Recurrence Relations for the Lerch Φ Function and Applications

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Abstract. In this paper we present a simple method for deriving recurrence relations and we apply it to obtain two equations involving the Lerch Phi function and sums of Bernoulli and Euler polynomials. Connections between these results and those obtained by H. M. Srivastava, M. L. Glasser and V. Adamchik [Z. Anal. Anwendungen 19 (2000), 831 – 846] are pointed out, emphasizing the usefulness of this approach with some meaningful examples.

Keywords: *Lerch Phi, Riemann Zeta, Dirichlet Beta, Euler polynomials, Bernoulli polynomials, series representations*

MSC 2000: Primary 11M35, secondary 11B68,11M06

1. Introduction

In recent years the properties of the Riemann Zeta function for positive integer values of its argument have received a lot of attention. Rapidly convergent series representation such as Euler's well-known one

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}$$

have been recently discovered by several authors in many different ways (see [6] for an exhaustive overview).

In this context, H. M. Srivastava, M. L. Glasser and V. S. Adamchik ([8]) found some interesting series representations of the values $\zeta(2n+1)$, $n \in \mathbb{N}$, studying different possible evaluations of the definite integral

$$I_s(\omega) = \int_0^{\frac{\pi}{\omega}} t^s \csc^2 t \, dt$$

$$= -\left(\frac{\pi}{\omega}\right)^s \cot\left(\frac{\pi}{\omega}\right) + \int_0^{\frac{\pi}{\omega}} t^{s-1} \cot t \, dt \qquad (\Re(s) > 1; \ \omega > 1) \, .$$

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At the end of their article, the authors demonstrate that every representation
they found can be derived from the unification formula

$$\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+p)\omega^{2k}}$$

$$= \frac{\pi i}{2(p+1)\omega} - \frac{1}{2}\log\left(1 - e^{\frac{2\pi i}{\omega}}\right) - \frac{p!}{2}\left(\frac{i\omega}{2\pi}\right)^{p}\zeta(p+1) \qquad (1.1)$$

$$+ \frac{1}{2}\sum_{k=1}^{p}\binom{p}{k}k!\left(\frac{i\omega}{2\pi}\right)^{k}\mathrm{Li}_{k}\left(e^{\frac{2\pi i}{\omega}}\right) \qquad (p \in \mathbb{N} : \ \omega \in \mathbb{R}, \ |\omega| > 1).$$

This paper presents a very simple method for deriving some recurrence
relations and shows how it can be used as a "short cut" to obtain two formulas
that generalize equation (1.1). The usefulness of the obtained results is then
shown by deducting in a simple and unified way many known (or equivalent to
known) evaluations and series representations for the Riemann Zeta function
and the Dirichlet Beta function.

2. Notations

This section presents the notation used in the next sections. The *Euler Gamma
function* $\Gamma(s)$ is defined as the analytic continuation of the integral

$$\Gamma(s) = \int_{0}^{\infty} t^{s-1}e^{-t}dt \qquad (s > 1)$$

and, for $n \in \mathbb{N}$, satisfies the property $\Gamma(n+1) = n!$. For $s \in \mathbb{C}$ let $\Re(s)$ be
the real part of s. The *Hurwitz Zeta function* $\zeta(s, b)$ is defined as the analytic
continuation of the series

$$\zeta(s, b) = \sum_{k=0}^{\infty}\frac{1}{(k+b)^{s}} \qquad (\Re(s) > 1, \ \Re(b) > 0),$$

and it reduces to the *Riemann Zeta function* $\zeta(s)$ in the case $b = 1$. Similarly,
the *Dirichlet Beta function* $\beta(s)$ is the analytic continuation of the series

$$\beta(s) = \sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k+1)^{s}} \qquad (\Re(s) > 0).$$

For $a \in \mathbb{C}$ with $|a| \leq 1$, taken $s \in \mathbb{C}$ satisfying $\Re(s) > 1$ if $a = 1$ and $\Re(s) > 0$
if $|a| = 1 \wedge a \neq 1$, the *Polylogarithm function* is defined by

$$\mathrm{Li}_{s}(a) = \sum_{k=1}^{\infty}\frac{a^{k}}{k^{s}} \qquad (|a| \leq 1)$$

and the *Lerch* Φ *function* is defined by

$$\Phi(a, s, b) = \sum_{k=0}^{\infty} \frac{a^k}{(k+b)^s} \qquad (\Re(b) > 0).$$

Furthermore we use *Bernoulli polynomials* $B_k(x)$ defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \qquad (|t| < 2\pi) \tag{2.1}$$

and *Euler polynomials* $E_k(x)$ defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!} \qquad (|t| < \pi). \tag{2.2}$$

