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Comments on the Michael Selection Problem in Hyperconvex Metric Spaces

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Abstract. Let X be a paracompact space, H a hyperconvex metric space, and $\Phi : X \multimap H$ a l.s.c. multimap with nonempty closed values. Then Φ admits a continuous selection under certain restrictions. Such selection results are applied to obtain fixed point theorems.

Keywords: *Hyperconvex metric space, C-space, Γ-set, LC-metric space, selection, nonexpansive map, locally-uniformly weak lower semicontinuous*

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1. Introduction

Recently, in [23], its author showed that a selection problem raised by Michael has an affirmative solution for hyperconvex metric spaces and that the lower semicontinuity of the involved multimap in the problem can be weakened. Moreover, as an application of the selection result in [23], a fixed point theorem for "locally-uniformly weak" lower semicontinuous multimaps was given.

The notion of hyperconvex metric spaces was introduced by Aronszajn and Panitchpakdi [1] in 1956. Later, in 1979, independently Sine [19] and Soardi [21] proved that a bounded hyperconvex metric space has the fixed point property for nonexpansive maps. Since then many interesting works appeared for hyperconvex metric spaces.

For a long period, the study of hyperconvex metric spaces concentrated on the relationship with nonexpansive maps (see [20]). On the other hand, Khamsi [9] established the Knaster-Kuratowski-Mazurkiewicz theorem (in short, KKM theorem) for hyperconvex metric spaces and applied it to obtain a Schauder type fixed point theorem. This line of study was followed by Kirk [12], Kirk

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and Shin [13], Kim and Shin [11], and Park [15, 16]. The present author obtained extensions or equivalent forms of the KKM theorem, a Fan-Browder type fixed point theorem, and other results for hyperconvex metric spaces in [15, 16]. Moreover, Kirk, Sims, and Yuan [14] established the KKM theorem, its equivalent formulations, fixed point theorems, and their applications for hyperconvex metric spaces. Further related results also appeared in [10, 17, 18].

However, some of the above-mentioned works are simple consequences of much more general results. In fact, Horvath [3 – 7] initiated the study of the KKM theory and fixed point theory for C-spaces, which are meaningful generalizations of convex spaces or convex subsets of topological vector spaces. Moreover, in [7], he found that hyperconvex metric spaces are particular type of C-spaces and gave a useful selection theorem on l.s.c. multimaps related to C-spaces. Later, this selection theorem was extended by Ben-El-Mechaiekh and Oudadess [2] following some ideas from the celebrated theory on continuous selections due to Michael.

Our principal aim in the present paper is to show that main results of [23] are simple consequences of a selection theorem in [2] and a fixed point theorem in [9, 15]. This simplifies considerably proofs in [23]. Some additional comments on [23] are also stated.

2. Preliminaries

A metric space (H, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$. It is known that the space $\mathfrak{C}(E)$ of all continuous real functions on a Stonian space E (that is, an extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathfrak{C}(E)$ for some Stonian space E. Therefore, $(\mathbb{R}^n, \|\cdot\|_\infty)$, l^∞, and L^∞ are concrete examples of hyperconvex metric spaces. Recently, there appeared a number of new examples.

Results of Aronszajn and Panitchpakdi [1, Theorem 1′] and Isbell [8, Theorem 1.1.] are combined in the following.

Theorem 1. *A hyperconvex metric space is complete and (freely) contractible.*

The concepts of C-spaces, LC-spaces, and LC-metric spaces were introduced and extensively studied by Horvath in a sequence of papers [3 – 7]:

A *C-space* (X, Γ) is a topological space X with a multimap $\Gamma : \langle X \rangle \multimap X$ from the set $\langle X \rangle$ of all nonempty finite subsets of X into the power set of X such that

1. for each $A \in \langle X \rangle$, $\Gamma(A) = \Gamma_A$ is n-connected for all $n \geq 0$; and
2. for all $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A subset $Y \subset X$ is said to be Γ-*convex* if $A \in \langle Y \rangle$ implies $\Gamma_A \subset Y$.

A C-space (X, Γ) is called an *LC-space* (or a *locally H-convex space* [22]) if X is a Hausdorff uniform space and there exists a basis $\{V_\lambda\}_{\lambda \in I}$ for the uniform structure such that for each $\lambda \in I$, $\{x \in X : D \cap V_\lambda[x] \neq \emptyset\}$ is Γ-convex whenever $D \subset X$ is Γ-convex, where

$$V_\lambda[x] = \{x' \in X : (x, x') \in V_\lambda\}.$$

For example, any nonempty convex subset X of a locally convex Hausdorff topological vector space is an LC-space with $\Gamma_A = \operatorname{co} A$, the convex hull of $A \in \langle X \rangle$.

