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Compactness and Sobolev-Poincaré Inequalities for Solutions of Kinetic Equations

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Abstract. In this paper, we prove a regularity result on the velocity averages of the solution of a kinetic equation whose data have a Sobolev regularity $W^{s,p}$, $0 < s < \frac{1}{p}$, $p \in [1, +\infty)$, in the velocity variable. Namely, the velocity averages have the same Sobolev regularity in the time-space variable.

Keywords. Kinetic equations, Sobolev spaces, Littlewood-Paley decomposition

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1. Introduction

We consider the Cauchy problem of the following kinetic equation:

$$\frac{\partial}{\partial t}\left(f(t,x,v)\right) + a(v)\cdot\nabla_x\left(f(t,x,v)\right) = g(t,x,v) \quad \text{in } \mathcal{D}'(\mathbb{R}_t^+ \times \mathbb{R}_x^N \times \mathbb{R}_v^N), \quad (1)$$

when the initial data is zero:

$$f(0,x,v) = 0 \qquad\qquad \forall (x,v) \in \mathbb{R}^{2N}.$$

We assume that a is a C^∞ function from $\mathbb{R}^N$ to $\mathbb{R}^N$ and that the source term g is in $L^1_{loc}(\mathbb{R}_t^+ \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$, so that the solution f of (1) exists and is unique in $L^1_{loc}(\mathbb{R}_t^+ \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ (see [13, p. 67]).

In addition to the usual time and space variables (t,x), kinetic models involve the velocity v as a third variable and yield to equations of the form (1). Let us point out that we consider in this paper only the case when the dimension N of the space variable is equal to the dimension of the velocity variable. For instance, famous kinetic equations such as Vlasov and Boltzmann equations are of the form (1) with $a(v) = v$ (see [3] for a description of these equations). The kinetic formulation of a scalar conservation law with entropy condition also

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yields to the problem (1), when the dimension of the space variable is equal to one. Indeed, it is shown in [11] that if $u \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x)$ is the entropy solution of the conservation law

$$\begin{cases} \dfrac{\partial}{\partial t}\big(u(t,x)\big) + \ \mathrm{div}_x\,\Big(A\big(u(t,x)\big)\Big) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N) \\ \qquad\qquad\qquad\qquad u(0,\cdot) = u_0, & u_0 \in L^\infty, \end{cases} \tag{2}$$

then the function f defined on $\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}_v$ by

$$f(t,x,v) := \begin{cases} 1 & \text{if } 0 < v \leq u((t,x) \\ -1 & \text{if } u(t,x) \leq v < 0 \\ 0 & \text{else} \end{cases} \tag{3}$$

satisfies the kinetic problem (1) where $a(v) = A'(v)$ and g is the derivative in v of the entropy measure. We refer to [13] for a review of kinetic formulations. Let us point out that our restriction to the case of a same dimension for space and velocity variables only allows us to study kinetic formulations of conservation laws in one space dimension.

It was observed in [8] that compactness and regularity results exist, not for the solution f of (1), but for velocity averages of f. For any $\phi \in C^\infty_c(\mathbb{R}^N)$, the velocity average ρ of f associated to ϕ is defined by

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,v)\phi(v)\,dv \quad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N. \tag{4}$$

For any $f \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$, the set of all velocity averages of f, denoted by $\mathcal{V}(f)$, is defined as follows:

$$\mathcal{V}(f) := \big\{\rho \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x) \,\big|\, \exists\,\phi \in C^\infty_c(\mathbb{R}^N) \text{ such that (4) holds}\big\}.$$

The velocity averages of the solution f of (1) are of physical interest : in the case of transport models, they may correspond to the density of particles, the moment density or the energy density (see [3]), whereas in the case of the kinetic formulation of the conservation law (2), the solution of the conservation law u is in $\mathcal{V}(f)$ where f is the solution of the associated kinetic problem. The main result obtained in [8] is a gain of regularity for the velocity averages of f : if f, g are in $L^2(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ and satisfy (1) with $a(v) = v$, then any velocity average of f is in $H^{\frac{1}{2}}(\mathbb{R}^+_t \times \mathbb{R}^N_x)$. Such results are called in the literature "kinetic averaging lemmas" (for a survey of them, see [3]). Among all of them, we may quote the result given in [5] (in a weaker form) and in [2] (in the present form).

Theorem 1.1 (DLM, B). *Let $f, g \in L^p(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ with $1 < p \leq 2$, satisfying (1) with $a(v) = v$ for all $v \in \mathbb{R}^N$. Then any velocity average ρ in $\mathcal{V}(f)$ is in $H^{1-1/p}_p(\mathbb{R}^{1+N})$.*

For any $s \in \mathbb{R}$, $H^s_p(\mathbb{R}^d)$, $1 < p < +\infty$ and $d \in \mathbb{N}^*$, denotes the fractional Sobolev space defined by

$$H^s_p(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) \,\middle|\, \mathcal{F}^{-1}\big((1+|\cdot|^2)^{\frac{s}{2}}\mathcal{F}f\big) \in L^p(\mathbb{R}^d) \right\},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform in $\mathbb{R}^d$, respectively. A norm on $H^s_p(\mathbb{R}^d)$ is given by

$$\|f\|_{H^s_p} = \|f\|_{L^p} + \big\|\mathcal{F}^{-1}\big((1+|\cdot|^2)^{\frac{s}{2}}\mathcal{F}f\big)\big\|_{L^p}.$$

Let us mention that the regularity result obtained in Theorem 1.1 also holds when f is a solution of (1) with $a \in C^\infty(\mathbb{R}, \mathbb{R}^N)$, as in the case of the kinetic formulation of a scalar conservation law in space dimension N, as soon as a satisfies the following non-degeneracy condition (see [11]) :

$$\begin{aligned} &\forall\, M > 0,\, \exists\, C > 0 \text{ such that } \forall \xi \in \mathbb{R}^N\!,\ u \in \mathbb{R} \text{ s.t. } |\xi| + |u| \le 1,\ \forall \varepsilon > 0 : \\ &\qquad\qquad \mathcal{L}^1\big(\{v \in [-M, M] \,|\, |a(v)\cdot\xi + u| \le \varepsilon\}\big) \le C\varepsilon. \end{aligned} \tag{5}$$

Recently, P.-E. Jabin and B. Perthame improved the embedding of Theorem 1.1, assuming some regularity in the variable v of f and g. This assumption has some relevance when the equation (1) comes from a conservation law : the function f defined by (3) for $u \in L^1_{loc}$ is in L^∞ and the derivative in v of f is a Radon measure in $\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}_v$. By interpolation, $f \in L^q(\mathbb{R}^+_t \times \mathbb{R}^N_x, W^{\gamma,q}(\mathbb{R}_v))$, for any $\gamma < \frac{1}{2}$ and $q < 2$. In [10], they proved the following result:

Theorem 1.2 (JP). *Let $f \in L^q(\mathbb{R}^+_t \times \mathbb{R}^N_x, W^{\gamma,q}(\mathbb{R}^N_v))$, $g \in L^p(\mathbb{R}^+_t \times \mathbb{R}^N_x, W^{\beta,p}(\mathbb{R}^N_v))$, with $1 < p, q \le 2$, $1 - \frac{1}{q} < \gamma \le \frac{1}{2}$ and $\beta \le \frac{1}{2}$, satisfying* (1). *If either $a(v) = v$ for all $v \in \mathbb{R}^N$, or $a \in C^\infty(\mathbb{R}, \mathbb{R}^N)$ satisfies* (5), *then any velocity average $\rho \in \mathcal{V}(f)$ is in $W^{s',r'}_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x)$ for any $s' < \theta$ and $r' < r$ with $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ and*

$$\theta = \frac{\alpha(\gamma, q)}{\alpha(\gamma, q) + 1 - \alpha(\beta, p)}\,, \tag{6}$$

where the function α is defined by

$$\alpha(\gamma, q) = \begin{cases} 1 + \gamma - \frac{1}{q} & \text{if } \gamma \le \frac{1}{2} \\ 2 - \gamma + 2\frac{\gamma - 1}{q} & \text{if } \gamma > \frac{1}{2}. \end{cases}$$

The space $W^{s,p}(\mathbb{R}^d)$, also called in the literature fractional Sobolev space, is defined for $0 < s < 1$, $1 \le p < +\infty$ and $d \in \mathbb{N}^*$ by

$$W^{s,p}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) \,\middle|\, \iint_{\mathbb{R}^{2d}} \frac{|f(x) - f(y)|^p}{|x - y|^{sp+d}}\, dx\, dy < +\infty \right\}.$$

A norm on $W^{s,p}(\mathbb{R}^d)$ is given by

$$\|f\|_{W^{s,p}} = \|f\|_{L^p} + \left(\iint_{\mathbb{R}^{2d}} \frac{|f(x) - f(y)|^p}{|x-y|^{sp+d}} \, dx \, dy \right)^{\frac{1}{p}}. \tag{7}$$

Actually, $W^{s,p}(\mathbb{R}^d) \subset H^s_p(\mathbb{R}^d)$, but, unless $p = 2$, the inverse inclusion is not true (see [14]).

The approach for proving the above velocity averaging lemmas is to write the velocity average ρ as a sum of two functions, for the first one, the assumption on f is used, whereas the assumption on g is used for the second one, and to conclude by a real interpolation argument.

Here, we want to address the question of existence of averaging lemmas without making further assumptions on f than to be in L^1_{loc}. The method described above can't be applied anymore. Actually, very few results are known if f is not assumed to be at least in L^p with $p > 1$. In [8], it is given an example of a sequence g_n bounded in L^1 such that the sequence of velocity averages of the solutions f_n of (1) (with g replaced by g_n) is not weakly compact in L^1_{loc}. Thus, some stronger assumption on g is needed to get regularity or compactness results for the velocity averages of f. For instance, in the case $N = 1$, it is shown in [8] that if a sequence (g_n) is bounded in $L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x \times \mathbb{R}_v)$ and uniformly integrable, then the sequence of any velocity average of the solution f_n of (1) (with g replaced by g_n) is compact in L^1_{loc}.

In the present work, we shall make the assumption that the source term g has some regularity in the velocity variable v, whereas the solution f of (1) is only known to be in L^1_{loc}. Such a situation appears for instance while considering blow-ups in studying the structure of the singular set of the entropic solution u of (2), with any L^∞ initial data. In [7], it is proved in the case $N = 1$ that the entropy measure μ of the problem (2) (in the kinetic formulation of (2), the source term $g = \partial_v \mu$) concentrates on the 1-rectifiable singular set of u, when assuming that the set of zeros of A'' is locally finite and that the solution of (2) is an entropic solution (i.e., satisfies a sign condition on entropy measures). Actually, it was already known in the general N-dimensional case, without restricting to the entropic solution of (2), that the singular set of u coincides, up to a $\mathcal{H}^N$-negligible set, with the set of points of $\mathbb{R}^+ \times \mathbb{R}^N$ where the upper N-dimensional density of μ is positive (see [12] for $N = 1$ and [6] for the general case). Here, $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure. To prove the concentration of μ on this set, in the case when $N = 1$, the approach in [7] is to show that μ "doesn't see" the points (t_0, x_0) where the upper 1-dimensional density of μ is 0 (i.e., $\limsup_{r \to 0} \frac{\mu(B_r(t_0,x_0))}{r} = 0$). The usual blow-up process consists of studying the limit of the rescaled functions $u_r(t,x) := u(t_0 + rt, x_0 + rx)$. But, when the sequence $\alpha_n := \frac{\mu(B_{r_n}(t_0,x_0))}{r_n}$ goes to 0 for some sequence $r_n \to 0$, then one has to divide by α_n the rescaled functions to get a non trivial equation at the limit. In this case, the sequence of functions $f_n(t,x,v) := \frac{f(t_0+r_nt, x_0+r_nx, v)}{\alpha_n}$, where f is defined from u by (3), satisfies (1) with a Radon measure g_n whose total variation is bounded independently of n. But, f_n is not bounded in L^∞, since $\alpha_n \to 0$. In [7], the authors prove that the second derivative in v of the entropy measure (i.e., the first derivative in v of g for the associated kinetic problem) is a Radon measure on $\mathbb{R}^+ \times \mathbb{R}^2$ (here the assumption that the solution is entropic is crucial, for more details see [7]). With this regularity in the velocity variable, they are able to prove a weak L^1 precompactness result for velocity averages which allows to obtain the concentration result. Our goal was to improve this weak L^1 compactness result. We manage to prove some Sobolev-Poincaré type inequality. Precisely our main result is the following:

Theorem 1.3. *Let $f \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ satisfying (1) and coming from the zero initial data. Let us assume that either $a(v) = v$ for all $v \in \mathbb{R}^N$, or $N = 1$ and $a \in C^\infty(\mathbb{R},\mathbb{R})$ satisfies (5). If $g \in L^p_{loc}\big(\mathbb{R}^+_t \times \mathbb{R}^N_x, W^{s,p}_{loc}(\mathbb{R}^N_v)\big)$ with $p \in [1,+\infty)$ and $s \in (0,\frac{1}{p})$, then any velocity average $\rho \in \mathcal{V}(f)$ is in $W^{s,p}_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x)$. Moreover, for any $T \in (0,+\infty)$ and I a compact of $\mathbb{R}^N$,*

$$\|\rho\|_{W^{s,p}([0,T]\times I))} \le C\|g\|_{L^p([0,\tilde{T}]\times J, W^{s,p}(K))}, \tag{8}$$

where $C, \tilde{T}$ are positive constants, J, K are compacts of $\mathbb{R}^N$ which only depend on s, p, T and on the support, the sup and Lipschitz norms of a, its inverse a^{-1} and ϕ, where ϕ is the C^∞_c function to which ρ is associated by (4).

