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# The Factorization Method for Real Elliptic Problems

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**Abstract.** The Factorization Method localizes inclusions inside a body from measurements on its surface. Without a priori knowing the physical parameters inside the inclusions, the points belonging to them can be characterized using the range of an auxiliary operator. The method relies on a range characterization that relates the range of the auxiliary operator to the measurements and is only known for very particular applications.
In this work we develop a general framework for the method by considering symmetric and coercive operators between abstract Hilbert spaces. We show that the important range characterization holds if the difference between the inclusions and the background medium satisfies a coerciveness condition which can immediately be translated into a condition on the coefficients of a given real elliptic problem. We demonstrate how several known applications of the Factorization Method are covered by our general results and deduce the range characterization for a new example in linear elasticity.

**Keywords.** Elliptic partial differential equations, inverse problems, factorization method

**Mathematics Subject Classification (2000).** 35R30, 65N21

## 1. Introduction

Several applications in medical imaging and nondestructive testing of materials lead to the problem of reconstructing physical parameters inside a body from measurements on its surface. A common goal is to detect areas where the parameters significantly differ from that of the known smooth background.

A fairly recent method to localize such inclusions is the Factorization Method, which was originally developed by Kirsch for problems in Inverse Scattering [14] and generalized to problems in Electrical Impedance Tomography by Brühl and Hanke [4, 3]. It was successfully expanded to problems in electrostatics [9], optical tomography [12], also with singular interfaces [2], and to different

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electrode models in Impedance Tomography [13, 10]. In [15], complex diffusion problems with matrix valued coefficients are treated.

The Factorization Method compares the boundary measurements with those obtained from a reference body without inclusions. By factorizing the deviation, the range of an auxiliary operator is determined. The inclusion is then characterized by the fact that certain singular functions belong to this range, if and only if the location of their singularity is inside the inclusion.

We illustrate these steps using a simpler version of the diffusion equation in [15]:

**Example 1.1** (Diffusion Example)**.** Let $B \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $T := \partial B$. Injection of a particle flux $g \in H^{-1/2}(T)$ into a reference body with diffusion and absorption parameters $\kappa, c \in L^\infty_+(B)$ leads to a particle density $u \in H^1(B)$ that solves

$$\begin{aligned} -\mathrm{div}\,(\kappa\,\mathrm{grad}\,u) + cu &= 0 \quad \text{in } B \\ \kappa\partial_\nu u &= g \quad \text{on } T. \end{aligned} \tag{1}$$

Measuring the particle density on $T$ for all values of $g$ is described by the Neumann-Dirichlet operator

$$\Lambda_0 : \; g \mapsto u_0|_T, \quad \text{where } u_0 \in H^1(B) \text{ solves (1).} \tag{2}$$

Now let $\Omega$ be another domain with smooth boundary $\Sigma := \partial\Omega$ and $\overline{\Omega} \subset B$ as

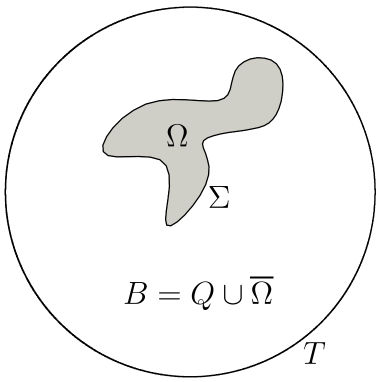


Figure 1: Sketch of geometry

in Figure 1. If $\Omega$ is an inclusion in the body $B$ with diffusion and absorption parameters $\kappa - \kappa_1, c - c_1 \in L^\infty_+(\Omega)$, the particle density $u \in H^1(B)$ solves

$$\begin{aligned} -\mathrm{div}\,((\kappa - \kappa_1\chi_\Omega)\,\mathrm{grad}\,u) + (c - c_1\chi_\Omega)u &= 0 \quad \text{in } B \\ \kappa\partial_\nu u &= g \quad \text{on } T, \end{aligned} \tag{3}$$

and boundary measurements are described by

$$\Lambda_1 : \; g \mapsto u_1|_T, \quad \text{where } u_1 \in H^1(B) \text{ solves (3).} \tag{4}$$

The problem of locating the inclusion from boundary measurements can now be formulated mathematically as calculating $\chi_\Omega$ from $\Lambda_1 - \Lambda_0$.

We introduce the auxiliary operator $L : H^{-1/2}(\Sigma) \to H^{1/2}(T)$, $\psi \mapsto u|_T$, where $u$ solves

$$\begin{aligned} -\operatorname{div}(\kappa \operatorname{grad} u) + cu = 0 \quad & \text{in } Q := B \setminus \overline{\Omega}, \\ \kappa \partial_\nu u = \begin{cases} \psi & \text{on } \Sigma \\ 0 & \text{on } T. \end{cases} & \end{aligned} \tag{5}$$

This corresponds to virtual measurements of a particle density resulting from a boundary flux from inside the inclusion. To see that the range of $L$ fully determines $\Omega$ let $z \in B$ and $u_z$ solve $\kappa \partial_\nu u_z = 0$ on $T$ and

$$-\operatorname{div}(\kappa \operatorname{grad} u) + cu = 0 \quad \text{in } B \setminus \{z\}. \tag{6}$$

Furthermore let $u_z$ have no continuation that solves (6) in $B$ (e. g. because of a too strong singularity in $z$). Then obviously $z \in \Omega$ implies $u_z|_T \in \mathcal{R}(L)$, since $u_z|_Q$ solves (5). If the solution is uniquely determined by Cauchy data on $T$, then also the converse is true hence

$$z \in \Omega \quad \text{ if and only if } \quad u_z \in \mathcal{R}(L). \tag{7}$$

The key result of the Factorization Method is that the range of $L$ (the virtual measurements) can be calculated from $\Lambda_1$ and $\Lambda_0$ (the real measurements) by

$$\mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2}) = \mathcal{R}(L). \tag{8}$$

Thus the inclusion is found by calculating $u_z$ for every point $z \in B$ and checking whether $u_z \in \mathcal{R}(L) = \mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2})$ (cf. [3] for a numerical implementation of such a range test using the Picard criterion).

A big challenge in the application of the Factorization Method is the need for the non-trivial mathematical relation (8). In the above mentioned works the proof of this range characterization is typically quite involved and apparently uses very particular properties of the considered problems. In this work we show that this is not necessary, the range characterization holds in fact under very weak assumptions on the considered problem. In particular we formulate an easy criterion how the physical properties of the inclusions must differ from the reference configuration. This criterion can immediately be translated into conditions on the coefficients of a given real elliptic equation.

In spite of the fact that – by considering the problems in a very general setting – the required spaces and operators for the Factorization Method lose their physical interpretation, the proofs become even simpler and more elementary than in the mentioned works.

In Section 2 we formulate a general setting for real elliptic problems in a domain with inclusions using abstract Hilbert spaces. To describe the geometry of the problem and to define boundary measurements in these abstract spaces we assume the existence of trace and restriction operators. Real elliptic problems are then defined by their corresponding symmetric and coercive bilinear forms.

In the third section we formulate and proof our main result: *The range characterization holds if the difference of the inclusion's bilinear form and the one of the background medium is coercive.* We also give an extension to insulating inclusions (cavities) and to the case where the difference is a compact perturbation of a coercive bilinear form.

Throughout the first three sections we use Example 1.1 to motivate our assumptions and to show how our general results apply to a given problem. In the last section we demonstrate how five other applications fit in our general framework. While four of them have been considered previously by different authors, the last one is (to our knowledge) a completely new application for the Factorization Method.

## 2. Notations and assumptions

**2.1. Spaces and Traces.** To apply the Factorization Method to a real elliptic equation, we do not only need the space of the solutions $H(B)$ of the equation on $B$ (like $H^1(B)$ for Example 1.1), but we also have to restrict the solutions to the inclusion $\Omega$, its complement $Q := B \setminus \overline{\Omega}$ and to the boundaries $T$ and $\Sigma$. For this we need appropriate function spaces $H(\Omega)$, $H(Q)$, $H(\Sigma)$ and $H(T)$, e.g. the spaces $H^1(\Omega)$, $H^1(Q)$, $H^{1/2}(\Sigma)$ and $H^{1/2}(T)$ with the well-known restriction and trace operators for the diffusion example.

**Remark 2.1.** The reader may skip the technical details and proceed directly to the main result in the next section using the following more intuitive ideas and notations:

Having in mind Figure 1 we denote by $E_{(\cdot)}$ and $\gamma_{(\cdot)\to(\cdot)}$ restriction and trace operators on the specified subsets and boundaries. We assume that two real elliptic differential operators $A_i$, describing a body with inclusions ($i = 1$) resp. without ($i = 0$), coincide on $Q$ but differ on $\Omega$. Their restrictions are denoted by $A_Q$ resp. $A_{\Omega,i}$, and the corresponding bilinear forms by $a_{(\cdot)}$.

In the general case $H(\cdot)$ might not be spaces of functions or distributions on $B$, $\Omega$, $Q$, $\Sigma$ and $T$. We will therefore proceed one step further and treat $B$, $\Omega$, $Q$,

$\Sigma$ and $T$ only as indices for abstract real Hilbert spaces with abstract restriction and trace operators. This generalization may hamper readability on the first glance, but has advantages when treating factor spaces (see Application 4.1) or the case $T \neq \partial B$ (see Application 4.4).