A triple $(X, d; \Gamma)$ is called an *LC-metric space* whenever (X, d) is a metric space and (X, Γ) is a C-space such that open balls are Γ-convex, and any neighborhoods $\{x \in X : d(x, Y) < r\}$ of a Γ-convex set $Y \subset X$ is also Γ-convex.

Horvath [7, Theorem 9] obtained the following

Theorem 2. *Any hyperconvex metric space H is a complete LC-metric space with $\Gamma_A = \bigcap\{B : B$ is a closed ball containing $A\}$ for each $A \in \langle H \rangle$.*

Note that Γ_A itself is hyperconvex. From now on, a hyperconvex metric space $(H, d; \Gamma)$ is simply denoted by H or (H, d). An *admissible* subset of H is a nonempty intersection of closed balls in H (see [9]). Moreover, in [23], a Γ-convex subset of H is said to be *sub-admissible*.

The following is due to Ben-El-Mechaiekh and Oudadess [2, Theorem 3].

Theorem 3. *Let X be paracompact, $(Y, d; \Gamma)$ a complete LC-metric space, $Z \subset X$ with $\dim_X Z \leq 0$, and $\Phi : X \multimap Y$ a lower semicontinuous (l.s.c.) multimap with nonempty closed values such that $\Phi(x)$ is Γ-convex for $x \notin Z$. Then Φ admits a continuous selection $f : X \to Y$ such that $f(x) \in \Phi(x)$ for all $x \in X$.*

Recall that $\dim_X Z \leq 0$ means that the covering dimension of Y is ≤ 0 for every set $Y \subset Z$ which is closed in X (see [2]).

It is known that if X is paracompact, (Y, Γ) is a C-space, and $\Phi : X \multimap Y$ is a multimap such that

1. $\Phi(x)$ is nonempty and Γ-convex for each $x \in X$; and
2. $\Phi^-(y) := \{x \in X : y \in \Phi(x)\}$ is open for each $y \in Y$ (hence Φ is l.s.c.),

then Φ admits a continuous selection (see Horvath [7, Theorem 3]).

A multimap Φ satisfying 1. and 2. is usually called a *Browder map*. Theorem 3 tells us that if (Y, Γ) is a complete LC-metric space, the above result holds for a slightly different class of multimaps.

3. Main results

Combining Theorems 1 – 3, we have the following result.

Theorem 4. *Let X be a paracompact space, (H,d) a hyperconvex metric space, $Z \subset X$ with $\dim_X Z \le 0$, and $\Phi : X \multimap H$ a l.s.c. multimap with nonempty closed values such that $\Phi(x)$ is Γ-convex for $x \notin Z$. Then Φ admits a continuous selection $f : X \to H$.*

Example 1. Recall that the set $\mathbb{R}$ of reals with the usual Euclidean metric is hyperconvex. Define a multimap $\Phi : \mathbb{R} \multimap \mathbb{R}$ by $\Phi(x) = \mathbb{R}$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$ and $\Phi(x)$ is any nonempty subset of $\mathbb{R}$ for each integer $x \in \mathbb{Z}$. Then $\Phi^-(y)$ is open for each $y \in \mathbb{R}$, and hence Φ is l.s.c. It can be seen that Φ has a continuous selection by observation.

Example 2. For L^∞, define a multimap $\Phi : \mathbb{R} \multimap L^\infty$ by $\Phi(x) = L^\infty$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$ and $\Phi(x)$ is any nonempty *closed* subset of L^∞ for each integer $x \in \mathbb{Z}$. Then $\Phi^-(y)$ is open for each $y \in L^\infty$, and hence Φ is l.s.c. Then Φ has a continuous selection by Theorem 4.

For $Z = \emptyset$, Theorem 4 reduces to the following

Corollary 1. [23, Theorem 2.3] *Let X be a paracompact topological space, (M,d) a hyperconvex metric space and Y a nonempty sub-admissible subset of M. Further, let $T : X \multimap Y$ be a multimap such that*:

(i) *For each $x \in X, T(x)$ is a nonempty closed sub-admissible subset of M.*

(ii) *T is lower semicontinuous.*

Then there exists a continuous function $f : X \to M$ such that $f(x) \in T(x)$ for all $x \in X$.

