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Non-Compact and Sharp Embeddings of Logarithmic Bessel Potential Spaces into Hölder-Type Spaces

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Abstract. In our recent paper [Compact and continuous embeddings of logarithmic Bessel potential spaces. Studia Math. 168 (2005), 229 – 250] we have proved an embedding of a logarithmic Bessel potential space with order of smoothness σ less than one into a space of $\lambda(\cdot)$-Hölder-continuous functions. We show that such an embedding is not compact and that it is sharp.

Keywords. Generalized Lorentz-Zygmund spaces, logarithmic Bessel potential spaces, Hölder-continuous functions, embeddings

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1. Introduction

In the recent paper [8] we have derived embeddings of Bessel potential spaces with smoothness $\sigma \in (0,1)$, modelled upon generalized Lorentz-Zygmund spaces, into spaces of $\lambda(\cdot)$-Hölder-continuous functions. Here we discuss non-compactness and sharpness of those embeddings.

To be more specific, we need some notation. Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ or $X \hookrightarrow\hookrightarrow Y$ if $X \subset Y$ and the natural embedding is continuous or compact, respectively.

Let $p, q \in (0, \infty]$, $m \in \mathbb{N}$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and let Ω be a measurable subset of $\mathbb{R}^n$ (with respect to n-dimensional Lebesgue measure). The *generalized*

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Lorentz-Zygmund (GLZ) space $L_{p,q;\alpha_1,\dots,\alpha_m}(\Omega)$ consists of all measurable (real or complex) functions f on Ω such that the quantity

$$\|f\|_{p,q;\alpha_1,\dots,\alpha_m} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \left(\prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \right) f^*(t) \right\|_{q,(0,\infty)}$$

is finite. Here $\ell_1, \dots, \ell_m$ are (logarithmic) functions defined on $(0,\infty)$ by

$$\ell_1(t) = \ell(t) = 1 + |\log t|, \quad \ell_j(t) = 1 + \log \ell_{j-1}(t) \quad (j > 1),$$

f^* denotes the *non-increasing rearrangement* of f given by

$$f^*(t) = \inf \{ \lambda > 0 \; ; \; \left| \{ x \in \Omega ; |f(x)| > \lambda \} \right|_n \le t \}, \quad t \ge 0,$$

$|G|_n$ stands for the n-volume of a measurable subset G of $\mathbb{R}^n$ and $\|\cdot\|_{q,(a,b)}$ is the usual L^q-(quasi-)norm on an interval $(a,b) \subseteq \mathbb{R}$. (For more details about the spaces $L_{p,q;\alpha_1,\dots,\alpha_m}(\Omega)$ see [2]–[7], [9], and [11].)

The *Bessel kernel* g_σ, $\sigma > 0$, is defined to be that function on $\mathbb{R}^n$ whose Fourier transform $\widehat{g_\sigma}$ is

$$\widehat{g_\sigma}(\xi) = (2\pi)^{-\frac{n}{2}} \left(1 + |\xi|^2\right)^{-\frac{\sigma}{2}}, \quad \xi \in \mathbb{R}^n,$$

where by the Fourier transform $\widehat{f}$ of a function f we mean

$$\widehat{f}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ixy} f(y) \,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Let $\sigma > 0$, $p \in (1,\infty)$, $q \in [1,\infty]$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$. The *logarithmic Bessel potential space* $H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$ is defined by

$$H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) := \left\{ u = g_\sigma * f ; \; f \in L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \right\},$$

and is equipped with the (quasi-)norm

$$\|u\|_{\sigma;p,q;\alpha_1,\dots,\alpha_m} := \|f\|_{p,q;\alpha_1,\dots,\alpha_m}. \tag{1}$$

(By $f * g$ we mean the convolution of functions f and g.)

