2 \equiv \sup_{t \geq 0} \left(\int_{\{y:\rho(x_0,y)\leq t\}} v(y) d\mu(y) \right)^{\frac{1}{q}} \left(\int_{\{y:\rho(x_0,y)\geq t\}} w^{1-p'}(y) d\mu(y) \right)^{\frac{1}{p'}} < \infty.$$

Moreover, there exist positive constants c_j , $j = 1, \dots, 4$, depending only on p and q such that $c_1 A_1 \leq \|H_{x_0}\| \leq c_2 A_1$ and $c_3 A_2 \leq \|H'_{x_0}\| \leq c_4 A_2$.

3. The main results

In this section we formulate the main results of this paper.

Theorem 3.1. *Let $1 < p \leq q < \infty$, $\frac{1}{p} - \frac{1}{q} \leq \alpha < 1$, $\alpha \neq \frac{1}{p}$. Suppose that $\alpha p - 1 < \beta < p - 1$ and $\lambda = q(\frac{1}{p} + \frac{\beta}{p} - \alpha) - 1$. Then the inequality*

$$\left(\int_X |I_\alpha f(x)|^q \rho(x_0, x)^\lambda d\mu(x) \right)^{\frac{1}{q}} \leq c \left(\int_X |f(x)|^p \rho(x_0, x)^\beta d\mu(x) \right)^{\frac{1}{p}}, \quad (1)$$

with the positive constant c independent of f and x_0 , $x_0 \in X$, holds if and only if

$$B \equiv \sup_{a \in X, r > 0} \frac{\mu B(a, r)}{r} < \infty. \quad (2)$$

Remark 3.2. It follows immediately from (2) that $\mu\{a\} = 0$ for all $a \in X$. Therefore for sufficiency of Theorem 3.1 we can omit the assumption that the measure μ has any atoms.

Remark 3.3. Note that if $1 < p < q < \infty$ and $0 < \alpha < \frac{1}{p} - \frac{1}{q}$, then from the two-weight inequality

$$\|K_\alpha f\|_{L_v^q(R)} \leq c \|f\|_{L_w^p(R)}, \tag{3}$$

for example, for the one-dimensional potential

$$K_\alpha f(x) = \int_R \frac{f(y)}{|x - y|^{1-\alpha}} dy,$$

it follows that $v^{\frac{1}{q}}(x)/w^{\frac{1}{p}}(x) = 0$ a.e. on R . Indeed, if (3) holds for some weight pair (v, w) , then putting the function $f = w^{1-p'}\chi_{(x-r, x+r)}$ in the inequality (3) we observe that

$$r^{\alpha - \frac{1}{p} + \frac{1}{q}} \left(\frac{1}{r} \int_{x-r}^{x+r} v \right)^{\frac{1}{q}} \left(\frac{1}{r} \int_{x-r}^{x+r} w^{1-p'} \right)^{\frac{1}{p'}} \leq c \tag{4}$$

for all $x \in R$ and $r > 0$. Passing r to 0 we see that (4) will not remain valid unless $v^{\frac{1}{q}}(x)/w^{\frac{1}{p}}(x) = 0$ a.e..

From Theorem 3.1 it is easy to obtain the following corollary for the operator

$$I_\alpha^{x_0} f(x) = \rho(x_0, x)^{-\alpha} \int_X \frac{f(y)}{\rho(x, y)^{1-\alpha}} d\mu(y).$$

Corollary 3.4. Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p}$. Then the inequality

$$\left(\int_X |I_\alpha^{x_0} f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \leq c \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, \tag{5}$$

where the positive constant c does not depend on x_0 and f , holds if and only if the measure μ satisfies the condition (2).