**Assumption and Definition 2.2.** Let $H(B)$, $H(Q)$, $H(\Omega)$, $H(T)$ and $H(\Sigma)$ be real Hilbert spaces with inner products $(\cdot,\cdot)_{H(\cdot)}$ and continuous linear operators

$$\begin{aligned}\gamma_{Q\to T} &: H(Q) \to H(T), \quad E_Q : H(B) \to H(Q)\\ \gamma_{Q\to\Sigma} &: H(Q) \to H(\Sigma), \quad E_\Omega : H(B) \to H(\Omega)\\ \gamma_{\Omega\to\Sigma} &: H(\Omega) \to H(\Sigma).\end{aligned}$$

We define continuous linear operators

$$\begin{aligned}\gamma_{B\to T} &: H(B) \to H(T), \quad \gamma_{B\to T} := \gamma_{Q\to T}E_Q\\ \gamma_{B\to\Sigma} &: H(B) \to H(\Sigma), \quad \gamma_{B\to\Sigma} := \gamma_{Q\to\Sigma}E_Q\end{aligned}$$

and make the following assumptions:

(V1) There exists $k > 0$ such that for all $u \in H(B)$

$$(u,u)_{H(B)} \leq k\left((E_Q u, E_Q u)_{H(Q)} + (E_\Omega u, E_\Omega u)_{H(\Omega)}\right),$$

thus together with the continuity of $E_Q$ and $E_\Omega$ the bilinear form

$$(u,v) \mapsto (E_Q u, E_Q v)_{H(Q)} + (E_\Omega u, E_\Omega v)_{H(\Omega)}$$

is an inner product on $H(B)$, which induces a norm equivalent to the original one.

(V2a) $\gamma_{Q\to\Sigma}E_Q = \gamma_{\Omega\to\Sigma}E_\Omega$.

(V2b) $E_\Omega$ and $E_Q$ possess continuous right inverses $E_\Omega^-$ resp. $E_Q^-$ such that for all $u \in H(\Omega)$ and $v \in H(Q)$

$$\begin{aligned}E_Q E_\Omega^- u = 0 \quad &\text{if} \quad \gamma_{\Omega\to\Sigma}u = 0\\ E_\Omega E_Q^- v = 0 \quad &\text{if} \quad \gamma_{Q\to\Sigma}v = 0.\end{aligned}$$

(V3) $\gamma_{Q\to\Sigma}$ and $\gamma_{\Omega\to\Sigma}$ have continuous right inverses $\gamma_{Q\to\Sigma}^-$ and $\gamma_{\Omega\to\Sigma}^-$.

As we already stated above in the diffusion example, a possible choice are the Hilbert spaces (compare Remark 3.6 for the choice of $H(T)$)

$$\begin{aligned}&H(B) := H^1(B), \quad H(T) := H^{1/2}(T)\\ &H(Q) := H^1(Q), \quad H(\Sigma) := H^{1/2}(\Sigma), \quad H(\Omega) := H^1(\Omega).\end{aligned}$$

$E_{(\cdot)}$ are the restrictions to the subsets $Q$ resp. $\Omega$, and $\gamma_{(\cdot)}$ are the traces on $T$ and $\Sigma$. Then assumption (V1) is trivial, (V2b) is a consequence of [16, Lemma 6.85] and [16, Theorem 6.88], (V3) can be found in [16, Theorem 6.108] and (V2a) is easily shown by approximation with a sequence of smooth functions.

**Lemma 2.3.**

(a) *Assume* (V1) *and* (V3) *hold, then* (V2a) *and* (V2b) *are equivalent to*

(V2*) *For every* $u_Q \in H(Q)$ *and* $u_\Omega \in H(\Omega)$ *there exists* $u \in H(B)$ *with* $u_Q = E_Q u$ *and* $u_\Omega = E_\Omega u$ *if and only if* $\gamma_{Q\to\Sigma} u_Q = \gamma_{\Omega\to\Sigma} u_\Omega$.

*In particular* $E_\Omega^-$ *and* $E_Q^-$ *can be chosen such that*

$$E_Q E_\Omega^- = \gamma_{Q\to\Sigma}^- \gamma_{\Omega\to\Sigma} \text{ and } E_\Omega E_Q^- = \gamma_{\Omega\to\Sigma}^- \gamma_{Q\to\Sigma}, \tag{9}$$

*which we will assume from now on.*

(b) *Choosing* $E_\Omega^-$ *and* $E_Q^-$ *according to (9) we have* $E_Q^- \gamma_{Q\to\Sigma}^- = E_\Omega^- \gamma_{\Omega\to\Sigma}^-$.

*Proof.* (a1): Let (V1), (V2a), (V2b) and (V3) hold. Let $u_Q \in H(Q)$ and $u_\Omega \in H(\Omega)$.

($\alpha$): If there exists $u \in H(B)$ with $u_Q = E_Q u$ and $u_\Omega = E_\Omega u$, then (V2a) yields $\gamma_{Q\to\Sigma} u_Q = \gamma_{Q\to\Sigma} E_Q u = \gamma_{\Omega\to\Sigma} E_\Omega u = \gamma_{\Omega\to\Sigma} u_\Omega$.

($\beta$): Conversely let $\gamma_{Q\to\Sigma} u_Q = \gamma_{\Omega\to\Sigma} u_\Omega$, then

$$u := E_Q^- u_Q - E_\Omega^- \left(E_\Omega E_Q^- u_Q - u_\Omega\right) \in H(B)$$

satisfies $E_\Omega u = u_\Omega$. Using (V2a) we derive

$$\gamma_{\Omega\to\Sigma} \left(E_\Omega E_Q^- u_Q - u_\Omega\right) = \gamma_{Q\to\Sigma} E_Q E_Q^- u_Q - \gamma_{\Omega\to\Sigma} u_\Omega = 0,$$

so (V2b) yields $E_Q E_\Omega^- \left(E_\Omega E_Q^- u_Q - u_\Omega\right) = 0$ and thus $E_Q u = u_Q$.

(a2): Now let (V1), (V2*) and (V3) hold, then (V2a) is obvious. To verify (V2b) we define $E_\Omega^- : H(\Omega) \to H(B)$ by $E_\Omega^- u_\Omega := u$, where $u \in H(B)$ solves $E_\Omega u = u_\Omega$ and $E_Q u = \gamma_{Q\to\Sigma}^- \gamma_{\Omega\to\Sigma} u_\Omega$. (Note that (V2*) yields the existence of this $u$ and (V1) yields that it depends continuously of $u_\Omega$.) $E_Q^-$ is defined analogously. Then it is easily seen that $E_\Omega^-$ and $E_Q^-$ satisfy (V2b) and (9).

(b): Using (9) we have

$$\begin{aligned} E_\Omega \left(E_Q^- \gamma_{Q\to\Sigma}^-\right) &= \gamma_{\Omega\to\Sigma}^- \gamma_{Q\to\Sigma} \gamma_{Q\to\Sigma}^- = \gamma_{\Omega\to\Sigma}^- = E_\Omega \left(E_\Omega^- \gamma_{\Omega\to\Sigma}^-\right) \\ E_Q \left(E_Q^- \gamma_{Q\to\Sigma}^-\right) &= \gamma_{Q\to\Sigma}^- = \gamma_{Q\to\Sigma}^- \gamma_{\Omega\to\Sigma} \gamma_{\Omega\to\Sigma}^- = E_Q \left(E_\Omega^- \gamma_{\Omega\to\Sigma}^-\right). \end{aligned}$$

With (V1) this yields $E_Q^- \gamma_{Q\to\Sigma}^- = E_\Omega^- \gamma_{\Omega\to\Sigma}^-$. □

**2.2. Bilinear forms and operators.** For operators between real Hilbert spaces we rigorously distinguish between the dual operator (denoted by $'$) and the adjoint operator (denoted by $^*$). The inner product on a real Hilbert space $H$ is denoted by $(\cdot,\cdot)$ and the dual pairing on $H' \times H$ by $\langle\cdot,\cdot\rangle$. They are related by the isometry $\iota_H : H \to H'$ that "identifies $H$ with its dual", i. e., $\langle \iota_H u, \cdot\rangle := (u, \cdot)$ for all $u \in H$.

A bilinear form $a:\ H \times H \to \mathbb{R}$ is called symmetric if $a(u,v) = a(v,u)$ for all $u, v \in H$ and coercive if there exists $\alpha > 0$ such that $a(u,u) \geq \alpha \|u\|^2$ for all $u \in H$. An operator $A \in \mathcal{L}(H, H')$ is called symmetric resp. coercive if the associated bilinear form $a:\ (u,v) \mapsto \langle Au, v\rangle$ is symmetric resp. coercive.