The $B_k(0)$ values, called Bernoulli numbers, are represented as B_k, while Euler numbers are the $2^k E_k(\frac{1}{2})$ values, represented as E_k.

Two important (and well known) results ([1]) that relate Bernoulli and Euler numbers with the values $\zeta(2k)$ and $\beta(2k+1)$, where $k \in \mathbb{N}$, are the relations

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!} \tag{2.3}$$

$$\beta(2k+1) = \frac{(-1)^k(\frac{\pi}{2})^{2k+1}E_{2k}}{2(2k)!}. \tag{2.4}$$

Finally, the basic notions of complex variable analysis will be used.

3. Holomorphic functions in a strip

In this section some simple propositions used for the results of Section 4 are given. We start with the following one.

Proposition 3.1. *Let α and β be positive real numbers and let $f(z)$ be an holomorphic function in the strip $S = \{z \in \mathbb{C} : \Re(z) \geq 0, \Im(z) \in (-\alpha, \beta)\}$ such that $f(z) = O(z^\nu)$, where $\nu < 0$, if $\Re(z) \to \infty$. Then:*

$$\int_0^\infty f(t)\,dt - \int_0^\infty f(t + i\varphi)\,dt = i\int_0^\varphi f(it)\,dt, \qquad \varphi \in (-\alpha, \beta). \tag{3.1}$$

Proof. Given $\varphi \in (-\alpha, \beta)$, take $R \in \mathbb{R}$, with $R > 0$, and consider the rectangular contour C of vertices 0, R, $R + i\varphi$ and $i\varphi$. For Cauchy theorem the integral of f over C must be zero; taking the limit for $R \to \infty$, the integral over the right vertical side tends to zero and, parameterizing the integrals over the other three sides, we get (3.1). ∎

For the sake of clarity and notation compactness we give the following definition.

Definition 3.2. Given a function $g(z)$, then $M(g)(z)$ is its *Mellin transform*, that is

$$M(g)(z) = \int_0^\infty t^{z-1} g(t)\, dt \tag{3.2}$$

for those values of z for which the integral exists. Furthermore, we indicate with $g_\varphi(z)$ its translated of a value $i\varphi$ in the domain, that is

$$g_\varphi(z) = g(z + i\varphi). \tag{3.3}$$

We now state the main proposition used in Section 4.

Proposition 3.3. *Let α and β be positive real numbers and let $g(z)$ be an holomorphic function in the strip $S = \{z \in \mathbb{C} \ : \ \Re(z) \geq 0, \Im(z) \in (-\alpha, \beta)\}$ such that $g(z) = o(z^{-k})$ for all $k \in \mathbb{N}$ if $\Re(z) \to \infty$. Then, for every integer $n \geq 0$ and $\varphi \in (-\alpha, \beta)$*

$$M(g)(n+1) - \sum_{k=0}^n \binom{n}{k} (i\varphi)^{n-k} M(g_\varphi)(k+1) = i \int_0^\varphi (it)^n g(it)\, dt \tag{3.4}$$

Moreover, if $g(z)$ is not holomorphic in $z = 0$, having a pole of order p, the equation holds for every $n \geq p$.

Proof. Let us consider $f(z)$ defined as $f(z) = z^n g(z)$, where n is chosen as specified in the above Proposition. Then $f(z)$ satisfies the hypothesis of Proposition 3.1 and, in view of (3.3),

$$f(z + i\varphi) = (z + i\varphi)^n g_\varphi(z).$$

For the binomial theorem, we have

$$f(z + i\varphi) = \sum_{k=0}^n \binom{n}{k} (i\varphi)^{n-k} z^k g_\varphi(z).$$

Substituting this in (3.1), in view of (3.2), noting that the Mellin transforms are well defined, we obtain (3.4). ∎

4. Recurrence relations for the Lerch Φ function

In this section the main result of the paper is given; after recalling an important Mellin transform we present two recurrence relations involving the Lerch Φ function, generalizing (1.1).