The result is given for the solution of the Cauchy problem with zero initial data. Actually, since the initial data and the source term play an equivalent role, the same result holds when the initial data is not zero but has the same regularity in v as the source term g.

Our result can be resumed as follows : for $1 \le p < +\infty$, the regularity $W^{s,p}$ in the variable v for g is transfered to the variables (t,x) for any velocity average of f, under the condition $s \in (0,\frac{1}{p})$. On the contrary of velocity averaging lemmas (the method has been described above), we don't need any stronger assumption on f than to be in L^1_{loc}. Moreover, we can treat the case $p = 1$, which is not possible as soon as the argument is based on interpolation, like it was for the previous velocity averaging lemmas. Actually, the regularity result yielded by our method is all the better since p is close to 1. For the case $p = 2$, our method does not improve the previous result of [5] and [2] obtained without the v-regularity assumption.

Now, let us compare our result to the one of [10] (Theorem 1.2) where the v-regularity of g is taken into account. Assume that $g \in L^p(\mathbb{R}^+_t \times \mathbb{R}^N_x, W^{\beta,p}(\mathbb{R}^N_v))$. On one hand, by Theorem 1.3, any velocity average of f is in $W^{\beta,p}_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x)$, if $\beta \in (0,\frac{1}{p})$. On the other hand, in [10] it is proved, under the assumption that $f \in L^q(\mathbb{R}_t^+ \times \mathbb{R}_x^N, W^{\gamma,q}(\mathbb{R}_v^N))$ with $1 < q \leq 2$, $1 - \frac{1}{q} < \gamma \leq \frac{1}{2}$, that any velocity average of f is in $W_{loc}^{s',r'}(\mathbb{R}^+ \times \mathbb{R}^N)$, where $s' < \theta$ is given by (6). When $1 < p < 2$ and $\beta \in \left(\frac{1}{2}, \frac{1}{p}\right)$,

$$\theta = \frac{\alpha}{\alpha - 1 + \beta - 2\frac{\beta-1}{p}}, \qquad \text{where } \alpha = 1 + \gamma - \tfrac{1}{q} \in (0, 1].$$

One can easily remark that the sign of $\theta - \beta$ is positive if $\alpha > \frac{2}{p} - 1$ and negative else. For instance, β will be greater than θ if $\gamma = \varepsilon$, $q = \frac{1}{1-2\varepsilon}$ with ε small enough. As expected, it is when the information on f is poor (γ is close to 0 and q is close to 1) that the method we use to obtain the regularity result of Theorem 1.3 is really efficient compared to the one used in [10].

In the context of the kinetic formulation of a conservation law, when a satisfies the following weaker non-degeneracy assumption:

$$\begin{gathered}\forall M > 0, \exists\, C > 0, \exists\, \delta \in (0,1] \text{ such that } \forall \xi \in \mathbb{R}^N, u \in \mathbb{R} \text{ s.t.} |\xi| + |u| \leq 1, \forall \varepsilon > 0: \\ \mathcal{L}^1\left(\left\{v \in [-M, M] \,\middle|\, |a(v) \cdot \xi + u| \leq \varepsilon\right\}\right) \leq C\varepsilon^\delta, \qquad (9)\end{gathered}$$

the method of [5], [2] and [10] yields to regularity results for velocity averages (which are weaker than the one quoted before) : if $f, g \in L^p$, $1 < p \leq 2$, satisfy (1), with a satisfying (9), then the velocity averages of f are in $H_p^{\delta(1-1/p)}(\mathbb{R}^2)$. The problem of the adaptation of our method to the case where a only satisfies the weaker assumption (9) in the one dimensional case remains open.

One of the main motivations of our work was the question of the concentration of the entropy measure on a N-rectifiable set in the problem of a scalar conservation law, which remains open for any space dimension $N \geq 2$ or in one space dimension but when the solution of the conservation law is not entropic. If we manage to improve the compactness result used in [7] to prove concentration, the restriction to the case of equal space and velocity dimensions prevents us from dealing with conservation laws in space dimension strictly greater than 1. Adapting the proof of Theorem 1.3 to the case when the space dimension is not equal to the velocity dimension is not straightforward. The main idea of the proof of Theorem 1.3 is to make the space variable slide into the velocity variable by a simple change of variables. But, if the dimensions are not the same, this exchange can't be done anymore. We then leave open the interesting question of the generalization of Theorem 1.3 when space and velocity dimensions are different.

In a second part of this paper, we insist on the fact that our result includes the case $p = 1$, which seems to us the most interesting case. We obtain that any velocity average $\rho \in \mathcal{V}(f)$ is in $W_{loc}^{s,1}(\mathbb{R}^+ \times \mathbb{R}^N)$ for any $s \in (0, 1)$ as soon as we assume g to be in $L_{loc}^1(\mathbb{R}^+ \times \mathbb{R}^N, W_{loc}^{1,1}(\mathbb{R}^N))$. The result is optimal in the sense that we can't hope to get ρ in $W_{loc}^{1,1}(\mathbb{R}^+ \times \mathbb{R}^N)$. Indeed, velocity averages of f

typically present N-dimensional singularities ("shock waves" in the case of the kinetic formulation of a conservation law), even if the source term g is smooth, so that they can't belong to $W^{1,1}_{loc}(\mathbb{R}^+ \times \mathbb{R}^N)$. The

appropriate space for velocity averages would be the space of BV functions. We may wonder if we can estimate the BV norm of ρ by the $L^1_{t,x}BV_v$ norm of the source term g. In dimension 2, the space of BV functions is continuously embedded in the space of L^2 functions. Then, a first step in the case when $N = 1$ would be to obtain an estimate of the L^2 norm of ρ. Actually, we show that a Poincaré-type inequality for the L^2 norm of velocity averages can't hold : we give a counter-example which consists of approaching by L^1 functions (g_ε) a Dirac mass in the two-dimensional (t, x) space (Proposition 3.1). Therefore, we can't hope to obtain any BV estimate for velocity averaging with the only assumption on the v-regularity of the source term.

We then restrict the set of source terms to Radon measures g in $\mathbb{R}^+ \times \mathbb{R}$ whose derivative in v, $\partial_v g$, is also a Radon measure and which also have the following property:

$$\limsup_{R\to 0} \frac{g\left(B_R(t,x) \times \mathbb{R}\right)}{R} < +\infty \quad \text{for a.e. } (t,x) \in (0,+\infty) \times \mathbb{R}\,. \tag{10}$$

The new assumption (10) is relevant since it is satisfied when the kinetic equation (1) comes from a conservation law with entropy condition (see [6] and [12]). But, even for this restricted set of source terms, which excludes Dirac masses in the (t, x) space, we will show, giving a second counter-example, that we can't get a BV estimate of the velocity average by the $L^1_{t,x}BV_v$ norm of g (Proposition 3.2).

Finally, still in the case $N = 1$, we establish a "weak" Poincaré-type inequality involving the $L^{2,\infty}$ norm of velocity averages.

Theorem 1.4. *Let $f \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^2_{x,v})$ satisfying* (1) *and coming from the zero initial data. If $a \in C^\infty(\mathbb{R},\mathbb{R})$ satisfies* (5) *and $g \in L^1\big(\mathbb{R}^+_t \times \mathbb{R}_x, BV(\mathbb{R}_v)\big)$, then any velocity average $\rho \in \mathcal{V}(f)$ is in $L^{2,\infty}(\mathbb{R}^+_t \times \mathbb{R}_x)$ and satisfies*

$$\|\rho\|^*_{2,\infty} \leq C\|g\|_{L^1_{t,x}BV_v}. \tag{11}$$

The paper is organized as follows: Section 2 is devoted to the proof of Theorem 1.3. First we consider the case of $a(v) = v$, $v \in \mathbb{R}^N$:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(f(t,x,v)\right) + v \cdot \nabla_x \left(f(t,x,v)\right) = g(t,x,v) \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N) \\ \qquad\qquad\qquad\qquad f(0,x,v) = 0. \end{cases} \tag{12}$$

We show the regularity of velocity averages of solutions of (12) in the space variable (Proposition 2.1), using the explicit formula of f in the case when g is smooth, and then using a density argument. Then, we explain briefly how to generalize the preceding proof to the problem (1) when $N = 1$ and a satisfies (5) (Proposition 2.2). Finally, we prove that space regularity implies time regularity for velocity averages of solutions of (1), using Fourier analysis arguments (Proposition 2.3).

In Section 3, we consider the particular case of $p = 1$. First, we give the example which contradicts the L^2 Poincaré-type inequality. Then, we give the second example where the assumption (10) holds and which contradicts a BV estimate for the velocity averages. We finish by proving Theorem 1.4.

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Before going into proofs, let us fix some notations: $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ will denote respectively the Fourier transform and the inverse Fourier transform of f. From time to time, the notation $\mathcal{F}f$ can be replaced by the simpler one $\hat{f}$. When we consider the partial Fourier tranform (resp. the inverse partial Fourier) in the variable X of f, we will write $\mathcal{F}_X f$ (resp. $\mathcal{F}_X^{-1} f$).

For all $\phi : \mathbb{R}^N \to \mathbb{R}$, $\mathrm{supp}(\phi)$ denotes the support of ϕ, $\|\phi\|_\infty$ denotes the sup norm of ϕ, $\mathrm{Lip}(\phi)$ denotes the Lipschitz constant of ϕ, and $\mathrm{Lip}_K(\phi)$ denotes the Lipschitz constant of the restriction of ϕ on K.

For any set $A \subset \mathbb{R}^N$, $\mathbf{1}_A$ denotes the indicator function of A and $\mathcal{L}^N(A)$ denotes the Lebesgue measure of A. $B(x, R)$ denotes the closed ball centered in x of radius R in $\mathbb{R}^N$, B_R denotes $B(0, R)$. The Euclidean norm in any $\mathbb{R}^N$ is always denoted by $|\cdot|$.

A function f belongs to $L^p_{loc}(\mathbb{R}^N)$, $1 \le p < +\infty$, if and only if, for any compact $K \subset \mathbb{R}^N$,

$$\|f\|_{L^p(K)} := \left(\int_K |f|^p \, dx \right)^{\frac{1}{p}} < +\infty.$$

A function f belongs to $W^{s,p}_{loc}(\mathbb{R}^N)$, $0 < s < 1$ and $1 \le p < +\infty$, if and only if for any compact $K \subset \mathbb{R}^N$

$$\|f\|_{W^{s,p}(K)} := \|f\|_{L^p(K)} + \left(\iint_{K^2} \frac{|f(x) - f(y)|^p}{|x - y|^{sp+N}} \, dx \, dy \right)^{\frac{1}{p}} < +\infty.$$

A multi-variable function f is in $L^p_{loc}(\mathbb{R}^m, W^{s,p}_{loc}(\mathbb{R}^N))$, $N, m \in \mathbb{N}^*$, $0 < s < 1$ and $1 \le p < +\infty$, if and only if for any compacts $K \subset \mathbb{R}^N$, $J \subset \mathbb{R}^m$,

$$\|f\|_{L^p(J, W^{s,p}(K))} := \left\| \|f(X, \cdot)\|_{W^{s,p}_Y(K)} \right\|_{L^p_X(J)} < +\infty.$$

2. Proof of Theorem 1.3

2.1. Space regularity (case $a(v) = v$). In the case when $a(v) = v$ for all $v \in \mathbb{R}^N$, we prove partially the regularity result of Theorem 1.3 : the following proposition only gives the regularity of velocity averages of the solution f of (12) in the space variable.