Theorem 3.1 can be also formulated in the following form:

Theorem 3.1'. Let n be a positive number. Suppose that $1 < p \leq q < \infty$, $\frac{n}{p} - \frac{n}{q} \leq \alpha < n$, $\alpha \neq \frac{n}{p}$, $\alpha p - n < \beta < n(p - 1)$ and $\lambda = q(\frac{n}{p} + \frac{\beta}{p} - \alpha) - n$. Then the inequality

$$\left(\int_X |J_\alpha f(x)|^q \rho(x_0, x)^\lambda d\mu(x) \right)^{\frac{1}{q}} \leq c \left(\int_X |f(x)|^p \rho(x_0, x)^\beta d\mu(x) \right)^{\frac{1}{p}}$$

for the operator

$$J_\alpha f(x) = \int_X \frac{f(y)}{\rho(x, y)^{n-\alpha}} d\mu(y),$$

with the positive constant c independent of f and x_0 , holds if and only if

$$\sup_{a \in X, r > 0} \frac{\mu B(a, r)}{r^n} < \infty.$$

From Corollary 3.4 we can derive

Proposition 3.5. *Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p}$. Then the operator I_α is bounded in $L^p(X)$ if*

$$D \equiv \sup_{a \in X; r > 0} \frac{\int_{B(a,r)} \rho(x_0, x)^\alpha d\mu(x)}{r} < \infty \tag{6}$$

for some point $x_0 \in X$. Further, if I_α is bounded in $L^p(X)$, then

$$D_1 \equiv \sup_{a \in X; r > 0} \frac{\int_{B(a,r)} \rho(a, x)^\alpha d\mu(x)}{r} < \infty. \tag{7}$$

We now apply the Theorems 3.1 and 3.1' to some special measure spaces. A non-negative Borel measure m on \mathbb{C} is called a *Radon measure* if m is finite on compact sets and

$$m(A) = \sup m(K) = \inf m(U)$$

for every Borel set A , where the supremum is taken over all compact sets $K \subset A$ and the infimum is over all open sets U containing A . We say that a Borel measure m on \mathbb{C} is a *Carleson measure* if m is a Radon measure and there exists a constant $C := C(m) \geq 0$ such that

$$m(D(z, \varepsilon)) \leq C\varepsilon$$

for all disks $D(z, \varepsilon) := \{\tau \in \mathbb{C} : |\tau - z| < \varepsilon\}$. For the definition and some examples of the Carleson measures see, e.g., [1, p. 185].

Proposition 3.6. *Let m be a Radon measure on \mathbb{C} . Suppose that $1 < p \leq q < \infty$, $\frac{1}{p} - \frac{1}{q} \leq \alpha < 1$, $\alpha \neq \frac{1}{p}$, $\alpha p - 1 < \beta < p - 1$ and $\lambda = q(\frac{1}{p} + \frac{\beta}{p} - \alpha) - 1$. Then the two-weight inequality*

$$\left(\int_{\mathbb{C}} |K^\alpha f(z)|^q |z - z_0|^\lambda dm(z) \right)^{\frac{1}{q}} \leq c \left(\int_{\mathbb{C}} |f(z)|^p |z - z_0|^\beta dm(z) \right)^{\frac{1}{p}}$$

for the operator

$$K^\alpha f(z) = \int_{\mathbb{C}} \frac{f(\zeta)}{|\zeta - z|^{1-\alpha}} dm(\zeta),$$

with the positive constant c independent of f and z_0 , $z_0 \in \mathbb{C}$, holds if and only if m is a Carleson measure.

Let γ be a simple locally rectifiable curve in the plane and let ν be a measure on γ given by

$$\nu(A) := |\gamma \cap A|,$$

where $|\gamma \cap A|$ is a length of $\gamma \cap A$. Then ν is a Carleson measure if and only if γ is a regular (Carleson) curve, i.e., there exists a positive constant c such that the inequality

$$\nu(\gamma \cap D(z, r)) \leq cr$$

holds for all $z \in \mathbb{C}$ and $r > 0$. For r smaller than half the diameter of γ , the reverse inequality

$$\nu(\gamma \cap D(z, r)) \geq r$$

holds for all $z \in \mathbb{C}$. Note that there exist nonregular curves (see, e.g., [1, pp. 5/6]).