4.1. An important transform. Given $a, b \in \mathbb{C}$ such that $|a| \leq 1$ and $\Re(b) > 0$, let us consider the function of the complex variable t

$$\eta(t) = \frac{e^{(1-b)t}}{e^t - a}.$$

Its Mellin transform (as defined in (3.2)) is

$$M(\eta)(s) = \int_0^\infty \frac{t^{s-1} e^{(1-b)t}}{e^t - a} \, dt.$$

We note that the integral exists if $\Re(s) > 1$ when $a = 1$ and if $\Re(s) > 0$ in the other cases. We can write

$$
\begin{aligned}
M(\eta)(s) &= \int_0^\infty \frac{t^{s-1} e^{-bt}}{1 - ae^{-t}} \, dt \\
&= \int_0^\infty t^{s-1} e^{-bt} \sum_{k=0}^\infty (ae^{-t})^k \, dt \\
&= \sum_{k=0}^\infty a^k \int_0^\infty t^{s-1} e^{-(k+b)t} \, dt \\
&= \sum_{k=0}^\infty \frac{a^k \Gamma(s)}{(k+b)^s}
\end{aligned}
$$

so that we have (cf., for example, [7, pg. 121, eq. (4)])

$$M(\eta)(s) = \Gamma(s)\Phi(a, s, b)$$

$$(|a| \leq 1; \ \Re(b) > 0; \ \Re(s) > 1 \text{ when } a = 1, \ \Re(s) > 0 \text{ otherwise}).$$

$$(4.1)$$

4.2. Lerch Φ and Bernoulli polynomials series. Referring to equation (3.4) let us consider, for $b \in \mathbb{C}$ with $\Re(b) > 0$, the function of the complex variable z

$$g(z) = \frac{e^{(1-b)z}}{e^z - 1},$$

for which we have

$$g_\varphi(z) = e^{-ib\varphi} \frac{e^{(1-b)z}}{e^z - e^{-i\varphi}}.$$

Using (4.1), considering that $\Phi(1, s, b) = \zeta(s, b)$, we easily find that

$$M(g)(n+1) = n! \, \zeta(n+1, b) \qquad \text{for } n \geq 1$$
$$M(g_\varphi)(k+1) = e^{-ib\varphi} k! \, \Phi(e^{-i\varphi}, k+1, b) \qquad \text{for } k \geq 0 \text{ and } 0 < |\varphi| < 2\pi.$$

Furthermore, from (2.1) we have

$$i \int_0^\varphi (it)^n g(it)\, dt = i \int_0^\varphi (it)^{n-1} \sum_{k=0}^\infty \frac{B_k(1-b)(it)^k}{k!}\, dt$$

$$= (i\varphi)^n \sum_{k=0}^\infty \frac{B_k(1-b)(i\varphi)^k}{k!\,(k+n)} \qquad (n \geq 1,\ |\varphi| < 2\pi),$$

so that (3.4) becomes (multiplying both sides by $(i\varphi)^{-n}$)

$$n!\,(i\varphi)^{-n}\zeta(n+1,b) - e^{-ib\varphi} \sum_{k=0}^n \binom{n}{k}(i\varphi)^{-k}k!\,\Phi(e^{-i\varphi}, k+1, b)$$

$$= \sum_{k=0}^\infty \frac{B_k(1-b)(i\varphi)^k}{k!\,(k+n)} \qquad (n \geq 1,\ 0 < |\varphi| < 2\pi, \Re(b) > 0) \tag{4.2}$$

Some applications of this result, as well as its connection with (1.1), will be discussed in Section 5.