Let $\mathcal{L}$ be the *class of all continuous functions* $\lambda : (0,\infty) \to (0,\infty)$ which are increasing on some interval $(0,\delta)$, with $\delta = \delta(\lambda) > 0$, and satisfy $\lim_{\lambda \to 0_+} \lambda(t) = 0$. Let $\lambda \in \mathcal{L}$ and let Ω be a domain in $\mathbb{R}^n$. The *space* $C^{0,\lambda(\cdot)}(\overline{\Omega})$ of $\lambda(\cdot)$-*Hölder-continuous functions* consists of all those functions $u \in C(\overline{\Omega})$ for which the norm

$$\|u\|_{C^{0,\lambda(\cdot)}(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)| + \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|u(x) - u(y)|}{\lambda(|x-y|)}$$

is finite. Here $C(\overline{\Omega})$ stands for the family of all functions which are bounded and *uniformly* continuous on Ω. (For more information about such spaces see [1] or [10].)

We write $A \lesssim B$ (or $A \gtrsim B$) if $A \le cB$ (or $cA \ge B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B, and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. If $p \in [1, \infty]$, the conjugate number p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$ with the understanding that $1' = \infty$ and $\infty' = 1$.

In [8] we have extended Theorem 4.9 of [5] (to the range $\sigma \in (0,1)$) and proved the following embedding.

Theorem 1. *Let* $0 < \sigma < 1$, $\frac{n}{\sigma} < p < \infty$, $1 < q < \infty$, $m \in \mathbb{N}$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ *and let*

$$\lambda(t) = t^{\sigma - \frac{n}{p}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(t), \quad t > 0.$$

Then

$$H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\mathbb{R}^n}). \tag{2}$$

The aim of this paper is to show that the embedding of $H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$ into $C^{0,\lambda(\cdot)}(\overline{\Omega})$, where Ω is a nonempty domain in $\mathbb{R}^n$, cannot be compact and that the embedding (2) is sharp with respect to the function λ.

2. Main result and proofs

Our main result reads as follows.

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. Let* $n \ge 2$ *and* $\Omega \subseteq \mathbb{R}^n$ *be a nonempty domain. Then the embedding*

$$H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\lambda(\cdot)}(\overline{\Omega}) \tag{3}$$

is not compact. Moreover, if a function $\mu \in \mathcal{L}$ *satisfies* $\frac{\mu}{\lambda} \in \mathcal{L}$, *then the embedding*

$$H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\mu(\cdot)}(\overline{\Omega})$$

does not hold.

To prove Theorem 2, we need some preliminary work. We modify the idea from [7] to construct suitable test functions. Assume that $\mathcal{G}$ is a function with the following properties:

$$\mathcal{G} \text{ is positive and continuous on } (0,1]; \tag{4}$$

$$t\,\mathcal{G}(t) \text{ is nonincreasing on } (0, r_0], \text{ where } r_0 \in (0,1] \text{ is a fixed number;} \tag{5}$$

$$\mathcal{G}(\tfrac{t}{2}) \lesssim \mathcal{G}(t),\ t \in (0,1] \tag{6}$$

(notice that the assumption (5) is stronger than (4.2) of [7]). We use mollifiers to assign to the function $\mathcal{G}$ a family of functions $\{\mathcal{G}_r\}$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\int_{\mathbb{R}} \varphi = 1$ and $\operatorname{supp} \varphi = [-1, 1]$. We define the function φ_ε, $\varepsilon > 0$, by

$$\varphi_\varepsilon(t) := \tfrac{1}{\varepsilon}\varphi\big(\tfrac{t}{\varepsilon}\big), \quad t \in \mathbb{R},$$

and we put

$$\psi := \chi_{[-2+\frac{1}{16}, \frac{3}{4}-\frac{1}{16}]} * \varphi_{\frac{1}{16}}.$$

Now, we extend $\mathcal{G}$ by zero outside the interval $(0, 1]$ and we define functions $\mathcal{G}_r$, $r \in (0, 1)$, by

$$\mathcal{G}_r(t) := \big((\chi_{[r,\infty)}\, \psi\, \mathcal{G}) * \varphi_{\frac{r}{4}}\big)(t)\ , \quad t \in \mathbb{R}. \tag{7}$$

For any $r \in (0, \frac{1}{4})$, let a_r be a positive number, let

$$h_r(x) := a_r\, \mathcal{G}_r\big(|x|\big), \quad x \in \mathbb{R}^n, \tag{8}$$

and

$$u_r(x) := x_1\, (g_\sigma * h_r)(x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{9}$$

Our first aim is to show that the functions u_r belong to the source space in (3). To this end, we shall need the following result.