**Example 2.4** (Diffusion Example, continued)**.** An equivalent variational definition for the Neumann-Dirichlet operator

$$\Lambda_1:\ H^{-1/2}(T) \to H^{1/2}(T)$$

is $\Lambda_1 g = u|_T$, where $u \in H^1(B)$ solves

$$a_1(u,v) = \langle g, v|_T\rangle_T \text{ for all } v \in H^1(B). \tag{10}$$

Due to our assumption on the coefficients the bilinear form

$$a_1(u,v) := \int_B \{(\kappa - \kappa_1\chi_\Omega)\nabla u \nabla v + (c - c_1\chi_\Omega)uv\}\ \mathrm{d}x$$

is continuous, coercive and symmetric on $H^1(B)$.

Lax-Milgram's theorem grants continuous invertibility of the operator

$$A_1 : H^1(B) \to (H^1(B))', \quad A_1 u := a_1(u,\cdot)$$

and thus the unique existence of a solution of (10) that depends continuously on $g$. Using this operator $\Lambda_1$, can be written as

$$\Lambda_1 = \gamma_{B\to T} A_1^{-1} \gamma'_{B\to T}. \tag{11}$$

An analogous statement holds for $\Lambda_0$ using the bilinear form

$$a_0(u,v) := \int_B \{\kappa \nabla u \nabla v + cuv\}\ \mathrm{d}x.$$

$a_0$ and $a_1$ share a common part on $Q$, i. e.,

$$a_0(u,v) = a_Q(u|_Q, v|_Q) + a_{\Omega,0}(u|_\Omega, v|_\Omega), \tag{12}$$
$$a_1(u,v) = a_Q(u|_Q, v|_Q) + a_{\Omega,1}(u|_\Omega, v|_\Omega), \tag{13}$$

with

$$\begin{aligned} a_Q(u,v) &:= \int_Q \{\kappa \nabla u \nabla v + cuv\}\ \mathrm{d}x \\ a_{\Omega,0}(u,v) &:= \int_\Omega \{\kappa \nabla u \nabla v + cuv\}\ \mathrm{d}x \\ a_{\Omega,1}(u,v) &:= \int_\Omega \{(\kappa - \kappa_1)\nabla u \nabla v + (c - c_1)uv\}\ \mathrm{d}x. \end{aligned}$$

To generalize the Factorization Method we postulate the existence of coercive bilinear forms $a_Q$, $a_{\Omega,0}$ and $a_{\Omega,1}$, then we use them to compose $a_0$ and $a_1$ according to (12) and (13) and finally we define (abstract) Neumann-Dirichlet operators according to (11).

**Assumption and Definition 2.5.** Let

$$\begin{aligned} a_Q : H(Q) \times H(Q) &\to \mathbb{R} \\ a_{\Omega,0},\ a_{\Omega,1} : H(\Omega) \times H(\Omega) &\to \mathbb{R} \end{aligned}$$

be continuous, coercive and symmetric bilinear forms. Define $a_i : H(B) \times H(B) \to \mathbb{R}$ $(i = 0, 1)$ by setting for all $u, v \in H(B)$

$$a_i(u, v) := a_Q(E_Q u, E_Q v) + a_{\Omega,i}(E_\Omega u, E_\Omega v). \tag{14}$$

Due to (V1) they are continuous, coercive and symmetric bilinear forms.

Let $H'(B)$ be the dual space of $H(B)$ and $\langle \cdot, \cdot \rangle_B$ the dual pairing on $H'(B) \times H(B)$ (an analogous notation is used for $Q$, $\Omega$, $T$ and $\Sigma$). The bilinear form $a_Q$ canonically induces the operator

$$A_Q : \ H(Q) \to H'(Q), \quad \langle A_Q u, \cdot \rangle_Q := a_Q(u, \cdot).$$

Due to the assumptions on $a_Q$, $A_Q$ is continuous, coercive and symmetric, in particular continuously invertible. The same holds for the analogously defined operators

$$A_{\Omega,i} : \ H(\Omega) \to H'(\Omega) \quad \text{and} \quad A_i : H(B) \to H'(B) \quad (i = 0, 1).$$

Equation (14) then becomes

$$A_i = E_Q' A_Q E_Q + E_\Omega' A_{\Omega,i} E_\Omega \quad (i = 0, 1). \tag{15}$$

The operators $\Lambda_i : \ H'(T) \to H(T)$ $(i = 0, 1)$ are now defined by setting

$$\Lambda_i := \gamma_{B \to T} A_i^{-1} \gamma_{B \to T}'. \tag{16}$$

## 3. The Range Characterization

### 3.1. Formulation of the main result.

**Theorem 3.1.** *Let the Assumptions and Definitions* 2.2 *and* 2.5 *hold. If* $a_{\Omega,0} - a_{\Omega,1}$ *is coercive, then*

$$\mathcal{R}((\Lambda_1 \iota_{H(T)} - \Lambda_0 \iota_{H(T)})^{1/2}) = \mathcal{R}(L),$$

*where* $L : \ H'(\Sigma) \to H(T)$ *is given by* $L := \gamma_{Q \to T} A_Q^{-1} \gamma_{Q \to \Sigma}'$.

In the Diffusion Example 1.1 and 2.4 this definition of $L$ obviously complies with (5) and we have coerciveness of $a_{\Omega,0} - a_{\Omega,1}$ if $\kappa_1, c_1 \in L_+^\infty(\Omega)$. Thus, under this condition Theorem 3.1 yields (8). (Note that $\iota_{H(T)}$ is typically omitted for convenience.)

**3.2. Proof of the main result.** We start with a short sketch of the proof ignoring any difference between dual and adjoint operators and identifying every space with its dual:

1. We show that $\Lambda_1 - \Lambda_0 = LFL'$, with a symmetric operator $F$. (This is the factorization the method is named after.)
2. Using an auxiliary result from functional analysis the coercivity of $a_{\Omega,0} - a_{\Omega,1}$ is shown to imply that $F$ is coercive and thus possesses a surjective square root $F^{1/2}$.
3. With another result from functional analysis the factorization $\Lambda_1 - \Lambda_0 = LF^{1/2}F^{1/2}L'$ yields that $\mathcal{R}((\Lambda_1 - \Lambda_0)^{1/2}) = \mathcal{R}(LF^{1/2})$. The assertion then follows from the surjectivity of $F^{1/2}$.

**Lemma 3.2.** $\Lambda_1 - \Lambda_0$ *can be factorized as*

$$\Lambda_1 - \Lambda_0 = LFL',$$

*where* $F : \ H(\Sigma) \to H'(\Sigma)$ *is given by*

$$F := (\gamma^-_{Q\to\Sigma})' A_Q E_Q \left(A_1^{-1} - A_0^{-1}\right) E'_Q A_Q \gamma^-_{Q\to\Sigma}.$$

*Proof.* We first observe that

$$\begin{aligned}
\left(A_1^{-1} - A_0^{-1}\right) E'_Q A_Q &= \left(A_1^{-1} - A_0^{-1}\right) E'_Q A_Q E_Q E_Q^- \\
&\overset{(15)}{=} \left(A_1^{-1}(A_1 - E'_\Omega A_{\Omega,1} E_\Omega) - A_0^{-1}(A_0 - E'_\Omega A_{\Omega,0} E_\Omega)\right) E_Q^- \\
&= \left(A_0^{-1} E'_\Omega A_{\Omega,0} - A_1^{-1} E'_\Omega A_{\Omega,1}\right) E_\Omega E_Q^- \\
&\overset{(9)}{=} \left(A_0^{-1} E'_\Omega A_{\Omega,0} - A_1^{-1} E'_\Omega A_{\Omega,1}\right) \gamma^-_{\Omega\to\Sigma} \gamma_{Q\to\Sigma}.
\end{aligned}$$

Thus

$$\left(A_1^{-1} - A_0^{-1}\right) E'_Q A_Q \gamma^-_{Q\to\Sigma} \gamma_{Q\to\Sigma} = \left(A_1^{-1} - A_0^{-1}\right) E'_Q A_Q.$$

By using this equation and its dual the assertion follows from the definitions of $\gamma_{B\to T}$, $F$ and $L$. □

A symmetric operator $A \in \mathcal{L}(H, H')$ on a real Hilbert space $H$ is called positive semidefinite, $A \geq 0$, if $\langle Av, v\rangle \geq 0$ for all $v$, consequently we write $A \geq B$ if $A - B \geq 0$. The operator $F$ is obviously symmetric and its definiteness essentially depends on the factor $A_1^{-1} - A_0^{-1}$. The following lemma relates the definiteness of $A_1^{-1} - A_0^{-1}$ to that of $A_0 - A_1 = E'_\Omega(A_{\Omega,0} - A_{\Omega,1})E_\Omega$.

**Lemma 3.3.** *Let $H$ be a real Hilbert space and $A, B \in \mathcal{L}(H, H')$ be symmetric. If $A$ is coercive, then*

$$BA^{-1}B - B \geq B - A. \tag{17}$$

*In particular this yields that if $B$ is also bijective, then*

$$A^{-1} - B^{-1} = B^{-1}(BA^{-1}B - B)B^{-1} \geq B^{-1}(B - A)B^{-1}.$$

*Proof.* Let $w \in H$ and set $v := A^{-1}Bw$.

$$\begin{aligned}
-\frac{1}{2}\langle Av, v\rangle &= \frac{1}{2}\langle Aw, w\rangle - \frac{1}{2}\langle A(w-v), (w-v)\rangle - \langle Av, w\rangle \\
&\leq \frac{1}{2}\langle Aw, w\rangle - \langle Av, w\rangle \\
&= \frac{1}{2}\langle Aw, w\rangle - \langle Bw, w\rangle,
\end{aligned}$$

hence $-\frac{1}{2}BA^{-1}B \leq \frac{1}{2}A - B$ which implies (17). □

Note that for real reflexive Banach spaces the definitions of symmetric, positive and coercive make sense and the lemma stays valid with the same proof.