4.3. Lerch Φ and Euler polynomials series. For $\Re(b) > 0$, let us now consider the function of the complex variable z

$$g(z) = \frac{e^{(1-b)z}}{e^z + 1},$$

so that we have

$$g_\varphi(z) = e^{-ib\varphi} \frac{e^{(1-b)z}}{e^z + e^{-i\varphi}}.$$

In view of (4.1),

$$M(g)(n+1) = n!\,\Phi(-1, n+1, b) \qquad\qquad \text{for } n \geq 0$$

$$M(g_\varphi)(k+1) = e^{-ib\varphi}k!\,\Phi(-e^{-i\varphi}, k+1, b) \qquad \text{for } k \geq 0 \text{ and } |\varphi| < \pi.$$

Moreover, from (2.2) we have

$$i \int_0^\varphi (it)^n g(it)\, dt = i \int_0^\varphi \frac{(it)^n}{2} \sum_{k=0}^\infty \frac{E_k(1-b)(it)^k}{k!}\, dt$$

$$= \frac{(i\varphi)^{n+1}}{2} \sum_{k=0}^\infty \frac{E_k(1-b)(i\varphi)^k}{k!\,(k+n+1)} \qquad (n \geq 0,\ |\varphi| < \pi).$$

Substituting in (3.4) and multiplying both sides by $(i\varphi)^{-n}$ we obtain

$$n!\,(i\varphi)^{-n}\Phi(-1, n+1, b) - e^{-ib\varphi} \sum_{k=0}^n \binom{n}{k}(i\varphi)^{-k}k!\,\Phi(-e^{-i\varphi}, k+1, b)$$

$$= \frac{1}{2} \sum_{k=0}^\infty \frac{E_k(1-b)(i\varphi)^{k+1}}{k!\,(k+n+1)} \qquad (n \geq 0,\ 0 < |\varphi| < \pi, \Re(b) > 0). \tag{4.3}$$

5. Applications of (4.2)

In this section some applications of (4.2) as well as its connection with (1.1) are presented.

5.1. Deduction of (1.1). Suppose $b = 1$ in (4.2). It can be easily shown that $\zeta(n, 1) = \zeta(n)$ and, if $k \geq 0$ and $0 < |\varphi| < 2\pi$, then $\Phi(e^{-i\varphi}, k + 1, 1) = e^{i\varphi} \operatorname{Li}_{k+1}(e^{-i\varphi})$. So, equation (4.2) reduces to

$$
n! \, (i\varphi)^{-n}\zeta(n+1) - \sum_{k=0}^{n} \binom{n}{k}(i\varphi)^{-k}k! \operatorname{Li}_{k+1}(e^{-i\varphi})
$$
$$
= \sum_{k=0}^{\infty} \frac{(i\varphi)^k B_k}{k! \, (k+n)} \qquad (n \geq 1, \ 0 < |\varphi| < 2\pi).
\tag{5.1}
$$

Moreover, it is well known (see [1]) that $B_1 = -\frac{1}{2}$, $B_{2k+1} = 0$ for $k > 0$ and, if $\varphi \neq 0$, $\operatorname{Li}_1(e^{-i\varphi}) = -\log(1 - e^{-i\varphi})$. Thus, taking $\varphi = -\frac{2\pi}{\omega}$, equation (5.1) becomes

$$
n! \left(\frac{i\omega}{2\pi}\right)^n \zeta(n+1) + \log(1 - e^{\frac{2\pi i}{\omega}}) - \sum_{k=1}^{n} \binom{n}{k}\left(\frac{i\omega}{2\pi}\right)^k k! \operatorname{Li}_{k+1}(e^{\frac{2\pi i}{\omega}})
$$
$$
= \sum_{k=0}^{\infty} \frac{(-1)^k (2\pi)^{2k} B_{2k}}{(2k)! \, (2k+n)\omega^{2k}} + \frac{\pi i}{(n+1)\omega} \qquad (n \geq 1, \ |\omega| > 1).
$$

Using (2.3) to express B_{2k} in terms of $\zeta(2k)$, multiplying by $-\frac{1}{2}$ and rearranging terms, we obtain

$$
\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+n)\omega^{2k}}
$$
$$
= \frac{\pi i}{2(n+1)\omega} - \frac{1}{2}\log\left(1 - e^{\frac{2\pi i}{\omega}}\right) - \frac{n!}{2}\left(\frac{i\omega}{2\pi}\right)^n \zeta(n+1)
\tag{5.2}
$$
$$
+ \frac{1}{2}\sum_{k=1}^{n} \binom{n}{k} k! \left(\frac{i\omega}{2\pi}\right)^k \operatorname{Li}_{k+1}\left(e^{\frac{2\pi i}{\omega}}\right) \qquad (n \geq 1, \ |\omega| > 1),
$$

which is (1.1) (and thus [8, eq. (5.11)]) with the correction of the index of the PolyLogarithm.