Proposition 2.1. *Let $f \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ satisfy (12). Let us assume that g is in $L^p_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x, W^{s,p}_{loc}(\mathbb{R}^N_v))$ with $1 \le p < +\infty$ and $s \in (0, \frac{1}{p})$. Then, any velocity average $\rho \in \mathcal{V}(f)$ is in $L^p_{loc}\big(\mathbb{R}^+_t, W^{s,p}_{loc}(\mathbb{R}^N_x)\big)$, and for any $T \in (0, +\infty)$ and I a compact of $\mathbb{R}^N$,*

$$\|\rho\|_{L^p((0,T]), W^{s,p}(I))} \le C\|g\|_{L^p([0,T]\times J, W^{s,p}(K))}, \tag{13}$$

where C is a positive constant, J, K are compacts of $\mathbb{R}^N$ which only depend on s, p, T and on supp ϕ, $\|\phi\|_\infty$ and $Lip(\phi)$, where ϕ is the C^∞_c function to which ρ is associated by (4).

Proof. First, let us assume that $g \in C^\infty(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$. Then, the solution f of (12) is given by the following formula:

$$f(t, x, v) = \int_0^t g(\tau, x - (t - \tau)v, v)\, d\tau \quad \forall (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^{2N}.$$

Then, for all $t > 0$ and for all $x \in \mathbb{R}^N$,

$$\begin{aligned}\rho(t, x) &= \int_{\mathbb{R}^N} \int_0^t g(\tau, x - (t - \tau)v, v)\phi(v)\, d\tau\, dv \\ &= \int_0^t \int_{\mathbb{R}^N} g\left(\tau, z, \tfrac{x-z}{t-\tau}\right) \phi\left(\tfrac{x-z}{t-\tau}\right) \frac{1}{(t-\tau)^N}\, dz\, d\tau\end{aligned}$$

using Fubini's theorem and doing the change of variables $z(v) = x - (t - \tau)v$.

For any $T \in (0, +\infty)$ and for any compact $I \subset \mathbb{R}^N$, we have to estimate the quantity

$$\begin{aligned}A &:= \int_0^T \iint_{I^2} \frac{|\rho(t, x) - \rho(t, y)|^p}{|x - y|^{sp+N}}\, dx\, dy\, dt \\ &= \int_0^T \iint_{I^2} \frac{dx\, dy\, dt}{|x - y|^{sp+N}} \\ &\quad \times \left| \int_0^t \int_{\mathbb{R}^N} \left[g\left(\tau, z, \tfrac{x-z}{t-\tau}\right) \phi\left(\tfrac{x-z}{t-\tau}\right) - g\left(\tau, z, \tfrac{y-z}{t-\tau}\right) \phi\left(\tfrac{y-z}{t-\tau}\right)\right] \frac{1}{(t-\tau)^N}\, dz\, d\tau \right|^p.\end{aligned}$$

Let us take $M > 0$ such that $\mathrm{supp}(\phi) \subset B_M$. For all $x \in I$ and for all $u \in [0, T]$, let us denote $B(x, uM)$ by $J_{u,x}$. If $z \notin J_{u,x}$, then $\phi\big(\frac{x-z}{u}\big) = 0$. Then, for any

$(x,y)\in I^2$ and $t>\tau\in[0,T]$, we can restrict the integral in z on the compact $J_{t-\tau,x}\cup J_{t-\tau,y}$. Since $\mathcal{L}^N(J_{t-\tau,x})=\mathcal{L}^N(J_{t-\tau,y})=C_0(t-\tau)^NM^N$, with $C_0>0$, using Hölder inequality for the integral over τ, and then on the integral over z, we have

$$\begin{aligned}
A &= \int_0^T\iint_{I^2}\frac{dx\,dy\,dt}{|x-y|^{sp+N}}\\
&\quad\times\left|\int_0^t\int_{J_{t-\tau,x}\cup J_{t-\tau,y}}\Big[g\big(\tau,z,\tfrac{x-z}{t-\tau}\big)\,\phi\big(\tfrac{x-z}{t-\tau}\big)-g\big(\tau,z,\tfrac{y-z}{t-\tau}\big)\,\phi\big(\tfrac{y-z}{t-\tau}\big)\Big]\frac{1}{(t-\tau)^N}dzd\tau\right|^p\\
&\le T^{p-1}\int_0^T\int_0^T\iint_{I^2}\frac{dx\,dy\,dt}{|x-y|^{sp+N}}\\
&\quad\times\left|\int_{J_{t-\tau,x}\cup J_{t-\tau,y}}\Big[g\big(\tau,z,\tfrac{x-z}{t-\tau}\big)\,\phi\big(\tfrac{x-z}{t-\tau}\big)-g\big(\tau,z,\tfrac{y-z}{t-\tau}\big)\,\phi\big(\tfrac{y-z}{t-\tau}\big)\Big]\frac{1}{(t-\tau)^N}\,dz\right|^p d\tau\\
&\le C_1\int_0^T\int_0^t\iint_{I^2}\frac{dx\,dy\,dt\,d\tau}{|x-y|^{sp+N}}(t-\tau)^{N(p-1)}\\
&\quad\times\int_{J_{t-\tau,x}\cup J_{t-\tau,y}}\left|g\big(\tau,z,\tfrac{x-z}{t-\tau}\big)\,\phi\big(\tfrac{x-z}{t-\tau}\big)-g\big(\tau,z,\tfrac{y-z}{t-\tau}\big)\,\phi\big(\tfrac{y-z}{t-\tau}\big)\right|^p\frac{1}{|t-\tau|^{Np}}\,dz,
\end{aligned}$$

where C_1 is a positive constant depending on M and T.

Let us set $J:=\{z\in\mathbb{R}^N\mid \operatorname{dist}(z,I)\le TM\}$, then J contains $J_{u,x}$ for all $(u,x)\in[0,T]\times I$. Using Fubini's theorem and doing the changes of variables $x'=\frac{x-z}{t-\tau}$ and $y'=\frac{y-z}{t-\tau}$, we have

$$\begin{aligned}
A &\le C_1\int_0^T\int_0^T\int_J dt\,d\tau\,dz\iint_{I^2}\frac{\big|g(\tau,z,\frac{x-z}{t-\tau})\phi(\frac{x-z}{t-\tau})-g(\tau,z,\frac{y-z}{t-\tau})\phi(\frac{y-z}{t-\tau})\big|^p}{|x-y|^{sp+N}|t-\tau|^N}\,dx\,dy\\
&\le C_1\int_0^T\int_0^T\int_J\frac{dt\,d\tau\,dz}{|t-\tau|^{sp}}\iint_{\mathbb{R}^{2N}}\frac{\big|g(\tau,z,x')\phi(x')-g(\tau,z,y')\phi(y')\big|^p}{|x'-y'|^{sp+N}}\,dx'dy'.
\end{aligned}$$

Now, let us remark that since g is in $W^{s,p}_{loc}(\mathbb{R}^N_v)$ and ϕ is a Lipschitz funtion with compact support in $\mathbb{R}^N$, then $g\phi$ is in $W^{s,p}(\mathbb{R}^N_v)$. Precisely one can easily show that

$$\iint_{\mathbb{R}^{2N}}\frac{|g(\tau,z,x')\phi(x')-g(\tau,z,y')\phi(y')|^p}{|x'-y'|^{sp+N}}\,dx'\,dy'\le C_2\|g(\tau,z,\cdot)\|^p_{W^{s,p}(K)},\tag{14}$$

where C_2 depends on $\|\phi\|_\infty$ and $\mathrm{Lip}(\phi)$, and where K is the support of ϕ. Since, by assumption $sp<1$, we then only have to estimate the L^1 norm of a convolution of L^1 functions:

$$A\le C_1C_2\int_0^T\!\!\int_0^T\!\!\int_J\frac{\|g(\tau,z,\cdot)\|^p_{W^{s,p}(K)}}{|t-\tau|^{sp}}\,dz\,dt\,d\tau\le C\|g\|^p_{L^p([0,T]\times J,W^{s,p}(K))},$$

where C depends on $T, s, p,$ supp ϕ, $\|\phi\|_\infty$ and Lip(ϕ). Hence (13) is proved for any $g \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^{2N})$.

Now, let $g \in L^p_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x, W^{s,p}_{loc}(\mathbb{R}^N_v))$. Let $T \in (0, +\infty)$ and I be any compact of $\mathbb{R}^N$. Let $\phi \in C^\infty_c(\mathbb{R}^N)$. We define as above compacts J and K. Let $\theta \in C^\infty_c(\mathbb{R}^{1+2N})$ such that supp $\theta \subset [0,T] \times J \times K$, $\|\theta\|_\infty =$ Lip$(\theta) = 1$, and let us extend g by 0 on $(-\infty, 0) \times \mathbb{R}^{2N}$.

Let $\xi \in C^\infty_c(\mathbb{R}^{1+2N})$ such that $\xi \geq 0$, supp $\xi \subset B_1$ and $\int \xi = 1$. Let us set $\xi_n(t,x,v) := n^{1+2N}\xi(nt, nx, nv)$ for all $(t,x,v) \in \mathbb{R}^{1+2N}$ and $g_n := g\,\theta * \xi_n$. Then g_n converges to $g\,\theta$ in $L^p(\mathbb{R}_t \times \mathbb{R}^N_x, W^{s,p}(\mathbb{R}^N_v))$, and

$$\|g_n\|_{L^p(\mathbb{R}_t \times \mathbb{R}^N_x, W^{s,p}(\mathbb{R}^N_v))} \leq \|g\,\theta\|_{L^p(\mathbb{R}_t \times \mathbb{R}^N_x, W^{s,p}(\mathbb{R}^N_v))}.$$

For all $n \in \mathbb{N}$, let f_n be the solution of (12) with g replaced by g_n. Since g_n is in $C^\infty(\mathbb{R}^{1+2N})$, then f_n is given by

$$f_n(t,x,v) = \int_0^t g_n(\tau, x - (t-\tau)v, v)\, d\tau \quad \forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^{2N}.$$

Since the L^p norm of g_n is uniformly bounded, then (f_n) is bounded in L^p and, up to a subsequence, (f_n) converges in the distributional sense to the function f satisfying (12). For all $n \in \mathbb{N}$, let ρ_n be the velocity average of f_n associated to ϕ by (4), then (ρ_n) converges in the distributional sense to ρ the velocity average of f associated to ϕ. But, since $g_n \in C^\infty(\mathbb{R}^{1+2N})$, we know from above that

$$\|\rho_n\|_{L^p([0,T], W^{s,p}(I))} \leq C\|g_n\|_{L^p([0,T]\times J, W^{s,p}(K))} \leq C\|g\,\theta\|_{L^p(\mathbb{R}^{1+N}, W^{s,p}(\mathbb{R}^N))},$$

where $C > 0$ depends on $T, s, p,$ supp(ϕ), $\|\phi\|_\infty$ and Lip(ϕ). Using the estimate (14) with θ instead of ϕ, we have

$$\|g\theta\|_{L^p(\mathbb{R}^{1+N}, W^{s,p}(\mathbb{R}^N))} \leq C_0\|g\|_{L^p([0,T]\times J, W^{s,p}(K))},$$

where C_0 is some positive constant.

Therefore, the weak limit ρ of ρ_n is in $L^p_{loc}(\mathbb{R}^+_t, W^{s,p}_{loc}(\mathbb{R}_x))$ and satisfies $\|\rho\|_{L^p([0,T], W^{s,p}(I))} \leq CC_0\|g\|_{L^p([0,T]\times J, W^{s,p}(K))}$. Proposition 2.1 is proved. □

2.2. Space regularity ($N = 1$). In the one dimensional case, we generalize the preceding result to the case when $a \in C^\infty$ satisfies the non-degeneracy condition (5). We obtain the same result as for the case $a(v) = v$.

Proposition 2.2. *Let $f \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x \times \mathbb{R}_v)$ satisfy (1). Let us assume that a satisfies (5) and $g \in L^p_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x, W^{s,p}_{loc}(\mathbb{R}_v))$ with $1 \leq p < +\infty$ and $s \in (0, \frac{1}{p})$. Then, any velocity average $\rho \in \mathcal{V}(f)$ is in $L^p_{loc}\big(\mathbb{R}^+_t, W^{s,p}_{loc}(\mathbb{R}_x)\big)$, and for any $T \in (0, +\infty)$ and I a compact of $\mathbb{R}$,*

$$\|\rho\|_{L^p([0,T]), W^{s,p}(I))} \leq C\|g\|_{L^p([0,T]\times J, W^{s,p}(K))},$$

where C is a positive constant, J, K are compacts of $\mathbb{R}$ which only depend on s, p, T and on the support, the sup and Lipschitz norms of a, its inverse a^{-1} and ϕ, where ϕ is the C_c^∞ function to which ρ is associated by (4).