We have the next statement for the operator

$$K_\gamma^\alpha f(z) = \int_\gamma \frac{f(t)}{|z - t|^{1-\alpha}} d\nu(t), \quad 0 < \alpha < 1.$$

Proposition 3.7. *Let γ be a regular curve with $\nu(\gamma) = \infty$. Suppose that $1 < p \leq q < \infty$, $\frac{1}{p} - \frac{1}{q} \leq \alpha < 1$, $\alpha \neq \frac{1}{p}$, $\alpha p - 1 < \beta < p - 1$ and $\lambda = q(\frac{1}{p} + \frac{\beta}{p} - \alpha) - 1$. Then there exists a positive constant c such that the inequality*

$$\left(\int_\gamma |K_\gamma^\alpha f(z)|^q |z - z_0|^\lambda d\nu(z) \right)^{\frac{1}{q}} \leq c \left(\int_\gamma |f(z)|^p |z - z_0|^\beta d\nu(z) \right)^{\frac{1}{p}},$$

holds for all $z_0 \in \gamma$ and f .

Now we consider the case of s -sets. Let Γ be a subset of R^n which is an s -set ($0 \leq s \leq n$) in the sense that there is a Borel measure μ on R^n such that

- (a) $\text{supp } \mu = \Gamma$;
- (b) there are positive constants c_1 and c_2 such that for all $x \in \Gamma$ and all $r \in (0, 1)$, $c_1 r^s \leq \mu(B(x, r) \cap \Gamma) \leq c_2 r^s$.

It is known (see [16, Theorem 3.4]) that μ is equivalent to the restriction of the Hausdorff s -measure \mathcal{H}_s to Γ . We shall thus identify μ with $\mathcal{H}_s|_\Gamma$.

Given $x \in \Gamma$, put $\Gamma(x, r) = B(x, r) \cap \Gamma$. Let

$$K_\Gamma^\alpha f(x) = \int_\Gamma \frac{f(y)}{|x - y|^{s-\alpha}} d\mathcal{H}_s, \quad 0 < \alpha < s.$$

Proposition 3.8. *Let $1 < p \leq q < \infty$, $\frac{s}{p} - \frac{s}{q} \leq \alpha < s$, $\alpha \neq \frac{s}{p}$, $\alpha p - s < \beta < s(p - 1)$ and $\lambda = q(\frac{s}{p} + \frac{\beta}{p} - \alpha) - s$. Then the inequality*

$$\left(\int_\Gamma |K_\Gamma^\alpha f(x)|^q |x - x_0|^\lambda d\mathcal{H}_s(x) \right)^{\frac{1}{q}} \leq c \left(\int_\Gamma |f(x)|^p |x - x_0|^\beta d\mathcal{H}_s(x) \right)^{\frac{1}{p}},$$

with the positive constant c independent of f and x_0 , holds.

Note that since the Cantor set in R^n is an s -set (see [16, 4.9]), where

$$s = \frac{\log(3^n - 1)}{\log 3},$$

we can obtain two-weighted estimate for potentials on a Cantor set in R^n . For the theorem of Sobolev type and other weighted results for fractional integrals defined on curves and s -sets see [10, 11].

4. Proof of the main results

We now are ready to prove the main results.

Proof of Theorem 3.1. Necessity. Let us put the function $f_{x_0,r}(x) = \chi_{B(x_0,r) \setminus B(x_0,r/2)}(x)$ in (1). Then it is easy to see that

$$\begin{aligned} & \left(\int_X |I_\alpha f_{x_0,r}(x)|^q \rho(x_0, x)^\lambda d\mu(x) \right)^{\frac{1}{q}} \\ & \geq \left(\int_{B(x_0,r) \setminus B(x_0,r/2)} (I_\alpha f_{x_0,r}(x))^q \rho(x_0, x)^\lambda d\mu(x) \right)^{\frac{1}{q}} \\ & \geq cr^{\frac{\lambda}{q} + \alpha - 1} (\mu(B(x_0, r) \setminus B(x_0, r/2)))^{1 + \frac{1}{q}}. \end{aligned}$$