Now we can use Lemma 3.3 to show coerciveness of $F$.

**Lemma 3.4.** *If $a_{\Omega,0} - a_{\Omega,1}$ is coercive, then also $F$ is coercive.*

*Proof.* Lemma 3.3 yields

$$A_1^{-1} - A_0^{-1} \geq A_0^{-1}\left(A_0 - A_1\right)A_0^{-1} = A_0^{-1}E_\Omega'(A_{\Omega,0} - A_{\Omega,1})E_\Omega A_0^{-1},$$

thus by setting $v := E_\Omega A_0^{-1}E_Q' A_Q \gamma_{Q\to\Sigma}^- \phi$ we obtain

$$\langle F\phi, \phi\rangle_\Sigma \geq \langle (A_{\Omega,0} - A_{\Omega,1})v, v\rangle_\Omega. \tag{18}$$

Given that $a_{\Omega,0} - a_{\Omega,1}$ and thus $A_{\Omega,0} - A_{\Omega,1}$ is coercive, the assertion follows if the operator $E_\Omega A_0^{-1}E_Q' A_Q \gamma_{Q\to\Sigma}^-$ possesses a continuous left inverse. Such a left inverse is given by $\gamma_{Q\to\Sigma}A_Q^{-1}(E_Q^-)'E_\Omega' A_{\Omega,0} + \gamma_{\Omega\to\Sigma}$, because

$$\begin{aligned}
&\left(\gamma_{Q\to\Sigma}A_Q^{-1}(E_Q^-)'E_\Omega' A_{\Omega,0} + \gamma_{\Omega\to\Sigma}\right)\left(E_\Omega A_0^{-1}E_Q' A_Q \gamma_{Q\to\Sigma}^-\right) \\
&\quad = \gamma_{Q\to\Sigma}A_Q^{-1}(E_Q^-)'\left(E_\Omega' A_{\Omega,0}E_\Omega\right)A_0^{-1}E_Q' A_Q\gamma_{Q\to\Sigma}^- + \gamma_{\Omega\to\Sigma}E_\Omega A_0^{-1}E_Q' A_Q \gamma_{Q\to\Sigma}^- \\
&\quad = \mathrm{id}_{H(\Sigma)} - \gamma_{Q\to\Sigma}E_Q A_0^{-1}E_Q' A_Q \gamma_{Q\to\Sigma}^- + \gamma_{\Omega\to\Sigma}E_\Omega A_0^{-1}E_Q' A_Q \gamma_{Q\to\Sigma}^- \\
&\quad = \mathrm{id}_{H(\Sigma)},
\end{aligned}$$

where we used (15) to replace $(E_\Omega' A_{\Omega,0}E_\Omega)$ and assumption (V2a) from Section 2.1 in the last equality. □

The following lemma can be found in an equivalent form in [8], for the sake of completeness we give an elementary proof.

**Lemma 3.5.** *Let $H_i$, $i = 1, 2$, be Hilbert spaces with norms $\|\cdot\|_i$ and $X$ be a third Hilbert space with scalar product $(\cdot,\cdot)_X$ and operators $A_i \in \mathcal{L}(X, H_i)$. Then $A_1^*A_1 = A_2^*A_2$ implies $\mathcal{R}(A_1^*) = \mathcal{R}(A_2^*)$.*

*Proof.* Of course it suffices to show $\mathcal{R}(A_1^*) \subseteq \mathcal{R}(A_2^*)$. Let $z \in \mathcal{R}(A_1^*)$, by orthogonal projection of its preimage we obtain $y_1 \in \mathcal{N}(A_1^*)^\perp$ with $A_1^* y_1 = z$.

As $\mathcal{N}(A_1^*)^\perp = \overline{\mathcal{R}(A_1)}$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $A_1 x_n \to y_1$. The convergence implies boundedness of $\{\|A_1 x_n\|_1 : n \in \mathbb{N}\}$ and along with

$$\|A_2 x_n\|_2 = (A_2^* A_2 x_n, x_n)_X = (A_1^* A_1 x_n, x_n)_X = \|A_1 x_n\|_1$$

we achieve boundedness of $\{\|A_2 x_n\|_2 \,:\, n \in \mathbb{N}\}$. Therefore $(A_2 x_n)_{n\in\mathbb{N}}$ possesses a weakly convergent subsequence $(A_2 x_{n_k})_{k\in\mathbb{N}}$ with weak limit $y_2 \in H_2$. Thus the subsequence $(A_2^* A_2 x_{n_k})_{k\in\mathbb{N}}$ converges weakly against $A_2^* y_2$, but as $A_2^* A_2 x_{n_k} = A_1^* A_1 x_{n_k}$ it also converges (strongly) against $A_1^* y_1 = z$. Hence $A_2^* y_2 = z$ and thus $z \in \mathcal{R}(A_2^*)$. □

Note that the lemma stays valid (with analogous proof) if the Hilbert space $X$ is replaced by a reflexive Banach space and $A_i^*$ by $A_i' \iota_{H_i}$ (cf. [15, Lemma 2.4] for the case of injective and compact operators).

Now we can prove our main result.

**Proof of Theorem 3.1.** We have

$$(\Lambda_1 - \Lambda_0)\iota_{H(T)} = L\iota_{H(\Sigma)}\iota_{H(\Sigma)}^{-1} F L' \iota_{H(T)}.$$

Setting $\tilde{F} := \iota_{H(\Sigma)}^{-1} F \in \mathcal{L}(H(\Sigma), H(\Sigma))$, we have for all $\phi_1, \phi_2 \in H(\Sigma)$:

$$\begin{aligned}(\tilde{F}\phi_1, \phi_2)_{H(\Sigma)} &= (\iota_{H(\Sigma)}^{-1} F\phi_1, \phi_2)_{H(\Sigma)} = \langle F\phi_1, \phi_2\rangle_\Sigma = \langle F\phi_2, \phi_1\rangle_\Sigma \\ &= (\iota_{H(\Sigma)}^{-1} F\phi_2, \phi_1)_{H(\Sigma)} = (\phi_1, \tilde{F}\phi_2)_{H(\Sigma)},\end{aligned}$$

so $\tilde{F}$ is selfadjoint. As

$$(\tilde{F}\phi_1, \phi_1)_{H(\Sigma)} = \langle F\phi_1, \phi_1\rangle_\Sigma \geq c \|\phi_1\|_{H(\Sigma)}^2,$$

$\tilde{F}$ is positive definite and bijective, thus it possesses a bijective selfadjoint square root $\tilde{F}^{1/2}$ (cf. e. g. [17, Theorem 12.33]).

Now setting $\tilde{L} := L\iota_{H(\Sigma)} \in \mathcal{L}(H(\Sigma), H(T))$, we have $\tilde{L}^* = L'\iota_{H(T)}$ and conclude that

$$(\Lambda_1 - \Lambda_0)\iota_{H(T)} = \tilde{L}\tilde{F}\tilde{L}^* = \tilde{L}\tilde{F}^{1/2}(\tilde{F}^{1/2})^*\tilde{L}^*,$$

hence $(\Lambda_1 - \Lambda_0)\iota_{H(T)}$ is selfadjoint and positive, thus possesses a selfadjoint square root $(\Lambda_1\iota_{H(T)} - \Lambda_0\iota_{H(T)})^{1/2}$ and Lemma 3.5 yields

$$\mathcal{R}((\Lambda_1\iota_{H(T)} - \Lambda_0\iota_{H(T)})^{1/2}) = \mathcal{R}(\tilde{L}\tilde{F}^{1/2}) = \mathcal{R}(L\iota_{H(\Sigma)}\tilde{F}^{1/2}).$$

Now the assertion follows from the surjectivity of $\iota_{H(\Sigma)}$ and $\tilde{F}^{1/2}$. □

### 3.3. Remarks and Extensions.

**Remark 3.6.** (a) Though we have chosen $\Lambda_0$ to belong to the domain without inclusion (and we will continue to do so in the applications), this choice was completely arbitrary. By simply interchanging $\Lambda_1$ and $\Lambda_0$ we can extend the assertion of Theorem 3.1:

If $a_{\Omega,1} - a_{\Omega,0}$ or $a_{\Omega,0} - a_{\Omega,1}$ is coercive then

$$\mathcal{R}(|\Lambda_1 \iota_{H(T)} - \Lambda_0 \iota_{H(T)}|^{1/2}) = \mathcal{R}(L),$$

where $|A| = (A^* A)^{1/2}$ for an operator $A \in \mathcal{L}(H)$ on a real Hilbert space $H$. (Note that here $|\Lambda_1 \iota_{H(T)} - \Lambda_0 \iota_{H(T)}|$ is either $(\Lambda_1 - \Lambda_0)\iota_{H(T)}$ or $(\Lambda_0 - \Lambda_1)\iota_{H(T)}$ as one of these expressions is positive definite.)