Proof. If $a \in C^\infty$ satisfies (5), then the derivative a' of a can't vanish. Indeed, let us asume that there exists $v_0 \in \mathbb{R}$ such that $a'(v_0) = 0$. Let us take M such that $v_0 \in (-M, M)$. By the Taylor formula,

$$\forall\, v \in [-M, M] : \quad |a(v) - a(v_0)| \leq C_1 (v - v_0)^2,$$

where $C_1 = \sup_{[-M,M]} |a''|$. Thus,

$$\left\{v \in [-M, M] \,\middle|\, (v - v_0)^2 \leq \tfrac{\varepsilon}{C_1}\right\} \subset \left\{v \in [-M, M] \,\middle|\, |a(v) - a(v_0)| \leq \varepsilon\right\}.$$

By (5), we have $\mathcal{L}^1\big(\big\{v \in [-M, M] \,\big|\, (v - v_0)^2 \leq \frac{\varepsilon}{C_1}\big\}\big) \leq C\varepsilon$ for all $\varepsilon > 0$. But, for ε small enough,

$$\mathcal{L}^1\Big(\Big\{v \in [-M, M] \,\Big|\, (v - v_0)^2 \leq \tfrac{\varepsilon}{C_1}\Big\}\Big) = \sqrt{\tfrac{\varepsilon}{C_1}},$$

which yields to a contradiction. Therefore, a' is either positive or negative, and for any compact K of $\mathbb{R}$, $\inf_K |a'| > 0$. Moreover, since a is either strictly increasing or strictly decreasing, then a is bijective and its inverse a^{-1} is in $C^1(\mathbb{R})$.

We first assume that $g \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$, and then we conclude by the same density argument as in the proof of Theorem 1.3. When g is in $C^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$, the solution f of (1) with zero initial data is given by

$$f(t, x, v) = \int_0^t g\left(\tau, x - (t - \tau)a(v), v\right)\, d\tau \quad \forall (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

Let $\phi \in C_c^\infty(\mathbb{R})$ and $\rho \in \mathcal{V}(f)$ associated to ϕ by (4), then for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$,

$$\rho(t, x) = \int_0^t \int_K g(\tau, x - (t - \tau)a(v), v)\phi(v)\, dv\, d\tau,$$

where K is the support of ϕ. Since $\inf_K |a'| > 0$, we can make the change of variable $z(v) = x - (t - \tau)a(v)$ and we get

$$\rho(t, x) = \int_0^t \int_{J_{t-\tau,x}} g\left(\tau, z, a^{-1}\left(\tfrac{x-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)} \frac{1}{t - \tau}\, dz\, d\tau,$$

where $J_{u,x} := x - ua(K)$ for all $(u, x) \in \mathbb{R}^+ \times \mathbb{R}$. Let us remark that for all $z \notin J_{t-\tau,x}$, $\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right) = 0$. For any $T \in (0, +\infty)$ and I a compact of $\mathbb{R}$, let us set

$$A := \int_0^T \iint_{I^2} \frac{|\rho(t, x) - \rho(t, y)|^p}{|x - y|^{sp+1}}\, dx\, dy.$$

Then,

$$\begin{aligned}
A \leq & \int_0^T \iint_{I^2} \frac{dx\, dy\, dt}{|x-y|^{sp+1}} \\
& \times \left| \int_0^t \int_{J_{t-\tau,x} \cup J_{t-\tau,y}} \left[\mathbf{1}_{J_{t-\tau,x}}(z) g\big(\tau, z, a^{-1}(\tfrac{x-z}{t-\tau})\big) \frac{\phi\big(a^{-1}(\frac{x-z}{t-\tau})\big)}{a'\big(a^{-1}(\frac{x-z}{t-\tau})\big)} \right.\right. \\
& \left.\left. - \mathbf{1}_{J_{t-\tau,y}}(z) g\left(\tau, z, a^{-1}\left(\tfrac{y-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{y-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{y-z}{t-\tau}\right)\right)} \right] \frac{dz\, d\tau}{t-\tau} \right|^p, \\
\leq & \, C_1 \int_0^T \iint_{I^2} \int_0^t \frac{dx\, dy\, dt}{|x-y|^{sp+1}} (t-\tau)^{p-1} \\
& \times \int_{J_{t-\tau,x} \cup J_{t-\tau,y}} \left| \mathbf{1}_{J_{t-\tau,x}}(z) g\left(\tau, z, a^{-1}(\tfrac{x-z}{t-\tau})\right) \frac{\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)} \right. \\
& \left. - \mathbf{1}_{J_{t-\tau,y}}(z) g\left(\tau, z, a^{-1}\left(\tfrac{y-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{y-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{y-z}{t-\tau}\right)\right)} \right|^p \frac{dz\, d\tau}{(t-\tau)^p},
\end{aligned}$$

using the Hölder inequality and noticing that $\mathcal{L}^1(J_{u,x}) = u\mathcal{L}^1\big(a(K)\big)$, for any (u,x) in $\mathbb{R}^+ \times \mathbb{R}$. Let us set $J := \big\{ z \in \mathbb{R} \,\big|\, \mathrm{dist}(z, I) \leq T\mathcal{L}^1\big(a(K)\big) \big\}$. Then, for all $x \in I$ and for all $\tau < t \in [0,T]$, $J_{t-\tau,x} \subset J$, and we have

$$\begin{aligned}
A \leq & \, C_1 \int_0^T \int_0^t \int_J \frac{dt\, d\tau\, dz}{(t-\tau)} \\
& \iint_{I^2} \left| \mathbf{1}_{J_{-(t-\tau),z}}(x) g\left(\tau, z, a^{-1}\left(\tfrac{x-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)} \right. \\
& \left. - \mathbf{1}_{J_{-(t-\tau),z}}(y) g\left(\tau, z, a^{-1}\left(\tfrac{y-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{y-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{y-z}{t-\tau}\right)\right)} \right|^p \frac{dx\, dy}{|x-y|^{sp+1}}.
\end{aligned}$$

Doing the changes of variables $x' = a^{-1}\left(\frac{x-z}{t-\tau}\right) \in K$ and $y' = a^{-1}\left(\frac{y-z}{t-\tau}\right) \in K$, we obtain

$$\begin{aligned}
A \leq & \, C_1 \int_0^T \int_0^t \int_J \frac{dt\, d\tau\, dz}{(t-\tau)} \\
& \iint_{K^2} \left| g(\tau, z, x') \frac{\phi(x')}{a'(x')} - g(\tau, z, y') \frac{\phi(y')}{a'(y')} \right|^p \frac{|a'(x')||a'(y')|(t-\tau)^2}{|a(x') - a(y')|^{sp+1}(t-\tau)^{sp+1}} dx' dy' \\
\leq & \, C_1 \, \mathrm{Lip}_K(a)^2 \int_0^T \int_0^T \int_J \frac{dt\, d\tau\, dz}{|t-\tau|^{sp}} \iint_{K^2} \frac{\left| g(\tau, z, x) \frac{\phi(x)}{a'(x)} - g(\tau, z, y) \frac{\phi(y)}{a'(y)} \right|^p}{|a(x) - a(y)|^{sp+1}} \, dx\, dy.
\end{aligned}$$

But, a' is locally Lipschitz and does not vanish, therefore $\frac{\phi}{a'}$ is locally Lipschitz and has compact support. Moreover, a^{-1} is Lipschitz on K so that for all

$x, y \in K$ we have $\frac{|x-y|}{|a(x)-a(y)|} \leq \mathrm{Lip}_K(a^{-1})$. Therefore, using the estimate (14), we have

$$A \leq C \, \|g\|_{L^p([0,T]\times J, W^{s,p}(K))},$$

where C depends on T, s, p, $\mathrm{supp}(\phi)$, $\mathrm{Lip}_K(a)$, $\mathrm{Lip}_K(a^{-1})$, $\left\|\frac{\phi}{a'}\right\|_\infty$ and $\mathrm{Lip}\left(\frac{\phi}{a'}\right)$. Proposition 2.2 is proved. □

2.3. Time regularity. We finish the proof of Theorem 1.3 showing that space regularity implies time regularity for the velocity averages. Let us stress on the fact that the following result does not require the v-regularity assumption on g or any further assumption on f. Also, let us remark that this result (quoted in the case when space and velocity dimensions are equal) holds even if the velocity dimension is not the same as space dimension.

Proposition 2.3. *Let $f \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ satisfying (1). Let us assume that $g \in L^p_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ with $1 \leq p < +\infty$. Let $s \in (0,1)$. If any velocity average $\rho \in \mathcal{V}(f)$ is in $L^p_{loc}(\mathbb{R}^+_t, W^{s,p}_{loc}(\mathbb{R}^N_x))$, then any velocity average $\rho \in \mathcal{V}(f)$ is in $W^{s,p}_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N_x)$.*

Proof. We extend f, g (resp. any function of $\mathcal{V}(f)$) by 0 on $(-\infty,0)\times\mathbb{R}^{2N}$ (resp. on $(-\infty,0)\times\mathbb{R}^N$). Let K be any compact of $\mathbb{R}^+\times\mathbb{R}^N$ and $\theta \in C^\infty_c(\mathbb{R}^{1+N})$ be such that $\theta \equiv 1$ on K. Let $\phi \in C^\infty_c(\mathbb{R}^N)$ and $\rho \in \mathcal{V}(f)$ be the velocity average of f associated to ϕ by (4). Let us show that, under the assumption of Proposition 2.3, $\rho_\theta := \rho\,\theta$ is in $W^{s,p}(\mathbb{R}^{1+N})$, this will prove that $\rho \in W^{s,p}_{loc}(\mathbb{R}^+_t \times \mathbb{R}^N)$.

Since f satisfies (1), then the following equality holds in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\int_{\mathbb{R}} f(t,x,v)\phi(v)\,dv\right) + \mathrm{div}_x\left(\int_{\mathbb{R}} f(t,x,v)a(v)\phi(v)\,dv\right) \\ &= \int_{\mathbb{R}} g(t,x,v)\phi(v)\,dv. \end{aligned} \tag{15}$$

Let us set $\tilde{\rho}(t,x) := \int_{\mathbb{R}} f(t,x,v)\phi(v)a(v)\,dv \in \mathcal{V}(f)$, $\tilde{g}(t,x) := \int_{\mathbb{R}} g(t,x,v)\phi(v)\,dv$. Multiplying (15) by θ, we get

$$\frac{\partial \rho_\theta}{\partial t} + \mathrm{div}_x(\tilde{\rho}_\theta) = G \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N), \tag{16}$$

where $\tilde{\rho}_\theta = \tilde{\rho}\theta$ and $G = \tilde{g}\theta - \rho\frac{\partial\theta}{\partial t} - \tilde{\rho}\,\mathrm{div}_x(\theta)$. Clearly, $G \in L^p(\mathbb{R}^{1+N})$. By assumption, ρ and $\tilde{\rho}$ are in $L^p_{loc}(\mathbb{R}^+_t, W^{s,p}_{loc}(\mathbb{R}^N_x))$. Thus, ρ_θ and $\tilde{\rho}_\theta$ are in $L^p(\mathbb{R}, W^{s,p}(\mathbb{R}^N))$.

To prove that $\rho_\theta \in W^{s,p}(\mathbb{R}^{1+N})$, we will use the characterization of $W^{s,p}(\mathbb{R}^d)$, for $d \in \mathbb{N}^*$, $0 < s < 1$ and $1 \leq p < +\infty$, in terms of Besov norms. Indeed, we have (see [14], Sections 2.2.2 and 2.5.7)

$$W^{s,p}(\mathbb{R}^d) = B^s_{p,p}(\mathbb{R}^d) \quad \forall s \in (0,1),\ \forall p \in [1,+\infty).$$

Let us recall how the Besov space $B^s_{p,p}(\mathbb{R}^d)$ is defined. The set Λ_d is defined to be the set of all sequences $(\eta_j)_{j\in\mathbb{N}}$ in $C^\infty(\mathbb{R}^d, \mathbb{R})$ such that

- supp $\eta_j \subset \left\{X \in \mathbb{R}^d \,\middle|\, 2^{j-1} \leq |X| \leq 2^{j+1}\right\}$ $\forall j \geq 1$
- supp $\eta_0 \subset \left\{X \in \mathbb{R}^d \,\middle|\, |X| \leq 2\right\}$
- $2^{j|\alpha|}\|D^\alpha \eta_j\|_\infty \leq C_\alpha$ $\forall \alpha = (\alpha_i)_{1\leq i\leq d} \in \mathbb{N}^d,\ |\alpha| = \sum_{1\leq i\leq d} \alpha_i$ $\forall j \in \mathbb{N}$
- $\sum_{j\in\mathbb{N}} \eta_j(X) = 1$ $\forall X \in \mathbb{R}^d$.