Lemma 1 (cf. Lemma 4.1 of [7])**.** *Let $r \in (0, \frac{1}{4})$ and let $\mathcal{G}_r$ be the functions defined by* (7)*, where $\mathcal{G}$ satisfies* (4)–(6)*. Then*

$$\mathcal{G}_r \in C_0^\infty(\mathbb{R})\,, \quad \operatorname{supp} \mathcal{G}_r \subset [\tfrac{r}{2}, 1] \quad \text{and} \quad \mathcal{G}_r \geq 0. \tag{10}$$

Moreover, there are positive constants C_1 and C_2 (independent of r and t) such that

$$\begin{aligned} \mathcal{G}_r(t) &\leq C_1\, \mathcal{G}(t)\, \chi_{[\frac{r}{2},1]}(t), && t \in (0, 1] \\ \mathcal{G}_r(t) &\geq C_2 \mathcal{G}(t), && t \in [2r, \tfrac{1}{2}]. \end{aligned} \tag{11}$$

We shall make use of the next assertions.

Lemma 2. *Let h belong to the Schwartz space $\mathcal{S}$, $\sigma \geq 0$, $j \in \{1, \ldots, n\}$ and let $\mathcal{R}_j$ be the Riesz transform. Then there exists a finite measure ν on $\mathbb{R}^n$ such that, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,*

$$x_j\, (g_\sigma * h)(x) = -\sigma\, (2\pi)^{-\frac{n}{2}}\, \big[g_\sigma * (\mathcal{R}_j(\nu * g_1 * h))\big](x) + \big[g_\sigma * (y_j\, h(y))\big](x).$$

Proof. The equality can be derived analogously to (4.48) in [7]. □

Lemma 3 (cf. Cor. 4.12 of [7]). *Let* $1 < p < \infty$, $1 \le q \le \infty$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ *and let* ν *be the measure from Lemma 2. Then, for all* $f \in L_{p,q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$,

$$\begin{aligned} \|g_\alpha * f\|_{p,q;\alpha_1,\ldots,\alpha_m} &\lesssim \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}, \quad \alpha \ge 0, \\ \|\mathcal{R}_j f\|_{p,q;\alpha_1,\ldots,\alpha_m} &\lesssim \|f\|_{p,q;\alpha_1,\ldots,\alpha_m}, \quad j = 1, \ldots, n, \\ \|\nu * f\|_{p,q;\alpha_1,\ldots,\alpha_m} &\lesssim \|f\|_{p,q;\alpha_1,\ldots,\alpha_m} \end{aligned}$$

We shall also need the following estimate.

Lemma 4. *Let* $n \ge 2$, $p > \frac{n}{n-1}$, $q \in [1, \infty]$, $\frac{1}{p} = \frac{1}{\widetilde{p}} - \frac{1}{n}$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. *Then, for all* $f \in L_{\widetilde{p},q;\alpha_1,\ldots,\alpha_m}(\mathbb{R}^n)$,

$$\|g_1 * f\|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \|f\|_{\widetilde{p},q;\alpha_1,\ldots,\alpha_m}.$$

Proof. The assumption $p > \frac{n}{n-1}$ and the equality $\frac{1}{p} = \frac{1}{\widetilde{p}} - \frac{1}{n}$ imply that $\widetilde{p} \in (1, n)$. Thus, the result follows on applying Theorem 3.1 of [7]. □

Lemma 5. *Let* $p, q \in (1, \infty)$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. *Let* g *be a positive function which is continuous in* $(0, 1]$ *and nonincreasing in some interval* $(0, r_0] \subset (0, 1]$. *Then, for all* $r \in (0, r_0)$,

$$\| g(|y|)\, \chi_{[r,1]}(|y|) \|_{p,q;\alpha_1,\ldots,\alpha_m} \lesssim \mathcal{V}_1(r) + \mathcal{V}_2(r),$$

where

$$\mathcal{V}_1(r) := \left\| t^{\frac{n}{p} - \frac{1}{q}} \left(\prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \right) g(t) \right\|_{q;(r,1)}$$

$$\mathcal{V}_2(r) := r^{\frac{n}{p}} \left(\prod_{j=1}^{m} \ell_j^{\alpha_j}(r) \right) g(r).$$

Proof. The estimate can be proved analogously to the estimate (4.3) in Lemma 4.1 of [4]. □

The next lemma provides the upper estimate of $\|u_r\|_{\sigma;p,q;\alpha_1,\ldots,\alpha_m}$, where u_r are the functions given by (9).