As explained by the authors in [8], equation (5.2) can be used to obtain several series representation for the ζ for odd integer values of its arguments.

Some interesting examples, obtained for $\omega = 2$, are ([8, eqs. (3.4) and (3.5)])

$$\zeta(3) = \frac{2\pi^2}{9}\left(\log 2 + 2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+3)}\right)$$

$$\zeta(3) = \frac{2\pi^2}{7}\left(\log 2 + 2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+2)}\right).$$

Combining these two equations we have ([3, eq. (2.19)] and [5, eq. (3.32) with $n = 1$])

$$\zeta(3) = -2\pi^2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+2)(2k+3)}$$

$$\log 2 = -\sum_{k=0}^{\infty}\frac{\zeta(2k)(4k+13)}{2^{2k}(2k+2)(2k+3)}.$$

On the other hand, when $\omega = 4$, more interesting results can be obtained. It is easy to see that

$$\mathrm{Li}_k(i) = \left(\frac{1-2^{k-1}}{2^{2k-1}}\right)\zeta(k) + i\beta(k) \qquad (k \geq 2).$$

Substituting this in (5.2), taking the real part of both sides (paying attention on the parity of n), it is possible to obtain the following results (calculations are omitted for brevity):

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\frac{2^{2k+1}(-1)^k(2k+1)!}{\pi^{2k+1}}\beta(2k+2)$$

$$+\sum_{k=1}^{n}\binom{2n+1}{2k}\frac{2^{2k}(2^{2k}-1)(-1)^k(2k)!}{2^{4k+1}\pi^{2k}}\zeta(2k+1) \qquad (5.3)$$

$$= -\frac{\log 2}{2} - 2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{4k}(2k+2n+1)} \qquad (n \geq 0)$$

and

$$(-1)^n\left(\frac{2}{\pi}\right)^{2n}(2n)!\,\zeta(2n+1)$$

$$+\sum_{k=1}^{n}\binom{2n}{2k-1}\frac{\pi^{1-2k}(-1)^{1-k}(2k-1)!}{2^{1-2k}}\beta(2k)$$

$$+\sum_{k=1}^{n}\binom{2n}{2k}\frac{2^{2k}(2^{2k}-1)(-1)^k(2k)!}{2^{4k+1}\pi^{2k}}\zeta(2k+1) \qquad (5.4)$$

$$= -\frac{\log 2}{2} - 2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{4k}(2k+2n)} \qquad (n \geq 1),$$

which are equivalent to [8, eqs. (3.26) and (3.27)] but where we have isolated the β function terms (which corresponds to express the Clausen's functions of [8, eqs. (2.18) and (2.19)], when $\omega = 4$, in terms of the β function instead of using [8, eqs. (3.7) and (3.16)]).

If $n = 0$ in (5.3), then we have ([8, eq. (2.23)])

$$\beta(2) = \text{Catalan} = -\frac{\pi}{4}\left(\log(2) + 4\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{4k}(2k+1)}\right),\qquad(5.5)$$

and substituting this representation in (5.4) with $n = 1$ we get

$$\zeta(3) = -\frac{2\pi^2}{35}\left(\log(2) + 4\sum_{k=0}^{\infty}\frac{\zeta(2k)(2k+3)}{2^{4k}(2k+1)(2k+2)}\right).\qquad(5.6)$$

The formula (5.6) is essentially the same as a known result [3, p. 192, eq. (3.21)]. Applying alternatively (5.3) and (5.4) with increasing values of n and using the representations found at every step we get, e.g.,

$$\beta(4) = -\frac{\pi^3}{10080}\left(183\log(2) + 12\sum_{k=0}^{\infty}\frac{\zeta(2k)(244k^2 + 732k + 479)}{2^{4k}(2k+1)(2k+2)(2k+3)}\right)$$

$$\zeta(5) = -\frac{\pi^4}{166005}\left(942\log(2) + 48\sum_{k=0}^{\infty}\frac{\zeta(2k)(628k^3 + 3140k^2 + 5111k + 2581)}{2^{4k}(2k+1)(2k+2)(2k+3)(2k+4)}\right).$$

5.2. Another application: case $\mathbf{b} = \frac{1}{2}$. We present here another possible use of equation (4.2) when $b = \frac{1}{2}$ and $\varphi = \pi$.