Let $(\phi_j)_{j\in\mathbb{N}} \in \mathcal{F}^{-1}\Lambda_d := \left\{(\mathcal{F}^{-1}\eta_j)_{j\in\mathbb{N}}\,;\, (\eta_j)_{j\in\mathbb{N}} \in \Lambda_d\right\}$. The Besov space $B^s_{p,p}(\mathbb{R}^d)$ is defined by

$$B^s_{p,p}(\mathbb{R}^d) = \left\{f \in L^p(\mathbb{R}^d) \,\middle|\, \sum_{j=0}^{+\infty} \left(2^{sj}\|\phi_j * f\|_{L^p}\right)^p < +\infty\right\},$$

and a norm on $B^s_{p,p}(\mathbb{R}^d)$ is given by

$$\|f\|_{B^s_{p,p}} = \left(\sum_{j=0}^{+\infty} \left(2^{sj}\|\phi_j * f\|_{L^p}\right)^p\right)^{\frac{1}{p}}. \tag{17}$$

The Besov space $B^s_{p,p}(\mathbb{R}^d)$ does not depend on the choice of the sequence $(\phi_j)_{j\in\mathbb{N}}$ in $\mathcal{F}^{-1}\Lambda_d$: another choice will yield to an equivalent norm. For any $(\phi_j)_{j\in\mathbb{N}}$ in $\mathcal{F}^{-1}\Lambda_d$, the norm (17) is equivalent to the norm (7), when $s \in (0,1)$ and $p \in [1,+\infty)$.

We can construct a sequence $(\phi_j)_{j\in\mathbb{N}} \in \mathcal{F}^{-1}\Lambda_d$ in the following way : let $\hat{\phi}$ in $C^\infty_c(\mathbb{R}^d)$ such that supp $\hat{\phi} \subset \left\{\frac{1}{2} \leq |X| \leq 2\right\}$ and such that $\hat{\phi}(2X) + \hat{\phi}(X) = 1$, if $|X| \in [\frac{1}{2}, 1]$. Then, the sequence $(\hat{\phi}_j)_{j\in\mathbb{N}}$, defined by

$$\hat{\phi}_j(X) = \begin{cases} \hat{\phi}(2^{-j}X) & \forall X \in \mathbb{R}^d,\ \forall j \geq 1 \\ 1 - \sum_{j=1}^{+\infty} \hat{\phi}(2^{-j}X) & \forall X \in \mathbb{R}^d, \quad j = 0 \end{cases} \tag{18}$$

is in Λ_d. Then, $(\phi_j)_{j\in\mathbb{N}}$ defined by $\phi_j := \mathcal{F}^{-1}\hat{\phi}_j$, $\forall j \in \mathbb{N}$ is in $\mathcal{F}^{-1}\Lambda_d$.

Let $\hat{\phi} \in C^\infty_c(\mathbb{R}^{1+N})$ such that supp $\hat{\phi} \subset \left\{(\tau,\xi) \in \mathbb{R}^{1+N},\ \frac{1}{2} \leq (\tau^2 + |\xi|^2)^{\frac{1}{2}} \leq 2\right\}$. We define the sequence $(\phi_j)_{j\in\mathbb{N}}$ from $\hat{\phi}$ by (18). Then, $\rho_\theta \in W^{s,p}(\mathbb{R}^{1+N})$ if and only if $\sum_{j=0}^{+\infty} \left(2^{sj}\|\phi_j * \rho_\theta\|_{L^p}\right)^p < +\infty$. Moreover, in this case, there exists $C > 0$ such that

$$\|\rho_\theta\|_{W^{s,p}} \leq C\left(\sum_{j=0}^{+\infty} \left(2^{sj}\|\phi_j * \rho_\theta\|_{L^p}\right)^p\right)^{\frac{1}{p}}.$$

Let us choose $(\psi_j)_{j\in\mathbb{N}} \in \mathcal{F}^{-1}\Lambda_N$. Since $\rho_\theta \in L^p(\mathbb{R}, W^{s,p}(\mathbb{R}^N))$, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} \sum_{j=0}^{+\infty} \left(2^{sj}\|\psi_j * \rho_\theta(t,\cdot)\|_{L^p}\right)^p\, dt \leq C\|\rho_\theta\|^p_{L^p_t W^{s,p}_x}.$$

By the monotone convergence theorem, we can pass the integral under the sum to get

$$\sum_{j=0}^{+\infty}\Big(2^{sj}\|\psi_j *_x \rho_\theta\|_{L^p_{t,x}}\Big)^p \le C\|\rho_\theta\|^p_{L^p_t W^{s,p}_x}. \tag{19}$$

The same inequality holds for $\tilde{\rho}_\theta$.

Now, let $\chi_0 \in C^\infty(\mathbb{R}^{1+N})$ such that $0 \le \chi_0 \le 1$, $\chi_0 \equiv 1$ on $\{|\tau| \le |\xi|\} \cup \{|\tau| \le \frac{1}{2}\}$, supp $\chi_0 \subset \{|\tau| \le \sqrt{3}|\xi|\} \cap \{|\tau| \le 1\}$ and

$$|D^\alpha \chi_0|(\tau,\xi) \le \frac{C}{(\tau^2+|\xi|^2)^{|\alpha|/2}} \quad \forall\,(\tau,\xi)\in\mathbb{R}^{1+N}, \forall\,\alpha\in\mathbb{N}^{1+N},\ |\alpha|\le K,$$

where K is an integer depending on the dimension $1{+}N$ in the following way : $K > \frac{N+1}{2}$. Let us set $\chi_1 := 1-\chi_0$. For any $j \ge 1$, we have

$$\begin{aligned}\mathcal{F}(\phi_j * \rho_\theta) &= \hat{\phi}_j(\tau,\xi)\hat{\rho}_\theta(\tau,\xi)\\ &= \chi_0(\tau,\xi)\hat{\phi}_j(\tau,\xi)\hat{\rho}_\theta(\tau,\xi) + \chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\hat{\rho}_\theta(\tau,\xi).\end{aligned}$$

Taking the Fourier transform in (t,x) of (16), we get $-i\tau\hat{\rho}_\theta(\tau,\xi) - i\xi\cdot\hat{\tilde{\rho}}_\theta(\tau,\xi) = \hat{G}(\tau,\xi)$. Since supp $\chi_1 \subset \{|\tau| \ge \frac{1}{2}\}$, then we have

$$\begin{aligned}\mathcal{F}(\phi_j * \rho_\theta) = \chi_0(\tau,\xi)\hat{\phi}_j(\tau,\xi)\hat{\rho}_\theta(\tau,\xi) &- \chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\frac{\hat{G}(\tau,\xi)}{i\tau}\\ &- \chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\frac{\xi\cdot\hat{\tilde{\rho}}_\theta(\tau,\xi)}{\tau}.\end{aligned}$$

Thus, $\phi_j * \rho_\theta = \omega_{j0} - \omega_{j1} - \omega_{j2}$ where

$$\begin{aligned}\omega_{j0} &= \mathcal{F}^{-1}\big(\chi_0(\tau,\xi)\hat{\phi}_j(\tau,\xi)\hat{\rho}_\theta(\tau,\xi)\big)\\ \omega_{j1} &= \mathcal{F}^{-1}\left(\chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\frac{\hat{G}(\tau,\xi)}{i\tau}\right)\\ \omega_{j2} &= \mathcal{F}^{-1}\left(\chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\frac{\xi\cdot\hat{\tilde{\rho}}_\theta(\tau,\xi)}{\tau}\right),\end{aligned}$$

and

$$\sum_{j=1}^{+\infty} 2^{sjp}\|\phi_j * \rho_\theta\|^p_{L^p} \le \sum_{j=0}^{+\infty} 2^{sjp}\|\omega_{j0}\|^p_{L^p} + \sum_{j=0}^{+\infty} 2^{sjp}\|\omega_{j1}\|^p_{L^p} + \sum_{j=0}^{+\infty} 2^{sjp}\|\omega_{j2}\|^p_{L^p}. \tag{20}$$

Let us recall some results on the theory of L^p multipliers which will be useful to estimate each term of the right hand side of (20).

Let $M_p(\mathbb{R}^{1+N})$ be the set of L^p multipliers in $\mathbb{R}^{1+N}$, i.e., the set of all functions m defined on $\mathbb{R}^{1+N}$ such that for all $f \in L^p(\mathbb{R}^{1+N})$, $\mathcal{F}^{-1}(m\,\mathcal{F}f) \in L^p(\mathbb{R}^{1+N})$.

The M_p norm is defined by $\|m\|_{M_p} := \inf\{\|\mathcal{F}^{-1}(m\,\mathcal{F}f)\|_{L^p} \mid \|f\|_{L^p}=1\}$. We recall a sufficient condition on a function m to ensure that $m \in M_p(\mathbb{R}^{1+N})$ (see for instance [4, Lemma 6.1.5]) : if $m \in W^{K,2}(\mathbb{R}^{1+N})$, where $K > \frac{1+N}{2}$, then m is in $M_p(\mathbb{R}^{1+N})$ and there exists a constant $C > 0$ such that

$$\|\mathcal{F}^{-1}(m\,\mathcal{F}f)\|_{L^p} \leq C\|m\|_{W^{K,2}}\|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^{1+N}),\ \forall\, 1 \leq p < +\infty.$$

With this result, we can show the following lemma.

Lemma 2.4. *Let $h \in C_c^\infty(\mathbb{R}^{1+N})$ such that supp $h \subset \big\{2^{-\alpha} \leq (\tau^2 + |\xi|^2)^{\frac{1}{2}} \leq 2^\alpha\big\}$ with $\alpha \in \mathbb{N}^*$. Let $h_j := h(2^j\,\cdot)$. Let $K \in \mathbb{N}$ such that $K > \frac{1+N}{2}$. Let $\tilde{m} \in C^\infty(\mathbb{R}^{1+N})$ such that*

$$|D^\alpha \tilde{m}|(\tau, \xi) \leq \frac{C}{(\tau^2 + |\xi|^2)^{|\alpha|/2}} \quad \forall\, (\tau, \xi) \in \mathbb{R}^{1+N},\ \forall \alpha \in \mathbb{N}^{1+N},\ |\alpha| \leq K. \tag{21}$$

Then, $m = \tilde{m}h_j \in M_p(\mathbb{R}^{1+N})$ and $\|m\|_{M_p} \leq C$ where C does not depend on j.

Proof of Lemma 2.4. First, let us remark that for any $f, m : \mathbb{R}^{1+N} \to \mathbb{R}$, for any $b > 0$,

$$\mathcal{F}^{-1}(m\,\mathcal{F}f)(X) = \mathcal{F}^{-1}\big(m(b\,\cdot)\mathcal{F}\big(f(b^{-1}\cdot)\big)(\cdot)\big)(bX) \quad \forall X \in \mathbb{R}^{1+N}.$$

Thus,

$$\begin{aligned}
\|\mathcal{F}^{-1}(m\,\mathcal{F}f)\|_{L^p} &= b^{-\frac{1+N}{p}} \big\|\mathcal{F}^{-1}\big(m(b\,\cdot)\mathcal{F}\big(f(b^{-1}\cdot)\big)(\cdot)\big)\big\|_{L^p} \\
&\leq b^{-\frac{1+N}{p}} \|m(b\,\cdot)\|_{W^{K,2}} \|f(b^{-1}\cdot)\|_{L^p} \\
&= \|m(b\,\cdot)\|_{W^{K,2}} \|f\|_{L^p}
\end{aligned}$$

as soon as $m \in W^{K,2}(\mathbb{R}^{1+N})$. Hence, for all $b > 0$, $\|m\|_{M_p} \leq \|m(b\,\cdot)\|_{W^{K,2}}$. Therefore, we have $\|m\|_{M_p} \leq \|\tilde{m}(2^j\cdot)h\|_{W^{K,2}}$ for all $j \geq 1$. It is not hard to see that (21) implies that $\|\tilde{m}(2^j\cdot)h\|_{W^{K,2}} \leq C$ where C does not depend on j. Thus, Lemma 2.4 is proved. ☐

Let us start with the term ω_{j0}:

$$\hat{\omega}_{j0}(\tau, \xi) = \chi_0(\tau, \xi)\hat{\phi}_j(\tau, \xi)\hat{\rho}_\theta(\tau, \xi).$$