Note that, in [23], its author deduced Corollary 1 from a proximate selection theorem [23, Theorem 2.1]. For a topological space X and a metric space (Y,d), the author of [23] defined a *quasi-lower semicontinuous* multimap $T : X \multimap Y$ and a *locally-uniformly weak lower semicontinuous* multimap $T : X \multimap Y$.

From the proof of Theorem 2.4 in [23], we get the following

Theorem 5. *Let X be a paracompact space, (H,d) a hyperconvex metric space, and $T : X \multimap H$ a locally-uniformly weak l.s.c. multimap with nonempty closed Γ-convex values. Then there exists a l.s.c. multimap $T_0 : X \multimap H$ with nonempty closed Γ-convex values such that $T_0(x) \subset T(x)$ for all $x \in X$.*

In fact, in the proof of [23, Theorem 2.4], for each $r > 0$, a multimap $T_r : X \multimap Y$ is defined. Let $T_0(x) := \bigcap_{r>0} T_r(x)$ for each $x \in X$. Then it is shown that $T_0 : X \multimap Y$ is the required selection of T.

Combining Corollary 1 and Theorem 5, we obtain

Corollary 2. [23, Theorem 2.4] *Let $X, (H, d)$, and T be the same as in Theorem 5. Then there exists a continuous selection $f : X \to H$ of T.*

Note that in view of Theorem 5, Corollaries 1 and 2 are actually equivalent. Recall the following fixed point theorem due to the present author et al. [9, 15].

Theorem 6. [15, Theorem 5] *Let H be a hyperconvex metric space, X a compact admissible subset of H, and $f : X \to H$ a continuous function. Then f has a fixed point if one of the following conditions holds for all $x \in \operatorname{Bd} X$ such that $x \neq f(x)$:*

(i) *There exists a $y \in X$ such that*

$$d(x, f(x)) > d(y, f(x)).$$

(ii) *There exists a $\beta \in (0, 1)$ such that*

$$X \cap B(f(x), \beta d(x, f(x)) \neq \emptyset.$$

(iii) *There exists an $\alpha \in (0, 1)$ such that*

$$X \cap B(x, \alpha d(x, f(x))) \cap B(f(x), (1 - \alpha) d(x, f(x))) \neq \emptyset.$$

(iv) $f(x) \in X$.

Corollary 3. *Let H be a hyperconvex metric space and X a compact admissible subset of H. Then every Browder map $\Phi : X \multimap H$ satisfying $\Phi(\operatorname{Bd} X) \subset X$ has a fixed point.*

Proof. Since X is paracompact and H is a C-space, Φ has a continuous selection $f : X \to H$ by [7, Theorem 3] mentioned above at the end of Section 2. Moreover, for $x \in \operatorname{Bd} X$, we have $f(x) \in \Phi(x) \subset \Phi(\operatorname{Bd} X) \subset X$. Therefore by Theorem 6 (iv), f has a fixed point $x_0 \in X$, that is, $x_0 = f(x_0) \in \Phi(x_0)$. ∎

For a Browder map $\Phi : X \multimap X$, Corollary 3 reduces to the Fan-Browder type fixed point theorem for hyperconvex metric spaces (see [18]).

Corollary 4. *Let H be a hyperconvex metric space, X a compact admissible subset of H, and $\Phi : X \multimap H$ a (locally-uniformly weak) l.s.c. multimap having nonempty closed sub-admissible values. Then Φ has a fixed point if one of the following conditions holds for all $x \in \operatorname{Bd} X$ such that $x \notin \Phi(x)$:*

(i) *There exists a $y \in X$ such that*

$$d(x, z) > d(y, z) \text{ for all } z \in \Phi(x).$$

(ii) *For each $z \in \Phi(x)$, there exists a $\beta \in (0, 1)$ such that*

$$X \cap B(z, \beta d(x, z)) \neq \emptyset.$$

(iii) *For each* $z \in \Phi(x)$*, there exists an* $\alpha \in (0,1)$ *such that*

$$X \cap B(x, \alpha d(x,z)) \cap B(z, (1-\alpha)d(x,z)) \neq \emptyset.$$

(iv) $\Phi(x) \subset X$.

Proof. By Corollaries 1 and 2, there exists a continuous selection $f : X \to H$ of Φ satisfying the requirements of Theorem 6. Then f has a fixed point. This completes our proof. ∎

In [23, Theorem 2.5 and Corollary 2.6], particular forms of Cases (iii) and (iv) of Corollary 4 were obtained for a sub-admissible subset X. Recall that every compact sub-admissible subset X of a hyperconvex metric space is admissible (see [24, Proposition 1.4]).