On the other hand,

$$\left(\int_X |f(x)|^p \rho(x_0, x)^\beta d\mu(x) \right)^{\frac{1}{p}} \leq c (\mu(B(x_0, r) \setminus B(x_0, r/2)))^{\frac{1}{p}} r^{\frac{\beta}{p}}.$$

Summarizing these estimates and taking into account that the inequality (1) is independent of x_0 and r , we have

$$(\mu(B(x_0, r) \setminus B(x_0, r/2)))^{1 + \frac{1}{q} - \frac{1}{p}} r^{\frac{\lambda}{q} + \alpha - 1 - \frac{\beta}{p}} \leq c.$$

By the condition of the theorem we have $\frac{\lambda}{q} + \alpha - 1 - \frac{\beta}{p} = -1 - \frac{1}{q} + \frac{1}{p}$. Consequently $\frac{1}{r} \mu(B(x_0, r) \setminus B(x_0, r/2)) \leq c$. The latter inequality yields

$$\mu B(x_0, r) = \sum_{k=-\infty}^0 \mu(B(x_0, 2^k r) \setminus B(x_0, 2^{k-1} r)) \leq c \sum_{k=-\infty}^0 2^k r = 2cr.$$

Sufficiency. Let $f \geq 0$. Let us introduce the following notation:

$$\begin{aligned} E_1(x) & \equiv \left\{ y : \rho(x_0, y) < \frac{\rho(x_0, x)}{2a_1} \right\} \\ E_2(x) & \equiv \left\{ y : \frac{\rho(x_0, x)}{2a_1} \leq \rho(x_0, y) \leq 2a_1 \rho(x_0, x) \right\} \\ E_3(x) & \equiv \{ y : \rho(x_0, y) > 2a_1 \rho(x_0, x) \}. \end{aligned}$$

We have

$$\begin{aligned} & \int_X \rho(x_0, x)^\lambda (I_\alpha f(x))^q d\mu(x) \\ & \leq c \int_X \rho(x_0, x)^\lambda \left(\int_{E_1(x)} f(y) \rho(x, y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ & \quad + c \int_X \rho(x_0, x)^\lambda \left(\int_{E_2(x)} f(y) \rho(x, y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ & \quad + c \int_X \rho(x_0, x)^\lambda \left(\int_{E_3(x)} f(y) \rho(x, y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ & \equiv I_1 + I_2 + I_3. \end{aligned}$$

It is easy to verify that if $\rho(x_0, y) < \frac{\rho(x_0, x)}{2a_1}$, then

$$\rho(x_0, x) \leq a_1 \rho(x_0, y) + a_0 a_1 \rho(x, y) \leq \frac{\rho(x_0, x)}{2} + a_1 a_0 \rho(x, y).$$

Hence $\frac{\rho(x_0, x)}{2a_1 a_0} \leq \rho(x, y)$. Consequently,

$$I_1 \leq c \int_X \rho(x_0, x)^{\lambda+(\alpha-1)q} (H_{x_0} f(x))^q d\mu(x).$$

Further, taking into account the inequality $\lambda < (1 - \alpha)q - 1$ we have

$$\begin{aligned} \int_{\rho(x_0, x) \geq t} \rho(x_0, x)^{\lambda+(\alpha-1)q} d\mu(x) &= \sum_{k=0}^{+\infty} \int_{2^k t \leq \rho(x_0, x) < 2^{k+1} t} \rho(x_0, x)^{\lambda+(\alpha-1)q} d\mu(x) \\ &\leq cB \sum_{k=0}^{+\infty} (2^k t)^{\lambda+(\alpha-1)q+1} \\ &= (1 - 2^{\lambda+(\alpha-1)q+1})^{-1} cB t^{\lambda+(\alpha-1)q+1}, \end{aligned}$$

where the positive constant c depends only on α , λ and q . Analogously by virtue of the condition $\beta < p - 1$ it follows that

$$\int_{\rho(x_0, x) \leq t} \rho(x_0, x)^{\beta(1-p')} \leq cB t^{\beta(1-p')+1}.$$