Lemma 6. *Let* $n \ge 2$, $p > \frac{n}{n-1}$, $q \in (1, \infty)$, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. *Then the functions* u_r, $r \in (0, r_0)$, *defined by* (9) *(with* $\mathcal{G}$ *given by* (4)–(6)*), satisfy*

$$\|u_r\|_{\sigma;p,q;\alpha_1,\ldots,\alpha_m} \lesssim a_r\big(\mathcal{W}_1(r/2) + \mathcal{W}_2(r/2)\big),$$

where

$$\mathcal{W}_1(r) := \left\| t^{\frac{n}{p} + 1 - \frac{1}{q}} \left(\prod_{j=1}^{m} \ell_j^{\alpha_j}(t) \right) \mathcal{G}(t) \right\|_{q;(r,1)}$$

$$\mathcal{W}_2(r) := r^{\frac{n}{p} + 1} \left(\prod_{j=1}^{m} \ell_j^{\alpha_j}(r) \right) \mathcal{G}(r).$$

Proof. Since $u_r \in \mathcal{S}$ (cf. (10) and the fact that $g_\sigma * f \in \mathcal{S}$ for $f \in \mathcal{S}$ and $\sigma > 0$), we can use Lemma 2 and the definition in (1) to get

$$\begin{aligned} &\|u_r\|_{\sigma;p,q;\alpha_1,\dots,\alpha_m} \\ &\quad \lesssim \|g_\sigma * \mathcal{R}_1(\nu * g_1 * h_r)\|_{\sigma;p,q;\alpha_1,\dots,\alpha_m} + \|g_\sigma * (y_1\, h_r(y))\|_{\sigma;p,q;\alpha_1,\dots,\alpha_m} \\ &\quad = \|\mathcal{R}_1(\nu * g_1 * h_r)\|_{p,q;\alpha_1,\dots,\alpha_m} + \|y_1\, h_r(y)\|_{p,q;\alpha_1,\dots,\alpha_m}. \end{aligned} \tag{12}$$

Applying Lemma 3, Lemma 4, (8) and (11) to the first term, we obtain

$$\begin{aligned} \|\mathcal{R}_1(\nu * g_1 * h_r)\|_{p,q;\alpha_1,\dots,\alpha_m} &\lesssim \|g_1 * h_r\|_{p,q;\alpha_1,\dots,\alpha_m} \\ &\lesssim \|h_r\|_{\widetilde{p},q;\alpha_1,\dots,\alpha_m} \\ &\lesssim a_r\, \|\mathcal{G}(|y|)\, \chi_{[\frac{r}{2},1]}(|y|)\|_{\widetilde{p},q;\alpha_1,\dots,\alpha_m}. \end{aligned}$$

Moreover, using Lemma 5 with $g = \mathcal{G}$ (observe that this function satisfies the assumptions of Lemma 5) and the identity $\frac{n}{\widetilde{p}} = \frac{n}{p} + 1$, we arrive at

$$\|\mathcal{G}(|y|)\, \chi_{[\frac{r}{2},1]}(|y|)\|_{\widetilde{p},q;\alpha_1,\dots,\alpha_m} \lesssim \mathcal{W}_1(r/2) + \mathcal{W}_2(r/2).$$

Consequently,

$$\|\mathcal{R}_1(\nu * g_1 * h_r)\|_{p,q;\alpha_1,\dots,\alpha_m} \lesssim a_r\big[\mathcal{W}_1(r/2) + \mathcal{W}_2(r/2)\big]. \tag{13}$$