It is easy to verify that $\zeta(n, \frac{1}{2}) = (2^n - 1)\zeta(n)$, and $\Phi(e^{-i\pi}, k+1, \frac{1}{2}) = 2^{k+1}\beta(k+1)$ if $k \geq 0$.

It is also easy to demonstrate that $B_k(\frac{1}{2}) = (2^{1-k}-1)B_k$; thus equation (4.2) reduces to

$$n!\,(i\pi)^{-n}(2^{n+1} - 1)\zeta(n+1) + i\sum_{k=0}^{n}\binom{n}{k}(i\pi)^{-k}k!\,2^{k+1}\beta(k+1)$$

$$= \sum_{k=0}^{\infty}\frac{(2^{1-2k} - 1)B_{2k}\,(-1)^k\pi^{2k}}{(2k)!\,(2k+n)}\qquad(n \geq 1),$$

and using (2.3) we have

$$n!\,(i\pi)^{-n}(2^{n+1} - 1)\zeta(n+1) + i\sum_{k=0}^{n}\binom{n}{k}(i\pi)^{-k}k!\,2^{k+1}\beta(k+1)$$

$$= \sum_{k=0}^{\infty}\frac{(2^{2k-1} - 1)\zeta(2k)}{2^{4k-2}(2k+n)}\qquad(n \geq 1).\qquad(5.7)$$

This equation allows us to obtain some relations. Taking the imaginary part of both side we have the formulas

$$\sum_{k=0}^{n} \frac{2^{2k}(-1)^k}{(2n-2k)! \ \pi^{2k}} \ \beta(2k+1) = 0 \qquad (n \geq 1) \tag{5.8}$$

and

$$\zeta(2n+2) = \frac{(-1)^n \pi^{2n+1}}{(2^{2n+2}-1)} \sum_{k=0}^{n} \frac{2^{2k+1}(-1)^k}{(2n-2k+1)! \ \pi^{2k}} \ \beta(2k+1) \quad (n \geq 0), \tag{5.9}$$

which, in view of (2.3) and (2.4), show to be equivalent to (cf., for example, [7, p. 64, eq. (48)])

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0 \qquad (n \geq 1) \tag{5.10}$$

and

$$\sum_{k=0}^{n-1} \binom{2n-1}{2k} E_{2k} = \frac{2^{2n}(2^{2n}-1)}{2n} B_{2n} \qquad (n \geq 1), \tag{5.11}$$

respectively, (or, viceversa, known (5.10) and (5.11), (5.8) and (5.9) are equivalent to (2.3) and (2.4)).

Now, taking the real part of both sides of (5.7) we have for $n \geq 0$

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \frac{2^{2k+1}(2k+1)!}{\pi^{2k+1}} (-1)^k \beta(2k+2) = \sum_{k=0}^{\infty} \frac{(2^{2k-1}-1)\zeta(2k)}{2^{4k-1}(2k+2n+1)} \tag{5.12}$$

and for $n \geq 1$

$$\zeta(2n+1) = \frac{(-1)^n \pi^{2n}}{(2n)!(2^{2n+1}-1)} \left(\sum_{k=0}^{\infty} \frac{(2^{2k-1}-1)\zeta(2k)}{2^{4k-2}(2k+2n)} \right.$$
$$\left. - \sum_{k=0}^{n-1} \binom{2n}{2k+1} \frac{2^{2k+2}(2k+1)!}{\pi^{2k+1}} (-1)^k \beta(2k+2) \right). \tag{5.13}$$

From equation (5.12), some interesting series representations for $\beta(2m)$, $m \in \mathbb{N}$ can be derived. When $n = 0$, for example, we have

$$\beta(2) = \pi \sum_{k=0}^{\infty} \frac{(2^{2k-1}-1)\zeta(2k)}{2^{4k}(2k+1)}, \tag{5.14}$$

which can also be obtained by combining 5.5 and the known (see [6, eq. (4.11), p. 586]) sum (6.4), while for $n = 1$ we have

$$\beta(4) = \frac{\pi^3}{6} \sum_{k=0}^{\infty} \frac{(2^{2k-1} - 1)(k + 2)\zeta(2k)}{2^{4k}(2k + 1)(2k + 3)}.$$