Summarizing these estimates we find that

$$\begin{aligned} & \sup_{t>0} \left\{ \left(\int_{\rho(x_0, x) \geq t} \rho(x_0, x)^{\lambda+(\alpha-1)q} d\mu(x) \right)^{\frac{1}{q}} \right. \\ & \quad \left. \times \left(\int_{\rho(x_0, x) \leq t} \rho(x_0, x)^{\beta(1-p')} d\mu(x) \right)^{\frac{1}{p'}} \right\} \leq cB^{\frac{1}{q} + \frac{1}{p'}}, \end{aligned}$$

with the positive constant c independent of x_0 and f . Here we used the condition $\lambda = q(\frac{1}{p} + \frac{\beta}{p} - \alpha) - 1$. Now the first part of Theorem C leads us to the inequality

$$I_1 \leq b_1 \left(\int_X \rho(x_0, y)^\beta (f(y))^p d\mu(y) \right)^{\frac{q}{p}},$$

where the positive constant b_1 is independent of x_0 and f .

Repeating these arguments for I_3 and using the second part of Theorem C we derive the next estimate:

$$I_3 \leq b_2 \left(\int_X \rho(x_0, y)^\beta (f(y))^p d\mu(y) \right)^{\frac{q}{p}},$$

with the positive constant b_2 independent of x_0 and f .

To estimate I_2 we consider the cases $\alpha < \frac{1}{p}$ and $\alpha > \frac{1}{p}$ separately.

The case $\alpha < \frac{1}{p}$. In this case the condition $\alpha \geq \frac{1}{p} - \frac{1}{q}$ implies $q \leq p^*$, where $p^* = p/(1 - \alpha p)$. First assume that $q < p^*$. In the sequel we use the notation

$$F_k \equiv \{x : 2^k \leq \rho(x_0, x) < 2^{k+1}\}$$

$$\bar{F}_k \equiv \left\{y : \frac{1}{a_1} 2^{k-2} \leq \rho(x_0, y) < a_1 2^{k+2}\right\}.$$

By Hölder’s inequality with respect to the exponent $\frac{p^*}{q}$ and Theorem B we find that

$$\begin{aligned} I_2 &= \int_X \rho(x_0, x)^\lambda \left(\int_{E_2(x)} f(y) \rho(x, y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ &= \sum_{k \in Z} \int_{F_k} \rho(x_0, x)^\lambda \left(\int_{E_2(x)} f(y) \rho(x, y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ &\leq \sum_{k \in Z} \left(\int_{F_k} \left(\int_{E_2(x)} f(y) \rho(x, y)^{\alpha-1} d\mu(y) \right)^{p^*} d\mu(x) \right)^{\frac{q}{p^*}} \\ &\quad \times \left(\int_{F_k} \rho(x_0, x)^{\frac{\lambda p^*}{p^*-q}} d\mu(x) \right)^{\frac{p^*-q}{p^*}} \\ &\leq c B^{\frac{p^*-q}{p^*}} \sum_{k \in Z} 2^{k(\lambda + \frac{p^*-q}{p^*})} \left(\int_X (I_\alpha(f \chi_{\bar{F}_k}))(x)^{p^*} d\mu(x) \right)^{\frac{q}{p^*}} \\ &\leq c \sum_{k \in Z} 2^{k(\lambda + \frac{p^*-q}{p^*})} \left(\int_{\bar{F}_k} f^p(y) d\mu(y) \right)^{\frac{q}{p}} \\ &\leq c \left(\int_X \rho(x_0, x)^\beta f^p(x) d\mu(x) \right)^{\frac{q}{p}}. \end{aligned}$$

If $q = p^*$, then $\lambda = \frac{\beta p^*}{p}$ and consequently using directly Theorem B we have

$$\begin{aligned} I_2 &\leq c \sum_{k \in Z} 2^{k\beta \frac{p^*}{p}} \int_{F_k} (I_\alpha(f \chi_{\bar{F}_k})(x))^{p^*} d\mu(x) \\ &\leq c \sum_{k \in Z} 2^{k\beta \frac{p^*}{p}} \left(\int_{\bar{F}_k} f(y)^p d\mu(y) \right)^{\frac{p^*}{p}} \\ &\leq c \left(\int_X \rho(x_0, y)^\beta f(y)^p d\mu(y) \right)^{\frac{p^*}{p}}. \end{aligned}$$