Finally, the author of [23] noted that his results are different from the corresponding results of Horvath [6, 7] on selection problems and fixed point problems. However, we found that Theorem 2 of Horvath is the original source of the whole results in this paper.

References

[1] Aronszajn, N. and P. Panitchpakdi: *Extensions of uniformly continuous transformations and hyperconvex metric spaces*. Pacific J. Math. 6 (1956), 405 – 439.

[2] Ben-El-Mechaiekh, H. and M. Oudadess: *Some selection theorems without convexity*. J. Math. Anal. Appl. 195 (1995), 614 – 618.

[3] Horvath, C. D.: *Points fixes et coincidences dans les espaces topologiques compacts contractiles* (in French). C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), 519 – 521.

[4] Horvath, C. D.: *Some results on multivalued mappings and inequalities without convexitity*. In: Nonlinear Analysis and Convex Analysis (Santa Barbara, Calif., 1985; eds.: B. L. Lin and S. Simons). Lecture Notes in Pure and Appl. Math. 107. New York: Dekker 1987, pp. 99 – 106.

[5] Horvath, C. D.: *Convexité généralisée et applications*. Méthodes Topologiques en Analyse Convexe (Montréal (QC) 1986; ed.: A. Granas), Sém. Math. Sup. 110. Montréal (QC): Press. Univ. Montréal 1990, pp. 81 – 99.

[6] Horvath, C. D.: *Contractibility and generalized convexity*. J. Math. Anal. Appl. 156 (1991), 341 – 357.

[7] Horvath, C. D.: *Extension and selection theorems in topological spaces with a generalized convexity structure*. Annal. Fac. Sci. Toulouse 2 (1993), 253 – 269.

[8] Isbell, J. R.: *Six theorems about injective metric spaces*. Comment. Math. Helv. 39 (1964), 65 – 76.

[9] Khamsi, M. A.: *KKM and Ky Fan theorems in hyperconvex metric spaces*. J. Math. Anal. Appl. 204 (1996), 298 – 306.

[10] Kim, J.-H. and S. Park: *Comments on some fixed point theorems in hyperconvex metric spaces*. J. Math. Anal. Appl. 291 (2004), 154 – 164.

[11] Kim, T.-H. and S.-S. Shin: *Fixed point theorems on metric spaces*. Comm. Appl. Nonlinear Anal. 4 (1997), 77 – 85.

[12] Kirk, W. A.: *Continuous mappings in compact hyperconvex metric spaces*. Numer. Funct. Anal. Optimiz. 17 (1996), 599 – 603.

[13] Kirk, W. A. and S. S. Shin: *Fixed point theorems in hyperconvex spaces*. Houston J. Math. 23 (1997), 175 – 188.

[14] Kirk, W. A., B. Sims, and G. X.-Z. Yuan: *The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications*. Nonlinear Anal. 39 (2000), 611 – 627.

[15] Park, S.: *Fixed point theorems in hyperconvex metric spaces*. Nonlinear Anal. 37 (1999), 467 – 472.

[16] Park, S.: *The Schauder type and other fixed point theorems in hyperconvex spaces*. Nonlinear Anal. Forum 3 (1998), 1 – 12.

[17] Park, S.: *Fixed points of lower semicontinuous multimaps in LC-metric spaces*. J. Math. Anal. Appl. 235 (1999), 142 – 150.

[18] Park, S. and B. Sims: *Remarks on fixed point theorems on hyperconvex spaces*. Nonlinear Funct. Anal. Appl. 5 (2000), 51 – 64.

[19] Sine, R. C.: *On linear contraction semigroups in sup norm spaces*. Nonlinear Anal. 3 (1979), 885 – 890.

[20] Sine, R. C.: *Hyperconvexity and nonexpansive multifunctions*. Trans. Amer. Math. Soc. 315 (1989), 755 – 767.

[21] Soardi, P.: *Existence of fixed points of nonexpansive mappings in certain Banach lattices*. Proc. Amer. Math. Soc. 73 (1979), 25 – 29.

[22] Tarafdar, E.: *Fixed point theorems in locally H-convex uniform spaces*. Nonlinear Anal. 29 (1997), 971 – 978.

[23] Wu, X.: *Michael selection problem in hyperconvex metric spaces*. Z. Anal. Anwendungen 22 (2003), 505 – 516.

[24] Wu, X., B. Thompson, and G. X. Yuan: *Fixed point theorems of upper semicontinuous multivalued mappings with applications in hyperconvex metric spaces*. J. Math. Anal. Appl. 276 (2002), 80 – 89.

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