(b) Theorem 3.1 can also be used to compare a domain with a certain inclusion to one where the inclusion has different physical properties.

(c) The operators $\gamma_{Q\to T}$ resp. $\gamma_{B\to T}$ do not have to be surjective, e. g. in the diffusion example one can choose $H(T) = L^2(T)$ or a space of functions defined only on part of the boundary.

In the diffusion example another important case is that of insulating inclusions (cavities), i. e., inclusions with zero boundary flux. In this case $\Lambda_1$ is given by $\Lambda_1 g = u|_T$, where $u$ solves

$$\operatorname{div}(\kappa \operatorname{grad} u) - cu = 0 \quad \text{in } Q$$
$$\kappa \partial_\nu u = \begin{cases} 0 & \text{on } \Sigma \\ g & \text{on } T. \end{cases}$$

The elliptic problem describing such a body with cavities is now only defined on the inclusion's complement $Q$. We can generalize this idea to our abstract setting by leaving out the bilinear form $a_{\Omega,1}$ in Assumption 2.5 and replacing the definition of $\Lambda_1$ by $\Lambda_1 = \gamma_{Q\to T} A_Q^{-1} \gamma'_{Q\to T}$. The range characterization can now be proven analogously to Theorem 3.1.

**Theorem 3.7.** *Let the Hilbert spaces $H(\cdot)$ be as in Assumption 2.2 and bilinear forms $a_Q$, $a_{\Omega,0}$ as in Assumption 2.5. Let $A_Q$ and $A_{\Omega,0}$ be the induced operators and define*

$$\Lambda_0 := \gamma_{B\to T} A_0^{-1} \gamma'_{B\to T} \quad \textit{and} \quad \Lambda_1 := \gamma_{Q\to T} A_Q^{-1} \gamma'_{Q\to T}.$$

*Then $\mathcal{R}((\Lambda_1 \iota_{H(T)} - \Lambda_0 \iota_{H(T)})^{1/2}) = \mathcal{R}(L)$, where $L$ is given as in Theorem 3.1.*

*Proof.* We have

$$\begin{aligned} \Lambda_1 - \Lambda_0 &= \gamma_{Q\to T} A_Q^{-1} \gamma'_{Q\to T} - \gamma_{B\to T} A_0^{-1} \gamma'_{B\to T} \\ &= \gamma_{Q\to T} \left( A_Q^{-1} - E_Q A_0^{-1} E'_Q \right) \gamma'_{Q\to T} \\ &= \gamma_{Q\to T} A_Q^{-1} (E_Q^-)' \left( E'_Q A_Q E_Q - E'_Q A_Q E_Q A_0^{-1} E'_Q A_Q E_Q \right) E_Q^- A_Q^{-1} \gamma_{Q\to T}. \end{aligned}$$

Using $E_Q' A_Q E_Q = A_0 - E_\Omega' A_{\Omega,0} E_\Omega$ twice, the parenthesized term can be written as

$$\begin{aligned} E_Q' A_Q E_Q - E_Q' A_Q E_Q A_0^{-1} E_Q' A_Q E_Q &= E_Q' A_Q E_Q A_0^{-1} E_\Omega' A_{\Omega,0} E_\Omega \\ &= E_\Omega' A_{\Omega,0} E_\Omega - E_\Omega' A_{\Omega,0} E_\Omega A_0^{-1} E_\Omega' A_{\Omega,0} E_\Omega, \end{aligned}$$

thus

$$\begin{aligned} &E_Q' A_Q E_Q - E_Q' A_Q E_Q A_0^{-1} E_Q' A_Q E_Q \\ &\qquad = E_\Omega' (E_\Omega^-)' \left( E_Q' A_Q E_Q - E_Q' A_Q E_Q A_0^{-1} E_Q' A_Q E_Q \right) E_\Omega^- E_\Omega, \end{aligned}$$

which yields the factorization

$$\begin{aligned} \Lambda_1 - \Lambda_0 &= \gamma_{Q\to T} A_Q^{-1} (E_Q^-)' E_\Omega' (E_\Omega^-)' \left( E_Q' A_Q E_Q - E_Q' A_Q E_Q A_0^{-1} E_Q' A_Q E_Q \right) \\ &\qquad E_\Omega^- E_\Omega E_Q^- A_Q^{-1} \gamma_{Q\to T} \\ &\overset{(9)}{=} \gamma_{Q\to T} A_Q^{-1} \gamma_{Q\to\Sigma}' (\gamma_{\Omega\to\Sigma}^-)' (E_\Omega^-)' \left( E_Q' A_Q E_Q - E_Q' A_Q E_Q A_0^{-1} E_Q' A_Q E_Q \right) \\ &\qquad E_\Omega^- \gamma_{\Omega\to\Sigma}^- \gamma_{Q\to\Sigma} A_Q^{-1} \gamma_{Q\to T} \\ &= L (\gamma_{\Omega\to\Sigma}^-)' (E_\Omega^-)' E_Q' A_Q \left( A_Q^{-1} - E_Q A_0^{-1} E_Q' \right) A_Q E_Q E_\Omega^- \gamma_{\Omega\to\Sigma}^- L' \\ &\overset{(9)}{=} L F L', \end{aligned}$$

where now $F := (\gamma_{Q\to\Sigma}^-)' A_Q \left( A_Q^{-1} - E_Q A_0^{-1} E_Q' \right) A_Q \gamma_{Q\to\Sigma}^-$. Applying Lemma 3.3 on $A_Q^{-1}$ and $E_Q A_0^{-1} E_Q'$ we have

$$E_Q A_0^{-1} E_Q' A_Q E_Q A_0^{-1} E_Q' - E_Q A_0^{-1} E_Q' \geq E_Q A_0^{-1} E_Q' - A_Q^{-1}$$

and thus

$$A_Q^{-1} - E_Q A_0^{-1} E_Q' \geq E_Q A_0^{-1} E_Q' - E_Q A_0^{-1} E_Q' A_Q E_Q A_0^{-1} E_Q'.$$

For $F$ this implies

$$\begin{aligned} F &\geq (\gamma_{Q\to\Sigma}^-)' A_Q \left( E_Q A_0^{-1} E_Q' - E_Q A_0^{-1} E_Q' A_Q E_Q A_0^{-1} E_Q' \right) A_Q \gamma_{Q\to\Sigma}^- \\ &= (\gamma_{Q\to\Sigma}^-)' A_Q E_Q A_0^{-1} \left( A_0 - E_Q' A_Q E_Q \right) A_0^{-1} E_Q' A_Q \gamma_{Q\to\Sigma}^- \\ &= (\gamma_{Q\to\Sigma}^-)' A_Q E_Q A_0^{-1} E_\Omega' A_{\Omega,0} E_\Omega A_0^{-1} E_Q' A_Q \gamma_{Q\to\Sigma}^-. \end{aligned}$$

In the proof of Lemma 3.4 we showed, that the mapping $E_\Omega A_0^{-1} E_Q' A_Q \gamma_{Q\to\Sigma}^-$ has a continuous left inverse. Thus the coercivity of $A_{\Omega,0}$ yields coercivity of $F$ and the rest of the proof is identical to the proof of Theorem 3.1. □

If in the diffusion example only the diffusion coefficient differs in the inclusion, that is if the absorption coefficient equals $c$ throughout $B$, then

$$\langle (A_{\Omega,0} - A_{\Omega,1}) u, u \rangle_\Omega = \int_\Omega \kappa_1 \nabla u \nabla u \, \mathrm{d}x.$$

Thus $A_{\Omega,0} - A_{\Omega,1}$ is not coercive anymore but $A_{\Omega,0} - A_{\Omega,1} + I'\iota_{L^2(\Sigma)} I$ is, where $I$ is the compact imbedding of $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$. The following extension of Theorem 3.1 enables us to also treat such compact perturbations of coercive operators.

**Theorem 3.8.** *Let the Assumptions* 2.2 *and* 2.5 *hold and define* $L$ *as in Theorem 3.1. Suppose that there exists a compact and symmetric operator* $K : H(\Omega) \to H'(\Omega)$, *such that* $A_{\Omega,0} - A_{\Omega,1} + K$ *is coercive. If we further assume that* $\gamma_{\Omega\to\Sigma}(A_{\Omega,1}^{-1} - A_{\Omega,0}^{-1})\gamma'_{\Omega\to\Sigma}$ *is injective, then*

$$\mathcal{R}(|\Lambda_1 \iota_{H(T)} - \Lambda_0 \iota_{H(T)}|^{1/2}) = \mathcal{R}(L).$$

*Proof.* We note that this proof has been inspired by the one of Theorem 3.3 in [15]. Lemma 3.2 yields the factorization $\Lambda_1 - \Lambda_0 = LFL'$, and by following the lines of the proof of Lemma 3.4 we immediately obtain coerciveness of $F + K_2$ with compact and symmetric $K_2 : H(\Sigma) \to H'(\Sigma)$. Thus $F$ is bijective if it is injective.