The case $\alpha > \frac{1}{p}$. In this case by Hölder's inequality with respect to the exponent p we get the following estimate

$$I_2 \leq \int_X \rho(x_0, x)^\lambda \left(\int_{E_2(x)} f^p(y) d\mu(y) \right)^{\frac{q}{p}} \left(\int_{E_2(x)} \rho(x, y)^{(\alpha-1)p'} d\mu(y) \right)^{\frac{q}{p'}} d\mu(x).$$

On the other hand, using (2) and the inequality $\alpha > \frac{1}{p}$ we observe that

$$\begin{aligned} \int_{E_2(x)} \rho(x, y)^{(\alpha-1)p'} d\mu(y) &\leq \int_0^\infty \mu \left(B(x_0, \rho(x_0, x)) \cap \left\{ y : \rho(x, y) < \lambda^{\frac{1}{(\alpha-1)p'}} \right\} \right) d\lambda \\ &\leq B \rho(x_0, x)^{1+(\alpha-1)p'} + B \int_{\rho(x_0, x)^{(\alpha-1)p'}}^\infty \lambda^{\frac{1}{(\alpha-1)p'}} d\lambda \\ &= cB \rho(x_0, x)^{1+(\alpha-1)p'}, \end{aligned}$$

where the positive constant c does not depend on x and x_0 . The latter estimate yields

$$\begin{aligned} I_2 &\leq cB^{\frac{q}{p'}} \sum_{k \in Z} \int_{F_k} \rho(x_0, x)^{\lambda + [(\alpha-1)p' + 1] \frac{q}{p'}} \left(\int_{E_2(x)} (f(y))^p d\mu(y) \right)^{\frac{q}{p}} d\mu(x) \\ &\leq cB^{\frac{q}{p'}} \sum_{k \in Z} \int_{F_k} \rho(x_0, x)^{\lambda + [(\alpha-1)p' + 1] \frac{q}{p'}} d\mu(x) \left(\int_{\bar{F}_k} (f(y))^p d\mu(y) \right)^{\frac{q}{p}} \\ &\leq cB^{\frac{q}{p'} + 1} \sum_{k \in Z} 2^{k(\lambda + [(\alpha-1)p' + 1] \frac{q}{p'} + 1)} \left(\int_{\bar{F}_k} (f(y))^p d\mu(y) \right)^{\frac{q}{p}} \\ &= c \sum_{k \in Z} 2^{k \frac{\beta q}{p}} \left(\int_{\bar{F}_k} (f(y))^p d\mu(y) \right)^{\frac{q}{p}} \\ &\leq c \left(\int_X \rho(x_0, y)^\beta (f(y))^p d\mu(y) \right)^{\frac{q}{p}}. \end{aligned}$$

Theorem 3.1 is completely proved. ■

Proof of Theorem 3.1'. The proof immediately follows applying Theorem 3.1 to the nonhomogeneous space (X, ρ_1, μ) , where $\rho_1(x, y) = \rho(x, y)^n$. ■

To prove Proposition 3.5 we need

Lemma 4.1. *Let $1 < p < \infty$. The operator $I_{\alpha, x_0, p}$ is bounded in $L^p(X)$ if*

$$D(x_0) \equiv \sup_{a \in X; r > 0} \frac{\int_{B(a, r)} \rho(x_0, x)^{\alpha p} d\mu(x)}{r} < \infty. \tag{8}$$

Proof. First note that the boundedness of $I_{\alpha, x_0, p}$ in $L^p(X)$ is equivalent to the weighted inequality

$$\begin{aligned} S &\equiv \int_X \left(\int_X f(y) \rho(x, y)^{\alpha-1} \rho(x_0, y)^{\alpha p} d\mu(y) \right)^p d\mu(x) \\ &\leq c \int_X (f(y))^p \rho(x_0, y)^{\alpha p} d\mu(y), \quad (f \geq 0). \end{aligned}$$