Furthermore, we use (8), (11) and Lemma 5 with $g(t) = t\,\mathcal{G}(t)$ to get

$$\begin{aligned} \|y_1\, h_r(y)\|_{p,q;\alpha_1,\dots,\alpha_m} &\le \|\, |y|\, h_r(y)\|_{p,q;\alpha_1,\dots,\alpha_m} \\ &\lesssim a_r\|\, |y|\, \mathcal{G}(|y|)\, \chi_{[\frac{r}{2},1]}(|y|)\|_{p,q;\alpha_1,\dots,\alpha_m} \\ &\lesssim a_r\big[\mathcal{W}_1(r/2) + \mathcal{W}_2(r/2)\big]. \end{aligned} \tag{14}$$

Finally, by (12), (13) and (14) we obtain the result. □

To prove the non-compactness of the embedding (3), we shall need the following assertion.

Lemma 7. *Let* $\sigma \in (0,n)$, $R \in (0, \frac{1}{4})$ *and let*

$$a_r \le C \quad \text{for all } r \in (0, \tfrac{1}{4}) \text{ with some } C \in (0,\infty). \tag{15}$$

Moreover, let the function $\mathcal{G}$ *from* (4)–(6) *and the numbers* a_r *satisfy*

$$a_r \int_{2r}^{\frac{R}{2}} t^{\sigma-1}\, \mathcal{G}(t)\, \mathrm{d}t \to \infty \quad \text{as} \quad r \to 0_+. \tag{16}$$

Then there exist $\varepsilon = \varepsilon(\sigma) \in (0, \frac{1}{2})$, $r_1 = r_1(R) \in (0, \frac{R}{4})$ *and a positive constant* c *(independent of* R *and* r_1*) such that for the functions* u_r *defined by* (9), (8) *and* (7),

$$\big|\, [u_r(x) - u_R(x)] - [u_r(0) - u_R(0)] \,\big| \ge c\, r\, a_r \int_{2r}^{\frac{R}{2}} t^{\sigma-1} \mathcal{G}(t)\, \mathrm{d}t \tag{17}$$

for every $r \in (0, r_1)$ *and* $x = (\varepsilon r, 0, \dots, 0) \in \mathbb{R}^n$.

Proof. The result immediately follows from Lemma 4.5 of [7]. □

Now, we are ready to prove the main result.

Proof of Theorem 2. We can suppose without loss of generality that

$$B := \{x \in \mathbb{R}^n; |x| \le 1\} \subset \Omega. \tag{18}$$

Let $r \in (0, \frac{1}{4})$. Take $\gamma < 0$ and put

$$\mathcal{G}(t) = t^{\gamma-1-\frac{n}{p}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(t), \quad t \in (0,1], \qquad \text{and} \quad a_r = r^{-\gamma}.$$

The function $\mathcal{G}$ satisfies (4)–(6). Thus, by Lemma 6,

$$\|u_r\|_{\sigma;p,q;\alpha_1,\dots,\alpha_m} \lesssim r^{-\gamma}\left[\left(\int_{\frac{r}{2}}^{1} t^{\gamma q}\,\tfrac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} + r^{\gamma}\right] \lesssim 1 \quad \text{for all } r \in (0, r_0), \tag{19}$$

where u_r are the functions given by (9). (Observe, that the assumptions $\sigma \in (0,1)$ and $n \ge 2$ yield $p > \frac{n}{\sigma} > n > \frac{n}{n-1}$.)

Taking $R \in (0, \frac{1}{4})$, we can see that the conditions (15) and (16) are satisfied and so, by Lemma 7, there exists $\varepsilon \in (0, \frac{1}{2})$ and $r_1 \in (0, \frac{R}{4})$ and a positive constant c (independent of R and r_1) such that

$$\begin{aligned}&\big|\,[u_r(x) - u_R(x)] - [u_r(0) - u_R(0)]\,\big| \\ &\qquad \ge c\, r^{1-\gamma} \int_{2r}^{\frac{R}{2}} t^{\sigma-1+\gamma-1-\frac{n}{p}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(t)\,\mathrm{d}t \approx r^{\sigma-\frac{n}{p}} \prod_{j=1}^{m} \ell_j^{-\alpha_j}(r) = \lambda(r)\end{aligned}$$