Since $\text{supp}(\chi_0\hat{\phi}_j) \subset \big\{2^{j-1} \leq (\tau^2 + |\xi|^2)^{\frac{1}{2}} \leq 2^{j+1}\big\} \cap \big\{|\tau| \leq \sqrt{3}|\xi|\big\} = D_0^+ \cup D_0^-$ (see Figure 1), then $\text{supp}(\chi_0\hat{\phi}_j) \subset \big\{2^{j-2} \leq |\xi| \leq 2^{j+1}\big\}$ and $\sum_{k=j-2}^{j+1} \hat{\psi}_k(\xi) = 1$ for all $\xi \in \text{supp}(\chi_0\hat{\phi}_j)$. Hence,

$$\hat{\omega}_{j0}(\tau, \xi) = \sum_{k=j-2}^{j+1} \chi_0(\tau, \xi)\hat{\phi}_j(\tau, \xi)\hat{\psi}_k(\xi)\hat{\rho}_\theta(\tau, \xi).$$

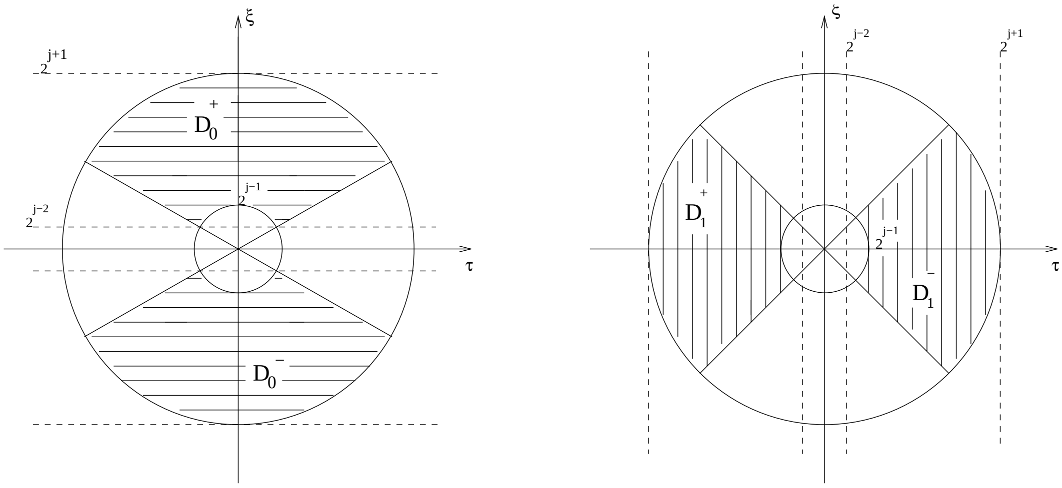

Figure 1: $\operatorname{supp}(\chi_0\hat{\phi}_j) \subset D_0^+ \cup D_0^-$, $\operatorname{supp}(\chi_1\hat{\phi}_j) \subset D_1^+ \cup D_1^-$

By Lemma 2.4 and by the assumption on χ_0, $m_0(\tau,\xi) := \chi_0(\tau,\xi)\hat{\phi}_j(\tau,\xi) \in M_p$ and

$$\|\omega_{j0}\|_{L^p}^p \leq \sum_{k=j-2}^{j+1} \|m_0\|_{M_p}^p \left\|\mathcal{F}_{t,x}^{-1}\left(\hat{\psi}_k(\xi)\hat{\rho}_\theta(\tau,\xi)\right)\right\|_{L^p_{t,x}}^p \leq C \sum_{k=j-2}^{j+1} \|\psi_k *_x \rho_\theta\|_{L^p_{t,x}}^p, \quad (22)$$

where C does not depend on j. Therefore, by (19) and (22),

$$\begin{aligned} \sum_{j=1}^{+\infty} 2^{sjp}\|\omega_{j0}\|_{L^p}^p &\leq C \sum_{j=1}^{+\infty} 2^{sjp} \sum_{k=j-2}^{j+1} \|\psi_k *_x \rho_\theta\|_{L^p_{t,x}}^p \\ &\leq C \sum_{j=1}^{+\infty} 2^{sjp}\|\psi_k *_x \rho_\theta\|_{L^p_{t,x}}^p \\ &\leq C\,\|\rho_\theta\|_{L^p_t W^{s,p}_x}^p. \end{aligned} \quad (23)$$

Next, let us consider ω_{j1}:

$$\hat{\omega}_{j1} = \chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\frac{\hat{G}(\tau,\xi)}{i\tau}.$$

Let $h \in C_c^\infty(\mathbb{R}^2)$ such that supp $h \subset \left\{\frac{1}{8} \leq (\tau^2 + |\xi|^2)^{\frac{1}{2}} \leq 4\right\}$ and such that $h \equiv 1$ on $\left\{\frac{1}{4} \leq (\tau^2 + |\xi|^2)^{\frac{1}{2}} \leq 2\right\}$. Let us define $h_j := h(2^j\,\cdot)$. In this way, $h_j \equiv 1$ on supp $\hat{\phi}_j$, and we have

$$\hat{\omega}_{j1}(\tau,\xi) = \chi_1(\tau,\xi)\frac{h_j(\tau,\xi)}{i\tau}\,\hat{\phi}_j(\tau,\xi)\hat{G}(\tau,\xi).$$

Since $G \in L^p(\mathbb{R}^{1+N})$, then $\mathcal{I}_s G := \mathcal{F}^{-1}_{t,x}\big((1+|\xi|^2+\tau^2)^{-\frac{s}{2}}\hat{G}\big) \in B^s_{p,p}(\mathbb{R}^{1+N})$ (see [14, Section 2.3.8]), and

$$\sum_{j=0}^{+\infty} 2^{sjp}\|\phi_j * \mathcal{I}_s G\|^p_{L^p} \le C\|G\|^p_{L^p}. \tag{24}$$

But, by Lemma 2.4, $m_1(\tau,\xi) = \chi_1(\tau,\xi)\frac{(1+|\xi|^2+\tau^2)^{\frac{s}{2}}}{i\tau}h_j(\tau,\xi) \in M_p$, and its norm does not depend on j. Indeed, as supp $\chi_1 \subset \{|\tau| \ge |\xi|\} \cap \{|\tau| \ge \frac{1}{2}\}$, one can easily show that $\tilde{m}_1(\tau,\xi) = \chi_1(\tau,\xi)\frac{(1+|\xi|^2+\tau^2)^{\frac{s}{2}}}{i\tau}$ satisfies (21). Since

$$\hat{\omega}_{j1} = m_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\mathcal{F}(\mathcal{I}_s G)(\tau,\xi),$$

then $\|\omega_{j1}\|^p_{L^p} \le C\|\phi_j * \mathcal{I}_s G\|^p_{L^p}$. By (24), we get

$$\sum_{j=1}^{+\infty} 2^{sjp}\|\omega_{j1}\|^p_{L^p} \le C\|G\|^p_{L^p}. \tag{25}$$

Finally, we consider ω_{j2}:

$$\hat{\omega}_{j2}(\tau,\xi) = \chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\frac{\xi\cdot\hat{\tilde{\rho}}_\theta(\tau,\xi)}{\tau}.$$

Since $\operatorname{supp}(\chi_1\hat{\phi}_j) \subset \{2^{j-1} \le (\tau^2+|\xi|^2)^{\frac{1}{2}} \le 2^{j+1}\} \cap \{|\tau| \ge |\xi|\} = D_1^+ \cup D_1^-$, then $\operatorname{supp}(\chi_1\hat{\phi}_j) \subset \{|\xi| \le 2^{j+1}\} \cup \{2^{j-2} \le |\tau| \le 2^{j+1}\}$ (see Figure 1), and $\sum_{k=0}^{j+1}\hat{\psi}_k(\xi) = 1$ for all $\xi \in \operatorname{supp}(\chi_1\hat{\phi}_j)$, and we have

$$\hat{\omega}_{j2}(\tau,\xi) = \sum_{k=0}^{j+1}\frac{1}{\tau}\chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)\,\hat{\psi}_k(\xi)\xi\cdot\hat{\tilde{\rho}}_\theta(\tau,\xi).$$

Let us set, for any function f defined on $\mathbb{R}$, $\mathcal{J}_1 f := \mathcal{F}^{-1}\big(\xi\hat{f}\big)$. By Lemma 6.2 of [4], $\|\mathcal{J}_1 f\|_{L^p} \le C2^k\|f\|_{L^p}$. Applying this result to the function $\psi_k * \tilde{\rho}_\theta(t,\cdot)$, we have for almost every $t \in \mathbb{R}$, $\|\mathcal{J}_1\psi_k *_x \tilde{\rho}_\theta(t,\cdot)\|^p_{L^p_x} \le C\,2^{kp}\|\psi_k *_x \tilde{\rho}_\theta(t,\cdot)\|^p_{L^p_x}$. Integrating in the variable t, we get

$$\|\mathcal{J}_1\psi_k *_x \tilde{\rho}_\theta\|^p_{L^p_{t,x}} \le C\,2^{kp}\|\psi_k *_x \tilde{\rho}_\theta\|^p_{L^p_{t,x}},$$

where $\mathcal{J}_1\psi_k *_x \tilde{\rho}_\theta = \mathcal{F}^{-1}_{t,x}\big(\hat{\psi}_k(\xi)\xi\cdot\hat{\tilde{\rho}}_\theta(\tau,\xi)\big)$.

Let us set $m_3(\tau,\xi) = \frac{1}{\tau}\chi_1(\tau,\xi)\hat{\phi}_j(\tau,\xi)$. Let h_j be the function defined above, let us recall that $h_j \equiv 1$ on supp $\hat{\phi}_j$, then $m_3(\tau,\xi) = \frac{1}{\tau}\,\tilde{m}(\tau,\xi)\hat{\phi}_j(\tau,\xi)$, where $\tilde{m}(\tau,\xi) := \chi_1(\tau,\xi)h_j(\tau,\xi)$. Since $\operatorname{supp}(h_j) \subset \{2^{j-3} \le (\tau^2+|\xi|^2)^{\frac{1}{2}} \le 2^{j+2}\}$, then $\operatorname{supp}(\chi_1 h_j) \subset \{2^{j-4} \le |\tau| \le 2^{j+2}\}$ and $\tilde{m}$ satisfies

$$|D^\alpha\tilde{m}(\tau,\xi)| \le \frac{2^{-j}C}{(\tau^2+|\xi|^2)^{|\alpha|/2}} \quad \forall\,(\tau,\xi) \in \mathbb{R}^{1+N}, \forall\,\alpha \in \mathbb{N}^{1+N}, |\alpha| \le K,$$

where C does not depend on j and $K > \frac{1+N}{2}$. By the same argument as Lemma 2.4, one can show that $m_3 = \tilde{m}\hat{\phi}_j \in M_p(\mathbb{R}^{1+N})$, with $\|m_3\|_{M_p} \le 2^{-j}C$. Then,

$$\|\omega_{j2}\|_{L^p}^p \le 2^{-j}C\sum_{k=0}^{j+1}\|\mathcal{J}_1\psi_k * \tilde{\rho}_\theta\|_{L^p}^p \le 2^{-j}C\sum_{k=0}^{j+1}2^{kp}\|\psi_k * \tilde{\rho}_\theta\|_{L^p}^p$$

and

$$\begin{aligned}\sum_{j=1}^{+\infty}2^{sjp}\|\omega_{j2}\|_{L^p}^p &\le C\sum_{j=1}^{+\infty}2^{(s-1)jp}\sum_{k=0}^{j+1}2^{kp}\|\psi_k * \tilde{\rho}_\theta\|_{L^p}^p \\ &\le C\sum_{k=0}^{+\infty}\left(\sum_{j=k-1}^{+\infty}2^{(s-1)jp}\right)2^{kp}\|\psi_k * \tilde{\rho}_\theta\|_{L^p}^p \\ &\le C\sum_{k=0}^{+\infty}2^{skp}\|\psi_k * \tilde{\rho}_\theta\|_{L^p}^p \\ &\le C\|\tilde{\rho}_\theta\|_{L_t^pW_x^{s,p}}^p.\end{aligned} \tag{26}$$

By (23), (25) and (26), we have

$$\sum_{j=1}^{+\infty}2^{sjp}\|\phi_j * \rho_\theta\|_{L^p}^p \le C\left(\|\rho_\theta\|_{L_t^pW_x^{s,p}}^p + \|\tilde{\rho}_\theta\|_{L_t^pW_x^{s,p}}^p + \|G\|_{L^p}^p\right). \tag{27}$$

By assumption, all the terms in the right hand side of (27) are finite. Moreover, $\|\phi_0 * \rho_\theta\|_{L^p} \le \|\phi_0\|_{L^1}\|\rho_\theta\|_{L^p}$, then $\sum_{j=0}^{+\infty}2^{sjp}\|\phi_j * \rho_\theta\|_{L^p}^p$ is finite and ρ_θ is in $W^{s,p}(\mathbb{R}^{1+N})$, which proves Proposition 2.3. □

Proof of Theorem 1.3. Theorem 1.3 is a consequence of the Propositions 2.1, 2.2 and 2.3. What we still have to prove is the inequality (8).