The latter inequality follows easily applying (8) and Corollary 3.4 to the space (X, ρ, μ_1) , $d\mu_1(x) \equiv \rho(x_0, x)^{\alpha p} d\mu(x)$, because

$$\begin{aligned} S &= \int_X \frac{\rho(x_0, x)^{\alpha p}}{\rho(x_0, x)^{\alpha p}} \left(\int_X f(y) \rho(x, y)^{\alpha-1} \rho(x_0, y)^{\alpha p} d\mu(y) \right)^p d\mu(x) \\ &\leq c \int_X (f(y))^p \rho(x_0, y)^{\alpha p} d\mu(y). \end{aligned} \quad \blacksquare$$

Proof of Proposition 3.5. Let us take a point $x_0 \in X$ and consider the operator

$$I_{\alpha, x_0, p} f(x) = \int_X \rho(x_0, x)^{\alpha(p-1)} \rho(x, y)^{\alpha-1} f(y) d\mu(y).$$

Further, due to condition (6) we have

$$S_1 \equiv \int_X \rho(x_0, x)^{\alpha-\alpha p} (I_\alpha f(x))^p d\mu(x) \leq c \int_X \rho(x_0, x)^{\alpha-\alpha p} (f(x))^p d\mu(x), \quad (f \geq 0).$$

Indeed, Lemma 4.1 with respect to the nonhomogeneous space (X, ρ, μ_2) , $d\mu_2(x) = \rho(x_0, x)^{\alpha-\alpha p} d\mu(x)$, yields

$$\begin{aligned} S_1 &= \int_X \rho(x_0, x)^{\alpha-\alpha p} \left(\int_X f(y) \rho(x, y)^{\alpha-1} \frac{\rho(x_0, y)^{\alpha p-\alpha}}{\rho(x_0, y)^{\alpha p-\alpha}} d\mu(y) \right)^p d\mu(x) \\ &\leq \int_X \rho(x_0, x)^{\alpha-\alpha p} (f(x))^p d\mu(x), \quad (f \geq 0). \end{aligned}$$

The latter inequality can be rewritten in the form

$$\int_X \rho(x_0, x)^{\alpha(1-p)} (I_\alpha (f \rho(x_0, \cdot)^{\frac{\alpha}{p}})(x))^p d\mu(x) \leq c \int_X (f(x))^p d\mu(x). \tag{9}$$

Consequently, using the notation from the proof of Theorem 3.1 we have

$$\begin{aligned} \int_X (I_\alpha f(x))^p d\mu(x) &\leq c \int_X \left(\int_{E_1(x)} f(y)\rho(x,y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ &\quad + c \int_X \left(\int_{E_2(x)} f(y)\rho(x,y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ &\quad + c \int_X \left(\int_{E_3(x)} f(y)\rho(x,y)^{\alpha-1} d\mu(y) \right)^q d\mu(x) \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Further, condition (6) implies

$$D_1(x_0) \equiv \sup_{r>0} \frac{\mu(B(x_0, r) \setminus B(x_0, r/2))}{r^{1-\alpha}} < \infty.$$

Besides, it is easy to check that

$$\begin{aligned} \sup_{t \geq 0} \left(\int_{\rho(x_0, x) \geq t} \rho(x_0, x)^{(\alpha-1)p} d\mu(x) \right)^{\frac{1}{p}} (\mu B(x_0, t))^{\frac{1}{p'}} &\leq c_1 D_1(x_0) \\ \sup_{t \geq 0} \left(\int_{\rho(x_0, x) \geq t} \rho(x_0, x)^{(\alpha-1)p'} d\mu(x) \right)^{\frac{1}{p'}} (\mu B(x_0, t))^{\frac{1}{p}} &\leq c_2 D_1(x_0). \end{aligned}$$

Let us show the first inequality.