Using (5.13), similar series representations for the values $\zeta(2m+1)$, $m \in \mathbb{N}$, can be obtained, for example

$$\zeta(3) = \frac{2\pi^2}{7} \sum_{k=0}^{\infty} \frac{(2^{2k-1} - 1)(2k + 3)\zeta(2k)}{2^{4k}(2k + 1)(2k + 2)} \qquad (5.15)$$

$$\zeta(5) = \frac{\pi^4}{186} \sum_{k=0}^{\infty} \frac{(2^{2k-1} - 1)(20k^2 + 80k + 83)\zeta(2k)}{2^{4k}(2k + 1)(2k + 3)(2k + 4)}.$$

In view of (5.14), the formula (5.15) is equivalent to a known result [3, p. 191, eq. (3.13)].

6. Applications of (4.3)

As a counterpart to Section 5 we now give some examples of application of (4.3).

6.1. Series of β: companion of (5.2). From equation (4.3), if $b = \frac{1}{2}$, a companion of equation (5.2) can be derived, in which the rule of the ζ function is played by the β function.

It is easy to see that $\Phi(-1, n + 1, \frac{1}{2}) = 2^{n+1}\beta(n + 1)$. Now, taking $\varphi = -\frac{\pi}{\omega}$, using (2.4) and remembering that $E_k = 2^k E_k(\frac{1}{2})$, we can rewrite the right hand side of (4.3) as

$$-2i \sum_{k=0}^{\infty} \frac{\beta(2k + 1)}{\omega^{2k+1}(2k + n + 1)},$$

where we have used the fact that $E_{2k+1} = 0$. Thus, multiplying both sides by $-\frac{1}{2}$, equation (4.3) can be rewritten as

$$\begin{aligned}
i \sum_{k=0}^{\infty} \frac{\beta(2k + 1)}{\omega^{2k+1}(2k + n + 1)} \\
= -n! \left(\frac{2\omega i}{\pi}\right)^n \beta(n + 1) \\
+ \frac{1}{2} e^{\frac{i\pi}{(2\omega)}} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{i\omega}{\pi}\right)^k k! \, \Phi\left(-e^{i\frac{\pi}{\omega}}, k + 1, \frac{1}{2}\right) \quad (n \geq 1, \ |\omega| < 1).
\end{aligned} \qquad (6.1)$$

6.2. Case $\omega = -2$ and Dirichlet L-series. As an example of application of equation (6.1), let us set $\omega = -2$ so that, multiplying both sides by -2, the left hand side becomes

$$i \sum_{k=0}^{\infty} \frac{\beta(2k+1)}{2^{2k}(2k+n+1)}.$$

With some further calculations, it is possible to demonstrate that for every $s > 1$

$$\Phi\left(i, s, \frac{1}{2}\right) = 2^{-2s}\left(\zeta\left(s, \frac{1}{8}\right) - \zeta\left(s, \frac{5}{8}\right) + i\zeta\left(s, \frac{3}{8}\right) - i\zeta\left(s, \frac{7}{8}\right)\right)$$

so that we have

$$e^{-i\frac{\pi}{4}}\Phi\left(i, s, \frac{1}{2}\right) = 2^{s-\frac{1}{2}}\left(L(s, \chi_1) - iL(s, \chi_2)\right), \tag{6.2}$$

where χ_1 and χ_2 are characters on $\mathbb{Z}/8$ satisfying

$$\chi_1(n) = \begin{cases} 1 & \text{if} \quad n = 1, 3 \\ -1 & \text{if} \quad n = 5, 7 \end{cases}, \qquad \chi_2(n) = \begin{cases} 1 & \text{if} \quad n = 1, 7 \\ -1 & \text{if} \quad n = 3, 5 \end{cases}.$$