Let $\phi \in C_c^\infty(\mathbb{R}^N)$ and $\rho \in \mathcal{V}(f)$ associated to ϕ by (4). Let $T \in (0,+\infty)$ and I be any compact of $\mathbb{R}^N$. Let $\theta \in C_c^\infty(\mathbb{R}^{1+N})$ such that $\theta \equiv 1$ on $[0,T]\times I$ and supp $\theta \subset [-1,T+1]\times I_1$, where I_1 is the set of points whose distance to I is less than 1. Let us extend g (resp. any function of $\mathcal{V}(f)$) by 0 on $(-\infty,0)\times\mathbb{R}^{2N}$ (resp. on $(-\infty,0)\times\mathbb{R}^N$). Then,

$$\|\rho\|_{W^{s,p}([0,T]\times I)} \le \|\rho_\theta\|_{W^{s,p}(\mathbb{R}^{1+N})} \le C\left(\sum_{j=0}^{+\infty}2^{sjp}\|\phi_j * \rho_\theta\|_{L^p}^p\right)^{\frac{1}{p}}.$$

Since ρ is extended by 0 on $(-\infty,0)\times\mathbb{R}^N$, then supp $\rho_\theta \subset [0,T+1]\times I_1$. Hence, $\|\rho_\theta\|_{L^pW^{s,p}} \le C\|\rho\|_{L^p([0,T+1],W^{s,p}(I_1))}$. The same inequality holds for $\tilde{\rho}$ defined from ρ as in the proof of Proposition 2.3. Moreover, by definition of G, we have $\|G\|_{L^p} \leq \|g\|_{L^p([0,T+1]\times I_1\times K)}$ where $K = \text{supp}(\phi)$. Therefore, by (27) and by Proposition 2.1 or 2.2 applied to the velocity averages ρ, $\tilde{\rho}$, to the positive constant $T+1$ and to the compact I_1, we have

$$\|\rho\|_{W^{s,p}([0,T]\times I)} \leq C\|g\|_{L^p([0,T+1]\times J, W^{s,p}(K))},$$

where $C > 0$, J compact of $\mathbb{R}^N$ which only depend on T, s, p and on the support, the sup and Lipschitz norms of a, a^{-1} and ϕ. Theorem 1.3 is proved. □

3. The case $p = 1$

3.1. Optimality of Theorem 1.3. If we assume that f satisfies (1) with g in $L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x, W^{s,1}_{loc}(\mathbb{R}_v))$, where $s \in (0,1)$, then, by Theorem 1.3, any velocity average of f is in $W^{s,1}_{loc}(\mathbb{R}^+_t\times \mathbb{R}_x)$. But, if we assume that g is in $L^1_{loc}(\mathbb{R}^+_t\times \mathbb{R}_x, W^{1,1}_{loc}(\mathbb{R}_v))$, can we obtain an estimation of the BV norm of velocity averages of f by the $L^1_{t,x}W^{1,1}_v$ norm of g? The following proposition proves that a Poincaré-type inequality in L^2 can't hold, i.e., we can't estimate the L^2 norm of the velocity average of f by the $L^1_{t,x}W^{1,1}_v$ norm of g. A fortiori, since $BV(\mathbb{R}^2)$ is continuously embedded in $L^2(\mathbb{R}^2)$ (see [1], Theorem 3.47), the answer to the above question is no.

Proposition 3.1. *There exists a family (g_ε) of C^∞functions, uniformly bounded in $L^1(\mathbb{R}^+\times\mathbb{R}, W^{1,1}(\mathbb{R}))$ and there exists $\phi \in C^\infty_c(\mathbb{R})$ such that the L^2 norm of ρ_ε, the velocity averages associated to ϕ of the solution f_ε of* (12) *with g_ε as source term, is not bounded independently on ε.*

Proof. Let $\chi \in C^\infty_c(\mathbb{R}^2)$ such that supp $\chi \subset [-1,1]^2$, $\chi \geq 0$, and $\int_{\mathbb{R}^2}\chi = 1$. Let $t_0 \geq 1$. For any $\varepsilon > 0$, let us set $\chi_\varepsilon(t,x) = \frac{1}{\varepsilon^2}\chi(\frac{t-t_0}{\varepsilon}, \frac{x}{\varepsilon})$. Then $\int_{\mathbb{R}^2}\chi_\varepsilon = 1$. Moreover, if $\psi \in L^1_{loc}(\mathbb{R}^2)$, then $\chi_\varepsilon * \psi \to \tilde{\psi}$ in L^1_{loc}, where $\tilde{\psi}(t,x) = \psi(t-t_0, x)$.

Let $h \in C^\infty_c(\mathbb{R})$, $h(v) = 1$ for all $v \in [-1,1]$, and supp $h \subset [-2,2]$. Let us consider the following family of source terms:

$$g_\varepsilon(t,x,v) = \chi_\varepsilon(t,x)h(v).$$

For all $\varepsilon > 0$, $g_\varepsilon \in C^\infty_c(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}))$ and the norm of g_ε in $L^1(\mathbb{R}^+_t\times\mathbb{R}_x, W^{1,1}(\mathbb{R}_v))$ does not depend on ε.

Let us take $\phi \in C^\infty_c(\mathbb{R})$ such that $\phi(v) = 1$ for all $v \in [-2,2]$. The velocity average associated to ϕ of the solution f_ε of (12) with g_ε as source term is given by

$$\begin{aligned}\rho_\varepsilon(t,x) &= \int_0^t\int_{\mathbb{R}} g_\varepsilon\left(\tau, z, \tfrac{x-z}{t-\tau}\right)\phi\left(\tfrac{x-z}{t-\tau}\right)\frac{1}{t-\tau}\,dz\,d\tau\\ &= \int_0^T\int_{\mathbb{R}}\chi_\varepsilon(\tau,z)\mathbf{1}_{\{t-\tau\geq 0\}}h\big(\tfrac{x-z}{t-\tau}\tfrac{1}{t-\tau}\big)\,dz\,d\tau\\ &= \chi_\varepsilon * \Phi(t,x),\end{aligned}$$

where $\Phi(t,x) = \mathbf{1}_{\{t\geq 0\}} h(\frac{x}{t})\frac{1}{t}$. $\Phi \in L^1_{loc}(\mathbb{R}^2)$, therefore $\rho_\varepsilon \to \tilde{\Phi}$ in L^1_{loc}. But, one can easily see that $\Phi \notin L^2_{loc}(\mathbb{R}^2)$. Thus, the L^2 norm of ρ_ε can't be bounded independently on ε. □

3.2. A second example. The first example described in the previous section consisted to choose a family of source terms (g_ε) which approaches a Dirac mass in the two-dimensional (t,x) space. When the kinetic equation (1) comes from a conservation law with entropy condition, the source term g is a Radon measure which can't concentrate on sets of Hausdorff dimension less than 1. Indeed, g satisfies

$$\sup_{R>0} \frac{g\left(B_R(t,x)\times\mathbb{R}\right)}{R} < +\infty,$$

for almost every $(t,x) \in (0,+\infty)\times\mathbb{R}$. But, even in this context, we can't obtain an estimation of the BV norm of velocity averages by the $L^1_{t,x}BV_v$ norm of the source term, as the following proposition shows.

Proposition 3.2. *There exists a family (g_ε) of C^∞functions, uniformly bounded in $L^1(\mathbb{R}^+_t \times \mathbb{R}_x, BV(\mathbb{R}_v))$, such that for almost every $(t,x) \in (0,+\infty)\times\mathbb{R}$,*

$$\exists\, C>0, \quad \sup_{R>0} \frac{1}{R}\int_{B_R(t,x)}\int_{\mathbb{R}} |g_\varepsilon(\tau,y,v)|\, dv\, dy\, d\tau \leq C,$$

and there exists a $\phi \in C^\infty_c(\mathbb{R})$ such that the BV norm of ρ_ε, the velocity average associated to ϕ of the solution f_ε of (12) *with g_ε as source term, is not bounded independently of ε.*

Proof. Let $\sigma \geq 0$. Let $\chi \in C^\infty_c(\mathbb{R})$ such that supp $\chi \subset [-1,1]$, $\chi \geq 0$ and $\int \chi = 1$. Let us set $\chi_\varepsilon(\cdot) := \frac{1}{\varepsilon}\chi(\frac{\cdot}{\varepsilon})$. Let h be the indicator function of the interval $[\sigma-1,\sigma+1]$ and let us set $h_\varepsilon := h * \chi_\varepsilon$. Then, $h_\varepsilon \to h$ in L^1 and $\|h_\varepsilon\|_{L^1} \leq \|h\|_{L^1}$. We consider the family (g_ε) defined by

$$g_\varepsilon(t,x,v) = \chi_\varepsilon(x-\sigma t)h_\varepsilon(v) \quad \forall (t,x,v) \in \mathbb{R}^+\times\mathbb{R}^2.$$

For all $\varepsilon>0$, $g_\varepsilon \in C^\infty(\mathbb{R}^+\times\mathbb{R}^2)$. For any $T\in(0,+\infty)$ and for any compact I of $\mathbb{R}$, $\|g_\varepsilon\|_{L^1([0,T]\times\mathbb{R},BV(\mathbb{R}))}$ is uniformly bounded in ε. For a.e. $(t,x)\in(0,+\infty)\times\mathbb{R}$, there exists a $C>0$ such that

$$\frac{1}{R}\int_{B_R(t,x)}\int_{\mathbb{R}} |g_\varepsilon(\tau,y,v)|\, dv\, dy\, d\tau \leq C \quad \forall R>0.$$

Let f_ε be the solution of (12) with g_ε as source term. Let $\phi \in C^\infty_c(\mathbb{R})$ such that $\phi \equiv 1$ on $[\sigma-2,\sigma+2]$ and let ρ_ε be the velocity average of f_ε associated to ϕ

by (1.4). Then, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$,

$$\begin{aligned}\rho_\varepsilon(t,x) &= \int_0^t \int_{\mathbb{R}} g_\varepsilon\left(\tau, z, \tfrac{x-z}{t-\tau}\right) \phi\left(\tfrac{x-z}{t-\tau}\right) \frac{1}{t-\tau}\, dz\, d\tau \\ &= \int_0^t \int_{\mathbb{R}} \chi_\varepsilon(z-\sigma\tau) h_\varepsilon\big(\tfrac{x-z}{t-\tau}\big) \frac{1}{t-\tau}\, dz\, d\tau \\ &= \int_0^t \int_{\mathbb{R}} \chi_\varepsilon(z) h_\varepsilon\big(\tfrac{x-z-\sigma\tau}{t-\tau}\big) \frac{1}{t-\tau}\, dz\, d\tau.\end{aligned}$$

The sequence (ρ_ε) converges in L^1_{loc} to ρ defined by

$$\rho(t,x) := \int_0^t h\big(\tfrac{x-\sigma\tau}{t-\tau}\big) \frac{d\tau}{t-\tau} \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Indeed, for any $T \in (0,+\infty)$,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} |\rho_\varepsilon - \rho| &\le \int_0^T \int_{\mathbb{R}} \int_0^t \left| \int_{\mathbb{R}} \chi_\varepsilon(z) h_\varepsilon\left(\tfrac{x-z-\sigma\tau}{t-\tau}\right) \frac{dz}{t-\tau} - h\left(\tfrac{x-\sigma\tau}{t-\tau}\right) \frac{1}{t-\tau} \right| d\tau\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}} \int_0^t \left| \int_{\mathbb{R}} \chi_\varepsilon(z) \Big[h_\varepsilon\left(\tfrac{x-z-\sigma\tau}{t-\tau}\right) - h\left(\tfrac{x-\sigma\tau}{t-\tau}\right) \Big] \frac{dz}{t-\tau} \right| d\tau\, dx\, dt,\end{aligned}$$

since $\int \chi_\varepsilon = 1$. Thus,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} |\rho_\varepsilon - \rho| \le{}& \int_0^T \int_{\mathbb{R}} \int_0^t \left| \int_{\mathbb{R}} \chi_\varepsilon(z) \Big[h\left(\tfrac{x-z-\sigma\tau}{t-\tau}\right) - h\left(\tfrac{x-\sigma\tau}{t-\tau}\right) \Big] \frac{dz}{t-\tau} \right| d\tau\, dx\, dt \\ &+ \|\chi_\varepsilon\|_\infty \int_0^T \int_0^T \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left| h_\varepsilon\left(\tfrac{x-z-\sigma\tau}{t-\tau}\right) - h\left(\tfrac{x-z-\sigma\tau}{t-\tau}\right) \right| \frac{dx}{t-\tau} \right] dz\, d\tau\, dt.\end{aligned}$$

The first term tends to 0 by the dominated convergence theorem and since the function $x \mapsto h\left(\frac{x-\sigma\tau}{t-\tau}\right) \frac{1}{t-\tau}$ is in L^1 for any $\tau < t$ and its L^1 norm does not depend on t, τ. The second term also tends to 0, again using the dominated convergence theorem and the convergence in L^1 of (h_ε) to h. But, the limit ρ of (ρ_ε) is not in $BV_{loc}(\mathbb{R}^+ \times \mathbb{R})$. Indeed, since $\frac{x-\sigma\tau}{t-\tau} \in [\tau-1, \tau+1]$ iff $|x-\sigma t| \le (t-\tau)$, then

$$\rho(t,x) = \int_0^{t-|x-\sigma t|} \frac{d\tau}{t-\tau} = \ln t - \ln|x-\sigma t| \notin BV_{loc}(\mathbb{R}^+ \times \mathbb{R}).$$

Therefore, the BV norm of (ρ_ε) can't be bounded independently of ε. □

3.3. Weak Poincaré-type inequality. We assume in this section that the derivative in v of g, $\partial_v g$, is a Radon measure in $\mathbb{R}^+ \times \mathbb{R}^2$ and we establish some weak Poincaré-type inequality in the sense that, instead of the L^2 norm, we estimate the $L^{2,\infty}$ norm of the velocity averages. Before stating the result, let us recall some useful facts about the Lorentz space $L^{2,\infty}(\mathbb{R}^2)$ (also called weak L^2 space). We refer to [9] for further details and proofs.