$$\begin{aligned} &\left(\int_{\rho(x_0, x) \geq t} \rho(x_0, x)^{(\alpha-1)p} d\mu(x) \right)^{\frac{1}{p}} (\mu B(x_0, t))^{\frac{1}{p'}} \\ &= \left(\sum_{k=0}^{\infty} \int_{2^k t \leq \rho(x_0, x) < 2^{k+1} t} \rho(x_0, x)^{(\alpha-1)p} d\mu(x) \right)^{\frac{1}{p}} (\mu B(x_0, t))^{\frac{1}{p'}} \\ &\leq c \left(\sum_{k=0}^{\infty} (t2^k)^{(\alpha-1)p} \mu\{x : 2^k t \leq \rho(x_0, x) < 2^{k+1} t\} \right)^{\frac{1}{p}} (\mu B(x_0, t))^{\frac{1}{p'}} \\ &\leq c(D_1(x_0))^{\frac{1}{p}} \left(\sum_{k=0}^{\infty} (t2^k)^{(\alpha-1)(p-1)} \right)^{\frac{1}{p}} \mu B(x_0, t)^{\frac{1}{p'}} \\ &= c(D_1(x_0))^{\frac{1}{p}} t^{\frac{\alpha-1}{p'}} (\mu B(x_0, t))^{\frac{1}{p'}} \\ &= c(D_1(x_0))^{\frac{1}{p}} t^{\frac{\alpha-1}{p'}} \left(\sum_{k=-\infty}^0 \mu(B(x_0, 2^k t) \setminus B(x_0, 2^{k-1} t)) \right)^{\frac{1}{p'}} \\ &\leq cD_1(x_0) t^{\frac{\alpha-1}{p'}} \left(\sum_{k=-\infty}^0 (t2^k)^{1-\alpha} \right)^{\frac{1}{p'}} \\ &= (1 - 2^{\alpha-1})^{-\frac{1}{p'}} cD_1(x_0). \end{aligned}$$

Analogously, the second inequality follows. Using now Theorem C we find

$$\begin{aligned}
 I_1 &\leq c \int_X \rho(x_0, x)^{(\alpha-1)p} \left(\int_{\{y:\rho(x_0,y)\leq\rho(x_0,x)\}} f(y) d\mu(y) \right)^p d\mu(x) \\
 &\leq c \int_X f(y)^p d\mu(y) \\
 I_3 &\leq c \int_X \left(\int_{\{y:\rho(x_0,y)\geq\rho(x_0,x)\}} \rho(x_0, y)^{\alpha-1} f(y) d\mu(y) \right)^p d\mu(x) \\
 &\leq c \int_X f(y)^p d\mu(y).
 \end{aligned}$$

To estimate I_2 we use (9). We have

$$\begin{aligned}
 I_2 &\equiv \int_X \left(\int_{E_2(x)} f(y) \rho(x, y)^{\alpha-1} d\mu(y) \right)^p d\mu(x) \\
 &= c \sum_{k \in Z} \int_{F_k} \rho(x_0, x)^{\alpha(1-p)} \left(\int_{E_2(x)} f(y) \rho(x_0, y)^{\frac{\alpha}{p}} \rho(x, y)^{\alpha-1} d\mu(y) \right)^p d\mu(x) \\
 &\leq c \sum_{k \in Z} \int_{\bar{F}_k} f(y)^p d\mu(y) \\
 &\leq c \int_X f(y)^p d\mu(y).
 \end{aligned}$$

For the necessity of Proposition 3.5 we put the function $f_a(x) = \chi_{B(a,r)}(x)$ in the inequality $\|I_\alpha f\|_{L^p(X)} \leq c \|f\|_{L^p(X)}$. Consequently, we observe that

$$\bar{D} \equiv \sup_{a \in X, r > 0} \frac{\mu B(a, r)}{r^{1-\alpha}} < \infty.$$

Further, it is clear that

$$\int_{B(a,r)} \rho(a, x)^\alpha d\mu(x) \leq r \bar{D}.$$

Proposition 3.5 is proved. ■

Acknowledgement. The second author expresses his gratitude to Professor Fulvio Ricci for his stimulating and valuable conversations. Both authors thank the referee for helpful remarks and suggestions.

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Received 14.05.2004; in revised form 12.02.2005