Considering the convergence of the L-series, we can say that equation (6.2) holds for every integer $s = k$ with $k \geq 1$. As a result, equation (6.1) becomes

$$n! \, (i\pi)^{-n} \, 2^{2n+1}\beta(n+1)$$

$$-\sum_{k=0}^{n}\binom{n}{k}k! \, (i\pi)^{-k}2^{2k+\frac{1}{2}}\left(L(k+1, \chi_1) - iL(k+1, \chi_2)\right)$$

$$= i\sum_{k=0}^{\infty}\frac{\beta(2k+1)}{2^{2k}(2k+n+1)} \qquad (n \geq 0).$$

Taking the imaginary part of both sides we get the two relations

$$\sum_{k=0}^{n-1}\binom{2n}{2k+1}\frac{(-1)^k 2^{4k+\frac{5}{2}}(2k+1)!}{\pi^{2k+1}} L(2k+2, \chi_1)$$

$$+\sum_{k=0}^{n}\binom{2n}{2k}\frac{(-1)^k 2^{4k+\frac{1}{2}}(2k)!}{\pi^{2k}} L(2k+1, \chi_2) \tag{6.3}$$

$$= \sum_{k=0}^{\infty}\frac{\beta(2k+1)}{2^{2k}(2k+2n+1)} \qquad (n \geq 0),$$

and

$$\frac{(-1)^{n-1}2^{4n+3}(2n+1)!}{\pi^{2n+1}}\beta(2n+2)$$

$$+\sum_{k=0}^{n}\binom{2n+1}{2k+1}\frac{(-1)^{k}2^{4k+5/2}(2k+1)!}{\pi^{2k+1}}\,L(2k+2,\chi_1)$$

$$+\sum_{k=0}^{n}\binom{2n+1}{2k}\frac{(-1)^{k}2^{4k+\frac{1}{2}}(2k)!}{\pi^{2k}}\,L(2k+1,\chi_2)$$

$$=\sum_{k=0}^{\infty}\frac{\beta(2k+1)}{2^{2k}(2k+2n+2)}\qquad(n\geq 0)$$

(see [9] for interesting more general, but not equivalent, recursions and series representation for Dirichlet L-series). If, for example, $n = 0$ in equation (6.3), we have

$$\sqrt{2}\,L(1,\chi_2)=\sum_{k=0}^{\infty}\frac{\beta(2k+1)}{2^{2k}(2k+1)}.$$

On the other hand one may verify that

$$L(1,\chi_2)=\frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\frac{1-(-1)^{n}}{n}\cos\left(\frac{n\pi}{4}\right)$$

so that, writing the cosines in exponential form, we can sum the series to obtain $L(1,\chi_2)=\log(1+\sqrt{2})/\sqrt{2}$. This gives us the result

$$\sum_{k=0}^{\infty}\frac{\beta(2k+1)}{2^{2k}(2k+1)}=\log(1+\sqrt{2}),$$

which is an interesting (presumably new) counterpart of the known sum ([6, eq. (4.11), p. 586])

$$\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+1)}=-\frac{1}{2}\log(2).\qquad(6.4)$$

These examples show some possible uses of equations (4.2) and (4.3), and it is clear that other chooses of the parameters b and φ will give more complicated, but maybe interesting, equalities and recurrence relations.

7. Reflection properties

It is well known that Bernoulli and Euler polynomials satisfy the property

$$B_k(1-x)=(-1)^k B_k(x)$$
$$E_k(1-x)=(-1)^k E_k(x).$$

So, calling $S_1(n, \varphi, b)$ the left hand side of (4.2), we have

$$S_1(n, \varphi, b) = \overline{S_1(n, \varphi, 1 - b)} = S_1(n, -\varphi, 1 - b)$$
$$(n \geq 1,\ 0 < |\varphi| < 2\pi,\ 0 < \Re(b) < 1)$$

and, calling $S_2(n, \varphi, b)$ the left hand side of (4.3), we have

$$S_2(n, \varphi, b) = -\overline{S_2(n, \varphi, 1 - b)} = -S_2(n, -\varphi, 1 - b)$$
$$(n \geq 1,\ 0 < |\varphi| < 2\pi,\ 0 < \Re(b) < 1)$$

We have found a relation between the values $\Phi(e^{i\varphi}, k, b)$ and $\Phi(e^{i\varphi}, k, 1 - b)$, $k = 1 \ldots n$, that does not include any infinite sum.

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