To show that injectivity of $\gamma_{\Omega\to\Sigma}(A_{\Omega,0}^{-1} - A_{\Omega,1}^{-1})\gamma'_{\Omega\to\Sigma}$ yields injectivity (and thus bijectivity) of $F$ we introduce auxiliary operators ($i = 0, 1$)

$$F_i := (\gamma^-_{Q\to\Sigma})' \left(A_Q E_Q A_i^{-1} E'_Q A_Q - A_Q\right) \gamma^-_{Q\to\Sigma}.$$

We have

$$\begin{aligned} A_Q E_Q A_i^{-1} E'_Q A_Q - A_Q &= A_Q E_Q A_i^{-1} \left(A_i - E'_\Omega A_{\Omega,i} E_\Omega\right) E^-_Q - A_Q \\ &= -A_Q E_Q A_i^{-1} E'_\Omega A_{\Omega,i} E_\Omega E^-_Q \\ &\overset{(9)}{=} -A_Q E_Q A_i^{-1} E'_\Omega A_{\Omega,i} \gamma^-_{\Omega\to\Sigma} \gamma_{Q\to\Sigma}, \end{aligned}$$

thus

$$\left(A_Q E_Q A_i^{-1} E'_Q A_Q - A_Q\right) \gamma^-_{Q\to\Sigma} \gamma_{Q\to\Sigma} = A_Q E_Q A_i^{-1} E'_Q A_Q - A_Q$$

and

$$F_i \gamma_{Q\to\Sigma} A_Q^{-1} \gamma'_{Q\to\Sigma} = (\gamma^-_{Q\to\Sigma})' A_Q E_Q A_i^{-1} E'_Q \gamma'_{Q\to\Sigma} - \mathrm{id}_{H'(\Sigma)}.$$

Furthermore

$$\begin{aligned} F_i \gamma_{\Omega\to\Sigma} A_{\Omega,i}^{-1} \gamma'_{\Omega\to\Sigma} &\overset{(9)}{=} (\gamma^-_{Q\to\Sigma})' A_Q E_Q A_i^{-1} E'_Q A_Q E_Q E^-_\Omega A_{\Omega,i}^{-1} \gamma'_{\Omega\to\Sigma} \\ &\quad - (\gamma^-_{Q\to\Sigma})' A_Q E_Q E^-_\Omega A_{\Omega,i}^{-1} \gamma'_{\Omega\to\Sigma} \\ &\overset{(15)}{=} (\gamma^-_{Q\to\Sigma})' A_Q E_Q A_i^{-1} \left(A_i - E'_\Omega A_{\Omega,i} E_\Omega\right) E^-_\Omega A_{\Omega,i}^{-1} \gamma'_{\Omega\to\Sigma} \\ &\quad - (\gamma^-_{Q\to\Sigma})' A_Q E_Q E^-_\Omega A_{\Omega,i}^{-1} \gamma'_{\Omega\to\Sigma} \\ &= - (\gamma^-_{Q\to\Sigma})' A_Q E_Q A_i^{-1} E'_\Omega \gamma'_{\Omega\to\Sigma} \\ &= - (\gamma^-_{Q\to\Sigma})' A_Q E_Q A_i^{-1} E'_Q \gamma'_{Q\to\Sigma}, \end{aligned}$$

so

$$F_i\left(\gamma_{Q\to\Sigma}A_Q^{-1}\gamma'_{Q\to\Sigma} + \gamma_{\Omega\to\Sigma}A_{\Omega,i}^{-1}\gamma'_{\Omega\to\Sigma}\right) = -\mathrm{id}_{H'(\Sigma)},$$

and along with the symmetry of the operators we conclude that $F_i$ is bijective with $F_i^{-1} = -\gamma_{Q\to\Sigma}A_Q^{-1}\gamma'_{Q\to\Sigma} - \gamma_{\Omega\to\Sigma}A_{\Omega,i}^{-1}\gamma'_{\Omega\to\Sigma}$. Thus

$$F = F_1 - F_0 = F_1\left(F_0^{-1} - F_1^{-1}\right)F_0 = F_1\gamma_{\Omega\to\Sigma}\left(A_{\Omega,1}^{-1} - A_{\Omega,0}^{-1}\right)\gamma'_{\Omega\to\Sigma}F_0,$$

and the injectivity (and hence bijectivity) of $F$ follows from the assumed injectivity of the operator $\gamma_{\Omega\to\Sigma}(A_{\Omega,1}^{-1} - A_{\Omega,0}^{-1})\gamma'_{\Omega\to\Sigma}$.

Just as in the proof of Theorem 3.1 we obtain $\tilde{\Lambda} := (\Lambda_1 - \Lambda_0)\iota_{H(T)} = \tilde{L}\tilde{F}\tilde{L}^*$, with bijective self-adjoint $\tilde{F}$ that is the sum of a coercive self-adjoint operator and a compact self-adjoint operator.

Let $E$ be the spectral decomposition of $\tilde{\Lambda}$ (cf., e. g., [17, Theorem 12.23]). Set

$$P_- := E((-\infty, 0)), \quad P_+ := E([0,\infty)),$$

then $|\tilde{\Lambda}| = \tilde{\Lambda}(P_+ - P_-)$.

Since $\langle \tilde{F}v, v\rangle < 0$ for all $v \in \mathcal{R}(\tilde{L}^*P_-)$, the space $\mathcal{R}(\tilde{L}^*P_-)$ is of finite dimension. Thus it is closed and there exists $c > 0$ such that

$$\langle \tilde{F}v, v\rangle < -c\,\|v\|^2 \qquad \text{for all } v \in \mathcal{R}(\tilde{L}^*P_-).$$

For $\langle \tilde{F}v, v\rangle \geq 0$ for all $v \in \mathcal{R}(\tilde{L}^*P_+)$, $\overline{\mathcal{R}(\tilde{L}^*P_+)} \cap \mathcal{R}(\tilde{L}^*P_-) = \{0\}$.

Since $\overline{\mathcal{R}(\tilde{L}^*P_+)} + \mathcal{R}(\tilde{L}^*P_-)$ is closed (cf., e. g., [17, Theorem 1.42]), there exists a continuous projection

$$Q_- : \ \overline{\mathcal{R}(\tilde{L}^*P_+)} + \mathcal{R}(\tilde{L}^*P_-) \to \mathcal{R}(\tilde{L}^*P_-) \qquad \text{with } \mathcal{N}(Q_-) = \overline{\mathcal{R}(\tilde{L}^*P_+)}$$

(cf., e. g., [17, Theorem 5.16]). By setting it to zero on the orthogonal complement, $Q_-$ can be extended to a continuous projection on $H(\Sigma)$. Setting $Q_+ := \mathrm{id} - Q_-$ on $H(\Sigma)$ we have $\tilde{L}^*P_- = Q_-\tilde{L}^*$ and $\tilde{L}^*P_+ = Q_+\tilde{L}^*$. Hence

$$|\tilde{\Lambda}| = \tilde{\Lambda}(P_+ - P_-) = \tilde{L}\tilde{F}(Q_+ - Q_-)\tilde{L}^*,$$

where $\tilde{F}(Q_+ - Q_-)$ is bijective, self-adjoint (since $P_+$ and $P_-$ commute with $\tilde{\Lambda}$) and positive. The assertion now follows from the arguments in the proof of Theorem 3.1. □

For the diffusion example with $c_1 = 0$ it remains to show the injectivity of $\gamma_{\Omega\to\Sigma}(A_{\Omega,1}^{-1} - A_{\Omega,0}^{-1})\gamma'_{\Omega\to\Sigma}$, which is just the difference of the Neumann-Dirichlet operators on the inclusion. Let $g$ be in its kernel, then Lemma 3.3 yields

$$A_{\Omega,1}^{-1} - A_{\Omega,0}^{-1} \geq A_{\Omega,0}^{-1}(A_{\Omega,0} - A_{\Omega,1})A_{\Omega,0}^{-1}$$

and since the kernel of $(A_{\Omega,0} - A_{\Omega,1})$ only contains constant functions, we have that the function $A_{\Omega,0}^{-1}\gamma'_{\Omega\to\Sigma}g$ is constant, which easily yields that $g = 0$.

## 4. Applications

We already showed that our general result yields the range characterization for the diffusion equation originally proven by Kirsch in [15]. In this section we will show how five other applications fit in our general framework. The range characterization for the first two are already known: the first (impedance tomography) was treated by Brühl [4], the second (optical tomography) was recently treated by Hyvönen [12], who also achieved a similar factorization result [13] for the third application (complete electrode model of impedance tomography). For the fourth application (electrostatics) our general results extend the results of Hähner [9], who treated the case of an insulating inclusion and the fifth (linear elasticity) is a new application for the factorization method.

**4.1. Impedance Tomography.** In impedance tomography an electric current $g$ is applied to the boundary $T$ of a body $B$ with an inclusion $\Omega$, and this leads to an electric potential $u$ that satisfies the equation (cf. e. g. [5])

$$\begin{aligned} \operatorname{div}\left((\kappa-\kappa_1\chi_\Omega)\operatorname{grad} u\right) &= 0 \qquad \text{in } B \\ \kappa\partial_\nu u &= g \qquad \text{on } T, \end{aligned} \tag{19}$$

where $\kappa$ is the conductivity coefficient in $B$ and $\kappa_1$ is the change of conductivity in the inclusion.