Let $f : \mathbb{R}^2 \to \mathbb{R}$. The distribution function of ρ is defined by

$$\lambda_f(\alpha) = \mathcal{L}^2 \left(\left\{ x \in \mathbb{R}^2 \,\middle|\, |f(x)| > \alpha \right\} \right) \quad \forall \alpha > 0.$$

The non-increasing rearrangement f^* of f is given by

$$f^*(y) = \inf\left\{ \alpha > 0 \,|\, \lambda_f(\alpha) \leq y \right\}.$$

The Lorentz space $L^{2,\infty}(\mathbb{R}^2)$ is the space of functions f such that

$$\|f\|^*_{2,\infty} := \sup_{y>0} \left(y^{\frac{1}{2}} f^*(y) \right) < +\infty.$$

This quantity only defines a quasi-norm on $L^{2,\infty}$, but it is possible to define a norm in $L^{2,\infty}$, denoted by $\|\cdot\|_{2,\infty}$, which satisfies

$$\|f\|^*_{2,\infty} \leq \|f\|_{2,\infty} \leq 2\|f\|^*_{2,\infty} \quad \forall f \in L^{2,\infty}(\mathbb{R}^2). \tag{28}$$

Precisely, this norm is defined in the following way : let us define for all $y > 0$,

$$f^{**}(y) = \sup_{E\subset\mathbb{R}^2, \mathcal{L}^2(E)\geq y} \frac{1}{\mathcal{L}^2(E)} \int_E |f|,$$

the norm $\|\cdot\|_{2,\infty}$ is given by $\|f\|_{2,\infty} := \|f^{**}\|^*_{2,\infty}$ for all $f \in L^{2,\infty}(\mathbb{R}^2)$. For any subset Ω of $\mathbb{R}^2$, $L^{2,\infty}(\Omega)$ is defined in the same way : the definition of the distribution function λ_f of a function f defined on Ω is replaced by

$$\lambda_f(\alpha) = \mathcal{L}^2 \left(\left\{ x \in \Omega \,\middle|\, |f(x)| > \alpha \right\} \right) \quad \forall \alpha > 0,$$

and the sup in the definition of f^{**} is taken over all $E \subset \Omega$.

Let us mention some properties, which will be useful in the following. First, we have

$$\|f\|^*_{2,\infty} = \sup_{y>0} \left(y^{\frac{1}{2}} f^*(y) \right) = \sup_{\alpha>0} \left(\alpha \left[\lambda_f(\alpha) \right]^{\frac{1}{2}} \right) \quad \forall f \in L^{2,\infty}(\mathbb{R}^2). \tag{29}$$

Secondly, we have,

$$\forall y > 0, \quad f^*(y) \leq f^{**}(y) \tag{30}$$

and

$$(f^{**})^* = f^{**}. \tag{31}$$

The last fact comes from the property of f^{**} to be non-negative and non-increasing. Let us remark that from the two last properties, one can obtain easily the first inequality in (28).

Proof of Theorem 1.4. Let us assume that $g \in C^\infty(\mathbb{R}^+\times \mathbb{R}^2)$ and $\phi \in C_c^\infty(\mathbb{R})$, then, as in the proof of Proposition 2.2, $\rho \in \mathcal{V}(f)$ associated to ϕ by (4) is given by

$$\rho(t,x) = \int_0^t \int_{\mathbb{R}} g\left(\tau, z, a^{-1}\left(\tfrac{x-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)} \frac{1}{t-\tau}\, dz\, d\tau.$$

Let us fix $y > 0$. By (30),

$$\rho^*(y) \le \rho^{**}(y) = \sup_{\substack{E\subset\mathbb{R}^+\times\mathbb{R}\\ y\le\mathcal{L}^2(E)<+\infty}} \frac{1}{\mathcal{L}^2(E)} \int_E |\rho(t,x)|\, dt\, dx.$$

For any $E \subset \mathbb{R}^+\times \mathbb{R}$ such that $y \le \mathcal{L}^2(E) < +\infty$,

$$\begin{aligned}
&\frac{1}{\mathcal{L}^2(E)} \int_E |\rho(t,x)|\, dt\, dx \\
&\quad \le \frac{1}{\mathcal{L}^2(E)} \int_E \int_0^T \int_{\mathbb{R}} \left| g\left(\tau, z, a^{-1}\left(\tfrac{x-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)} \frac{1}{t-\tau} \right| dz\, d\tau\, dt\, dx \\
&\quad = \int_0^T \int_{\mathbb{R}} \left[\frac{1}{\mathcal{L}^2(E)} \int_E \left| g\left(\tau, z, a^{-1}\left(\tfrac{x-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)} \frac{1}{t-\tau} \right| dt\, dx \right] dz\, d\tau.
\end{aligned}$$

Let us show that the function

$$\Psi(t,x) = \left| g\left(\tau, z, a^{-1}\left(\tfrac{x-z}{t-\tau}\right)\right) \frac{\phi\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)}{a'\left(a^{-1}\left(\frac{x-z}{t-\tau}\right)\right)} \frac{1}{t-\tau} \right|$$

is in $L^{2,\infty}(\mathbb{R}^+\times \mathbb{R})$ for all $(\tau, z) \in [0,T] \times \mathbb{R}$.

Since $BV(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$, then there exists $C > 0$ such that $\|g(\tau,z,\cdot)\|_\infty \le C\|g(\tau,z,\cdot)\|_{BV}$, for any $(\tau,z) \in [0,T]\times\mathbb{R}$. Let $K :=$ supp ϕ. Let $M > 0$ such that $a(K) \subset [-M,M]$, then $\phi\left(\left(a^{-1}(\frac{x-z}{t-\tau})\right)\right) = 0$ as soon as $|x-z| > M|t-\tau|$. Therefore, one have to take into account only the points (t,x) such that $|x-z| \le M|t-\tau|$ and for these points, we have

$$\frac{1}{|t-\tau|} \le \frac{\sqrt{M^2+1}}{(|x-z|^2+|t-\tau|^2)^{\frac12}}.$$

Hence,

$$0 \le \Psi(t,x) \le C\, \frac{\|g(\tau,z,\cdot)\|_\infty \|\frac{\phi}{a'}\|_\infty}{(|x-z|^2+|t-\tau|^2)^{\frac12}}$$

and

$$\{(t,x) \mid \Psi(t,x) \ge \alpha\} \subset \left\{(t,x) \mid (|x-z|^2+|t-\tau|^2)^{\frac12} \le \tfrac{C}{\alpha}\|g(\tau,z,\cdot)\|_\infty \left\|\tfrac{\phi}{a'}\right\|_\infty\right\}$$

for all $\alpha > 0$. Thus, $\lambda_\Psi(\alpha) \leq \frac{C}{\alpha^2}\|g(\tau,z,\cdot)\|_\infty^2 \|\frac{\phi}{a'}\|_\infty^2$. Then, by (29), $\Psi \in L^{2,\infty}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\|\Psi\|_{2,\infty}^* \leq C\,\|g(\tau,z,\cdot)\|_\infty \left\|\tfrac{\phi}{a'}\right\|_\infty \leq C\,\|g(\tau,z,\cdot)\|_{BV} \left\|\tfrac{\phi}{a'}\right\|_\infty. \tag{32}$$

For any $E \subset \mathbb{R}^+ \times \mathbb{R}$ such that $y \leq \mathcal{L}^2(E) < \infty$,

$$\frac{1}{\mathcal{L}^2(E)} \int_E \Psi(t,x)\,dt\,dx \leq \Psi^{**}(y). \tag{33}$$

Since $\Psi \in L^{2,\infty}(\mathbb{R}^+ \times \mathbb{R})$, then, by (28) and (31), for all $y > 0$,

$$y^{\frac{1}{2}}\Psi^{**}(y) \leq \|\Psi\|_{2,\infty} \leq 2\|\Psi\|_{2,\infty}^*. \tag{34}$$

Therefore, by (33) and (34), for any $y > 0$, for any $E \subset \mathbb{R}^+ \times \mathbb{R}$ such that $y \leq \mathcal{L}^2(E) < +\infty$,

$$\begin{aligned}\frac{1}{\mathcal{L}^2(E)} \int_E |\rho(t,x)|\,dt\,dx &\leq Cy^{-\frac{1}{2}} \left\|\tfrac{\phi}{a'}\right\|_\infty \int_0^T \int_{\mathbb{R}} \|g(\tau,z,\cdot)\|_{BV}\,dz\,d\tau \\ &= Cy^{-\frac{1}{2}} \left\|\tfrac{\phi}{a'}\right\|_\infty \|g\|_{L^1_{t,x}BV_v}.\end{aligned}$$

Hence, for any $y > 0$, $y^{\frac{1}{2}}\rho^*(y) \leq C\|\frac{\phi}{a'}\|_\infty \|g\|_{L^1_{t,x}BV_v}$ and (11) is proved when g is C^∞.

Now, let $g \in L^1(\mathbb{R}_t^+ \times \mathbb{R}_x, BV(\mathbb{R}_v))$ and $\phi \in C_c^\infty(\mathbb{R})$. Let $f \in L^1(\mathbb{R}^+ \times \mathbb{R}^2)$ be the solution of (1) and $\rho \in \mathcal{V}(f)$ associated to ϕ by (4). Let $\xi \in C_c^\infty(\mathbb{R}^3)$ such that supp $\xi \subset [-1,1]^3$, $\xi \geq 0$ and $\int \xi = 1$. Let us set $\xi_n := n^3\xi(n\,\cdot)$ and $g_n := \xi_n * g$ for all $n \in \mathbb{N}$, where g is extended by 0 on $(-\infty,0) \times \mathbb{R}^2$. Then, $g_n \to g$ in $L^1(\mathbb{R}^2, BV(\mathbb{R}))$ and $\|g_n\|_{L^1_{t,x}BV_v} \leq C\|g\|_{L^1_{t,x}BV_v}$. For all $n \in \mathbb{N}$, let f_n be the solution of (1) with g_n as source term, and let $\rho_n \in \mathcal{V}(f_n)$ associated to ϕ. Up to a subsequence, f_n (resp. ρ_n) converges to f (resp. ρ) in the sense of distributions (as explained at the end of the proof of Proposition 2.1). Since g_n is in $C^\infty(\mathbb{R}^3)$, then

$$\|\rho_n\|_{2,\infty}^* \leq C\|g_n\|_{L^1_{t,x}BV_v} \leq C\|g\|_{L^1_{t,x}BV_v}.$$

Therefore, (ρ_n) is bounded in $L^{2,\infty}(\mathbb{R}^+ \times \mathbb{R})$ and the limit ρ is in $L^{2,\infty}(\mathbb{R}^+ \times \mathbb{R})$. Moreover, by the lower semicontinuity of the norm $\|\cdot\|_{2,\infty}$ and by (28) we have

$$\|\rho\|_{2,\infty}^* \leq \|\rho\|_{2,\infty} \leq \|\rho_n\|_{2,\infty} \leq C\|\rho_n\|_{2,\infty}^* \leq C\|g\|_{L^1_{t,x}BV_v}.$$

Theorem 1.4 is proved. □

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