for every $r \in (0, r_1)$ and $x = (\varepsilon r, 0, \dots, 0)$. Consequently, for any fixed $R \in (0, \frac{1}{4})$ and every sufficiently small positive r,

$$\|u_r - u_R\|_{C^{0,\lambda(\cdot)}(\overline{\Omega})} \ge \frac{\big|[u_r(x) - u_R(x)] - [u_r(0) - u_R(0)]\big|}{\lambda(\varepsilon r)} \ge c\,\frac{\lambda(r)}{\lambda(\varepsilon r)} \ge c_0, \tag{20}$$

where c and c_0 are positive constants independent of R and r.

Finally, consider the sequence of functions $\{u_{1/k}\}_{k=k_0}^{\infty}$ with k_0 sufficiently large. By (19), this sequence is bounded in $H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n)$ however, in view of (20), it has no Cauchy subsequence in $C^{0,\lambda(\cdot)}(\overline{\Omega})$. Therefore, the embedding (3) is not compact.

To prove sharpness, suppose that there is a function $\mu \in \mathcal{L}$ such that $\frac{\mu}{\lambda} \in \mathcal{L}$ and $H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\mu(\cdot)}(\overline{\Omega})$ for some nonempty domain Ω in $\mathbb{R}^n$. Take a ball $B \subset \Omega$. Then

$$H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow C^{0,\mu(\cdot)}(\overline{B}). \tag{21}$$

Moreover, by Lemma 4.15 (iv) of [5], the condition $\frac{\mu}{\lambda} \in \mathcal{L}$ implies that

$$C^{0,\mu(\cdot)}(\overline{B}) \hookrightarrow\hookrightarrow C^{0,\lambda(\cdot)}(\overline{B}).$$

Combining this embedding with (21), we arrive at

$$H^\sigma L_{p,q;\alpha_1,\dots,\alpha_m}(\mathbb{R}^n) \hookrightarrow\hookrightarrow C^{0,\lambda(\cdot)}(\overline{B}),$$

which contradict the non-compactness of the embedding (3) with $\Omega = B$. $\square$

References

[1] Adams, R. A.: *Sobolev spaces.* New York: Academic Press 1975.

[2] Edmunds, D. E., Gurka, P. and Opic, B., Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces. *Indiana Univ. Math.* J. 44 (1995), 19 – 43.

[3] Edmunds, D. E., Gurka, P. and Opic, B., Double exponential integrability, Bessel potentials and embedding theorems. *Studia Math.* 115 (1995), 151 – 181.

[4] Edmunds, D. E., Gurka, P. and Opic, B., Sharpness of embeddings in logarithmic Bessel-potential spaces. *Proc. Roy. Soc. Edinburgh* 126A (1996), 995 – 1009.

[5] Edmunds, D. E., Gurka, P. and Opic, B., On embeddings of logarithmic Bessel potential spaces. *J. Funct. Anal.* 146 (1997), 116 – 150.

[6] Edmunds, D. E., Gurka, P. and Opic, B., Norms of embeddings of logarithmic Bessel potential spaces. *Proc. Amer. Math. Soc.* 126 (1998), 2417 – 2425.

[7] Edmunds, D. E., Gurka, P. and Opic, B., Optimality of embeddings of logarithmic Bessel potential spaces. *Quart. J. Math. Oxford Ser.* (2) 51 (2000), 185 – 209.

[8] Edmunds, D. E., Gurka, P. and Opic, B., Compact and continuous embeddings of logarithmic Bessel potential spaces. *Studia Math.* 168 (2005), 229 – 250.

[9] Evans, W. D., Opic, B. and Pick, L., Interpolation of operators on scales of generalized Lorentz–Zygmund spaces. *Math. Nachr.* 182 (1996), 127 – 181.

[10] Kufner, A., John, O. and Fučík, S., *Function spaces.* Prague: Academia 1977.

[11] Opic, B. and Pick, L., On generalized Lorentz–Zygmund spaces. *Math. Inequal. Appl.* 2 (1999), 391 – 467.

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