Our task is to identify $\Omega$ by measuring the potential $u$ on the boundary for different applied currents for a body with and without inclusion, that is we assume knowledge of the operators $\Lambda_i : \; g \to u_i|_T$, where $u_i$ solves (19) with $(i = 1)$ the term $\kappa_1\chi_\Omega$ and without $(i = 0)$.

We assume $B, \Omega \subset \mathbb{R}^n$ to be bounded domains with $\overline{\Omega} \subset B$ and $C^1$-boundaries $T := \partial B$, $\Sigma := \partial\Omega$, where $Q := B \setminus \overline{\Omega}$ is connected and $\Omega$ consists of $m$ connected components $\Omega_j$ with $C^1$-boundaries $\Sigma_j$ $(j = 1, \dots, m)$.

Let the conductivity coefficients satisfy

$$\kappa \in L^\infty_+(B), \quad \kappa-\kappa_1 \in L^\infty_+(\Omega).$$

The appropriate solution spaces are

$$\begin{aligned} H(B) &:= H^1(B)/\operatorname{span}\{\mathbf{1}_B\} \\ H(Q) &:= H^1(Q)/\operatorname{span}\{\mathbf{1}_Q\} \\ H(T) &:= L^2(T)/\operatorname{span}\{\mathbf{1}_T\} \\ H(\Omega) &:= H^1(\Omega)/\operatorname{span}\{\mathbf{1}_{\Omega_j} : j = 1, \dots, m\} \\ H(\Sigma) &:= H^{1/2}(\Sigma)/\operatorname{span}\{\mathbf{1}_{\Sigma_j} : j = 1, \dots, m\}, \end{aligned}$$

where $\mathbf{1}_{\mathcal{X}}$ denotes the constant function $u(x) = 1\ \forall x \in \mathcal{X}$. These quotient spaces are isomorphic to the orthogonal complements of their factors in the corresponding $H^1$ resp. $L^2$ spaces and inherit their Hilbert space structure, e. g.

$$(u + \operatorname{span}\{\mathbf{1}_B\}, v + \operatorname{span}\{\mathbf{1}_B\})_{H(B)} = (P_{\mathbf{1}_B^\perp} u, P_{\mathbf{1}_B^\perp} v)_{H^1(B)},$$

where $P_{\mathbf{1}_B^\perp}$ is the orthogonal projection on $\operatorname{span}\{\mathbf{1}_B\}^\perp$.

The restriction and trace operators from the diffusion example (see our remark below Assumption and Definition 2.2) can be canonically restricted to these spaces, e. g.

$$E_\Omega(u + \operatorname{span}\{\mathbf{1}_B\}) := u|_\Omega + \operatorname{span}\{\mathbf{1}_{\Omega_j} \mid j = 1, \ldots, m\}.$$

Assumption (V1) then follows from Poincaré's inequality. (V2a),(V2b) and (V3) are easily carried over from the diffusion example.

The bilinear forms for the variational formulation of (19) are

$$a_Q(u,v) := \int_Q \kappa \nabla u \nabla v \,\mathrm{d}x \qquad \text{for } u, v \in H(Q) \tag{20}$$

$$a_{\Omega,0}(u,v) := \int_\Omega \kappa \nabla u \nabla v \,\mathrm{d}x \qquad \text{for } u, v \in H(\Omega) \tag{21}$$

$$a_{\Omega,1}(u,v) := \int_\Omega (\kappa - \kappa_1) \nabla u \nabla v \,\mathrm{d}x \quad \text{for } u, v \in H(\Omega), \tag{22}$$

where we used the canonical restriction of the gradient to the above factor spaces.

With our assumptions to the coefficients it is well known that these satisfy Assumption 2.5 and that $a_{\Omega,0} - a_{\Omega,1}$ is coercive if $\kappa_1 \in L^\infty_+(\Omega)$. Thus Theorem 3.1 gives the range characterization from [4]. (The case of an insulating inclusion is covered by Theorem 3.7.)

**4.2. Optical Tomography.** The propagation of near-infrared light through a strongly scattering medium can be modeled by the diffusion equation with Robin boundary condition (cf. [11, 12]):

$$\begin{aligned} -\operatorname{div}\left((\kappa - \kappa_1\chi_\Omega) \operatorname{grad} u\right) + (c - c_1\chi_\Omega) u = 0 \qquad & \text{in } B \\ u + \kappa \partial_\nu u = g \qquad & \text{on } T. \end{aligned} \tag{23}$$

For this example, the coefficients and solution spaces can be chosen as in the diffusion example. For the sake of using the bilinear forms from the diffusion example we left out some constants in the Robin boundary condition.

Again our measurements are given by the operators $\Lambda_i :\ g \to u_i|_T$, where $u_i$ solves (23) with $(i = 1)$ the term $\kappa_1\chi_\Omega$ and without $(i = 0)$. Note that it makes no difference for $\Lambda_1 - \Lambda_0$ if we map $g$ to $u|_T$ or to the physically more relevant $(u - \kappa\partial_\nu u)|_T$.

The Robin boundary conditions are treated by adding a boundary integral term in the variational formulation of (23), $a_{\Omega,0}$ and $a_{\Omega,1}$ stay the same as in the diffusion example, $a_Q$ changes to

$$a_Q(u,v) := \int_Q \{\kappa\nabla u\nabla v + cuv\}\,\mathrm{d}x + \int_T u|_T v|_T\,\mathrm{d}x.$$

Obviously Assumption 2.5 is still satisfied. $a_{\Omega,0} - a_{\Omega,1}$ is coercive if we assume $\kappa_1, c_1 \in L^\infty_+(\Omega)$ (resp. a compact perturbation of a coercive bilinear form if we allow $c_1$ to vanish). Thus Theorem 3.1 (resp. Theorem 3.8) gives the range characterization from [12]. (Note that the additional assumption of Theorem 3.8 is the same as in the diffusion case and has already been verified in Section 3.3.)

**4.3. Complete electrode model of impedance tomography.** An extension of the model of impedance tomography is to assume that electric currents are induced into a domain $B$ by attaching perfectly conducting electrodes to pieces $T_k$ of the boundary $T$ with positive contact impedance $z$. This extends the equations (19) to the equations of the so-called complete electrode model of impedance tomography (cf. [18, 13])

$$\begin{aligned} \operatorname{div}\left((\kappa - \kappa_1\chi_\Omega)\operatorname{grad} u\right) &= 0 && \text{in } B\\ \kappa\partial_\nu u &= g && \text{on } T_0\\ u + z\kappa\partial_\nu u &= U_k\mathbf{1}_{T_k} && \text{on } T_k\\ \textstyle\int_{T_k}\kappa\partial_\nu u &= I_k. \end{aligned} \tag{24}$$

In (24) $I_k$ is the current that is applied through the $k$-th electrode and $U_k$ is the measured potential of the $k$-th electrode. Thus the operators that represent the measurements are extended to

$$\Lambda_i:\ (g, I_1, \ldots, I_K) \mapsto \left(u_i|_{T_0}, U_1^{(i)}, \ldots, U_K^{(i)}\right),$$

with $\Lambda_1$ denoting the case with inclusion (i. e., with the term $\kappa_1\chi_\Omega$) and $\Lambda_0$ without the inclusion (i. e. without this term).

We make the same assumptions on $B$, $\Omega$, $Q := B \setminus \overline{\Omega}$, $\kappa$ and $\kappa - \kappa_1$ as in Section 4.1. Furthermore let $T$ be divided into $K+1$ open pieces $T = \overline{T_0} \cup \overline{T_1} \cup \ldots \cup \overline{T_k}$ and $z \in L^\infty_+(T)$.

The solution spaces are extended by the unknown potentials $(U_k)_k \in \mathbb{R}^K$:

$$
\begin{aligned}
H(B) &:= (H^1(B) \times \mathbb{R}^K)/\mathrm{span}\{(\mathbf{1}_B, 1, \dots, 1)\} \\
H(Q) &:= (H^1(Q) \times \mathbb{R}^K)/\mathrm{span}\{(\mathbf{1}_Q, 1, \dots, 1)\} \\
H(T) &:= (L^2(T_0) \times \mathbb{R}^K)/\mathrm{span}\{(\mathbf{1}_{T_0}, 1, \dots, 1)\} \\
H(\Omega) &:= H^1(\Omega)/\mathrm{span}\{\mathbf{1}_{\Omega_j} : j = 1, \dots, m\} \\
H(\Sigma) &:= H^{1/2}(\Sigma)/\mathrm{span}\{\mathbf{1}_{\Sigma_j} : j = 1, \dots, m\},
\end{aligned}
$$

where the extended spaces and their factor spaces are equipped with the usual Hilbert space structure and the restriction and trace operators are appropriately extended, e. g.

$$
\begin{aligned}
\gamma_{B \to T} : (u, U_1, \dots, U_K) &+ \mathrm{span}\,\{(\mathbf{1}_B, 1, \dots, 1)\} \\
&\mapsto \quad (u|_{T_0}, U_1, \dots, U_K) + \mathrm{span}\,\{(\mathbf{1}_{T_0}, 1, \dots, 1)\}\,.
\end{aligned}
$$

Assumptions (V1), (V2a), (V2b) and (V3) are easily shown as in Section 4.1. The bilinear forms $a_{\Omega,0}$ and $a_{\Omega,1}$ are chosen as in Section 4.1, $a_Q$ is the canonical restriction of

$$
((u, (U_k)_k), (v, (V_k)_k)) \mapsto \int_Q \kappa \nabla u \nabla v \,\mathrm{d}x + \sum_{k=1}^{K} \int_{T_k} \frac{1}{z}(U_k - u)(V_k - v)\,\mathrm{d}x.
$$

to the factor space $H(Q)$.

With the above assumptions on the coefficients they satisfy Assumption 2.5 and $a_{\Omega,0} - a_{\Omega,1}$ is coercive if $\kappa_1 \in L^\infty_+(\Omega)$. Thus Theorem 3.1 yields the desired range characterization (cf. [13] for a similar result).

**4.4. Electrostatics.** The electrostatic potential $u$ of charges arranged along a closed surface $T$ with density $g(y)$, $y \in T$, solves

$$
\begin{aligned}
\mathrm{div}\,((\kappa - \kappa_1 \chi_\Omega)\,\mathrm{grad}\,u) &= 0 \qquad \text{in } \mathbb{R}^3 \setminus T \\
[\kappa \partial_\nu u]_T &= g \qquad \text{on } T,
\end{aligned}
\tag{25}
$$

where again $\kappa$ is the conductivity coefficient in $\mathbb{R}^3$ and $\kappa_1$ is the change of conductivity in an inclusion $\Omega$ that is surrounded by $T$. $[\kappa \partial_\nu u]_T$ denotes the jump in the normal derivative on $T$. In order to localize the inclusion we measure the potential on $T$ for different charge configurations, thus our measurements are given by the operators $\Lambda_i :\ g \to u_i|_T$, where $u_i$ solves (25) with $(i = 1)$ the term $\kappa_1 \chi_\Omega$ and without $(i = 0)$.

We assume $\Omega \subset B := \mathbb{R}^3$ to be a bounded domain with $C^1$-boundary $\Sigma := \partial\Omega$, where $Q := \mathbb{R}^3 \setminus \overline{\Omega}$ is connected and $\Omega$ consists of $m$ connected components $\Omega_j$ with $C^1$-boundaries $\Sigma_j$ $(j = 1, \dots, m)$. Let further $T$ be the smooth surface of a simply connected bounded domain containing $\overline{\Omega}$ (note that $T \neq \partial B$), $\kappa \in L^\infty_+(\mathbb{R}^3)$ and $\kappa - \kappa_1 \in L^\infty_+(\Omega)$.

The appropriate solution spaces are (cf. [7, Chapter XI, B, §2])

$$\begin{aligned}
H(B) &:= \left\{u \;:\; (1+|x|^2)^{-\frac{1}{2}} u \in L^2(\mathbb{R}^3), \partial_i u \in L^2(\mathbb{R}^3),\ i = 1,2,3\right\}\\
H(Q) &:= \left\{u \;:\; (1+|x|^2)^{-\frac{1}{2}} u \in L^2(Q), \partial_i u \in L^2(Q),\ i = 1,2,3\right\}\\
H(T) &:= L^2(T)\\
H(\Omega) &:= H^1(\Omega)/\mathrm{span}\{\mathbf{1}_{\Omega_j} : j = 1,\dots,m\}\\
H(\Sigma) &:= H^{1/2}(\Sigma)/\mathrm{span}\{\mathbf{1}_{\Sigma_j} : \ j = 1,\dots,m\}.
\end{aligned}$$

Assumptions (V1)–(V3) stay valid if the restriction and trace operators are chosen as in Application 4.1. Also the bilinear forms are chosen as in (20)–(22). ($a_Q$ is now defined by the integral over the unbounded domain $Q = \mathbb{R}^3 \setminus \overline{\Omega}$.) In the spaces given above they satisfy Assumption 2.5 (cf. [7]).

Again $a_{\Omega,0} - a_{\Omega,1}$ is coercive if $\kappa_1 \in L^\infty_+(\Omega)$. Thus Theorem 3.1 yields the desired range characterization and thus extends the result in [9] (which is covered by Theorem 3.7).

**4.5. Linear elasticity.** We conclude our work with a (at least to our knowledge) new application for the Factorization Method, namely an inverse problem in linear elasticity (cf. [1] where the problem of detecting cavities is treated with level set methods). If forces $g$ are applied to the boundary $T$ of an elastic body $B$, they result in a displacement $u$, that satisfies in the state of equilibrium the equations of linear elasticity (cf. [6, Sect. 6.3])

$$\begin{aligned}
\mathrm{div}\,((\lambda - \lambda_1\chi_\Omega)(\mathrm{tr}\, e(u))I + 2(\mu - \mu_1\chi_\Omega)e(u)) &= 0 \qquad \text{in } B\\
(\lambda(\mathrm{tr}\, e(u))I + 2\mu e(u))\,\nu &= g \qquad \text{on } T,
\end{aligned} \tag{26}$$

where $\lambda$ and $\mu$ are the Lamé-constants, $\nu$ is the unit normal on $T$, $e(u)_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, $I_{ij} = \delta_{ij}$ $(i,j = 1,2,3)$, tr denotes the trace of a matrix and the divergence of a matrix is the vector whose components are the divergences of the transposes of the row vectors of the matrix.

We try to localize the inclusion $\Omega$ by applying different forces $g$ to the body and measuring the resulting displacements on the boundary. Thus our measurements are given by the operators $\Lambda_i :\ g \to u_i|_T$, where $u_i$ solves (26) with $(i = 1)$ the terms $\lambda_1\chi_\Omega$, $\mu_1\chi_\Omega$ $(\Lambda_1)$ and without $(i = 0)$.

Again we assume $B, \Omega \subset \mathbb{R}^3$ to be bounded domains with $\overline{\Omega} \subset B$ and $C^1$-boundaries $T := \partial B$, $\Sigma := \partial\Omega$, where $Q := B \setminus \overline{\Omega}$ is connected and $\Omega$ consists of $m$ connected components $\Omega_j$ with $C^1$-boundaries $\Sigma_j$ $(j = 1,\dots,m)$.

Let the Lamé-constants satisfy

$$\lambda, \mu \in L^\infty_+(B), \quad \lambda - \lambda_1, \mu - \mu_1 \in L^\infty_+(\Omega).$$

If we define for a connected set $\mathcal{X} \subset \mathbb{R}^3$ the space of rigid deformations on $\mathcal{X}$

$$N_{\mathcal{X}} := \{u : \ \mathcal{X} \to \mathbb{R}^3, \ u(x) = a + b \wedge x, \ \forall x \in \mathcal{X}, \ a, b \in \mathbb{R}^3\}$$

(note that this is the space of all $u$ with $e(u) = 0$), then the appropriate solution spaces are

$$\begin{aligned}
&H(B) := H^1(B;\mathbb{R}^3)/N_B, \quad H(\Omega) := H^1(\Omega;\mathbb{R}^3)/\bigoplus_{j=1}^{m} N_{\Omega_j}\\
&H(Q) := H^1(Q;\mathbb{R}^3)/N_Q, \quad H(\Sigma) := H^{1/2}(\Sigma;\mathbb{R}^3)/\bigoplus_{j=1}^{m} N_{\Sigma_j}\\
&H(T) := L^2(T;\mathbb{R}^3)/N_T.,
\end{aligned}$$

Assumptions (V1), (V2a), (V2b) and (V3) are shown analogously to the diffusion example (with Korn's inequality taking the place of Poincaré's inequality). The bilinear forms for (26) are

$$\begin{aligned}
a_Q(u,v) &:= \int_Q \big(\lambda \operatorname{tr} e(u) \operatorname{tr} e(v) + 2\mu e(u) : e(v)\big)\,\mathrm{d}x\\
a_{\Omega,0}(u,v) &:= \int_\Omega \big(\lambda \operatorname{tr} e(u) \operatorname{tr} e(v) + 2\mu e(u) : e(v)\big)\,\mathrm{d}x\\
a_{\Omega,1}(u,v) &:= \int_\Omega \big((\lambda - \lambda_1) \operatorname{tr} e(u) \operatorname{tr} e(v) + 2(\mu - \mu_1) e(u) : e(v)\big)\,\mathrm{d}x,
\end{aligned}$$

where $B : C$ denotes the inner product $\sum_{ij} b_{ij}c_{ij}$ for matrices $B$, $C$ and we used the canonical restriction of $e(\cdot)$ to the above factor spaces.

With the above assumptions on the coefficients it is well-known that the bilinear forms satisfy Assumption 2.5 and that $a_{\Omega,0} - a_{\Omega,1}$ is coercive if $\lambda_1 \geq 0$ and $\mu_1 \in L^\infty_+(\Omega)$. Thus under this conditions Theorem 3.1 gives the desired range characterization. Again the case of cavities is covered by Theorem 3.7.

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