A Parabolic Integro-Differential Identification Problem in a Barrelled Smooth Domain

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Abstract. We consider the problem of recovering a space- and time-dependent kernel in a parabolic integro-differential equation. The related domain is assumed to be smooth and provided with two bases. Global existence and uniqueness results are proved.

Keywords. Inverse problem, memory kernel, parabolic equation

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1. Introduction

Linear heat flow in materials with memory is governed by parabolic equations of integro-differential type involving time-dependent (in case of inhomogeneity also space-dependent) memory kernels [17]. These kernels are often unknown in practice. Identification problems to determine kernels depending only on time in parabolic integro-differential equations were studied, e.g., in [8, 9, 11 – 14]. Problems to identify space- and time-dependent kernels in parabolic integro-differential equations for cylindrical domains were studied in [1 – 5] and [15]. In [6, 7] analogous problems were dealt with for spherical coronae and kernels with spherical symmetries. In [3 – 5] and [15] only *local (in time)* existence and uniqueness results were established, while in the more recent papers [1, 2] *global* results were obtained.

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In this paper we generalize the *global* existence and uniqueness results of [1, 2] from cylinders to more general domains, which however are smooth. Namely, we consider the problem of identifying a space- and time-dependent kernel in a parabolic integro-differential equation in a barrelled C^2-domain (cf. Definition (2.1) and Condition (2.3)). The C^2-smoothness enables to use proper semigroups.

In Section 2 we formulate the parabolic identification problem. In Sections 3 and 4 we transform it into a form admitting an abstract formulation, which is given in Section 5. Sections 6 and 7 contain auxiliary results. The main solvability results for the abstract problem and the parabolic identification problem are contained in Section 8.

Proving the main results we use the contraction argument in spaces of Hölder-continuous abstract functions C^β, $\beta > 0$, endowed with norms with exponential weights. This technique, due to Janno and Wolfersdorf [9, 10], was so far used in L^p- and C-spaces. The extension to C^β is not straightforward, requiring additional operations with $C^{\beta'}$, $\beta' \in (0, \beta)$. We have to deal with the Hölder-continuity because of certain semigroup properties.

2. Formulation of the problem

Let Ω be a 3-dimensional bounded connected open set with a C^2-boundary, admitting the following representation:

$$\Omega = \left\{ x = (x', y) \in \mathbb{R}^3 \,:\, |x'| < \rho\big(\tfrac{x'}{|x'|}, y\big), \, y \in [0, l] \right\}, \tag{2.1}$$

where $l > 0$ and $\rho \in C(\mathcal{O} \times [0, l]) \cap C^2(\mathcal{O} \times (0, l))$ and $-x_2 D_{x_1}\rho + x_1 D_{x_2}\rho \in C(\mathcal{O} \times [0, l])$, where $\mathcal{O}$ is the 2-dimensional unit sphere. A further fundamental requirement concerning ρ will be listed in the formula (3.1) below. We denote by Γ and Γ_l the boundary and the lateral surface of Ω, respectively. Further, let us assume that the (Lebesgue) measures of the sections

$$\Omega(y) = \left\{ x' \in \mathbb{R}^2 \,:\, (x', y) \in \Omega \right\}, \quad y \in [0, l], \tag{2.2}$$

of Ω are bounded away from 0, i.e.,

$$m_2(\Omega(y)) \geq m > 0 \quad \forall y \in [0, l]. \tag{2.3}$$

The open set Ω will be called a *barrelled* domain of class C^2. Let $\Gamma(y)$ stand for the boundary of $\Omega(y)$, $y \in [0, l]$.

Let us pose the following inverse problem: *given $a : [0, l] \to \mathbb{R}$, $f : [0, T] \times \Omega \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $u_\Gamma : [0, T] \times \Omega \to \mathbb{R}$, $g_1 : [0, T] \times [0, l] \to \mathbb{R}$ and $g_0 : [0, T] \to \mathbb{R}$, find $h : [0, T] \times [0, l] \to \mathbb{R}$ and $u : [0, T] \times \Omega \to \mathbb{R}$ such that*

$$D_t u(t, x) - \text{div}\big(a(y)\nabla u(t, x)\big) - \int_0^t \text{div}\big(h(t - s, y)\nabla u(s, x)\big)ds = f(t, x)$$

$$t \in [0, T], \ x = (x', y) \in \Omega, \tag{2.4}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega \tag{2.5}$$

$$u(t, x) = u_\Gamma(t, x), \qquad t \in [0, T], \ x \in \Gamma \tag{2.6}$$

$$\Phi[u(t, \cdot)](y) = g_1(t, y), \qquad t \in [0, T], \ y \in [0, l] \tag{2.7}$$

$$\Psi[u(t, \cdot)] = g_0(t), \qquad t \in [0, T], \tag{2.8}$$

where

$$\Phi[w](y) = \int_{\Omega(y)} \lambda(x)w(x', y)dx', \quad \Psi[w] = \int_\Omega \mu(x)w(x)dx \tag{2.9}$$

and $w : \Omega \to \mathbb{R}$, $\lambda, \mu : \Omega \to \mathbb{R}$ being given weight functions. Equation (2.4) describes the heat flow in domain Ω filled by material with memory, which is inhomogeneous in y-direction, and u stands for the temperature.

Introducing the new unknowns

$$\widehat{u} = u - u_\Gamma, \tag{2.10}$$

we transform problem (2.3) – (2.7) into the following one involving a homogeneous boundary condition:

$$D_t \widehat{u}(t, x) - \text{div}\big(a(y)\nabla \widehat{u}(t, x)\big)$$

$$- \int_0^t \text{div}\big(h(t - s, y)\nabla(\widehat{u}(s, x) + u_\Gamma(s, x))\big) ds = \widehat{f}(t, x), \tag{2.11}$$

$$t \in [0, T], \ \ x = (x', y) \in \Omega,$$

$$\widehat{u}(0, x) = \widehat{u}_0(x), \qquad x \in \Omega \tag{2.12}$$

$$\widehat{u}(t, x) = 0, \qquad t \in [0, T], \ x \in \Gamma \tag{2.13}$$

$$\Phi[\widehat{u}(t, \cdot)](y) = \widehat{g}_1(t, y), \qquad t \in [0, T], \ y \in [0, l] \tag{2.14}$$

$$\Psi[\widehat{u}(t, \cdot)] = \widehat{g}_0(t), \qquad t \in [0, T], \tag{2.15}$$

where

$$f(t, x) = f(t, x) - D_t u_\Gamma(t, x) + \text{div}\big(a(y)\nabla u_\Gamma(t, x)\big) \tag{2.16}$$

$$\widehat{u}_0(x) = u_0(x) - u_\Gamma(0, x) \tag{2.17}$$

$$\widehat{g}_1(t, y) = g_1(t, y) - \Phi[u_\Gamma(t, \cdot)](y) \tag{2.18}$$

$$\widehat{g}_0(t) = g_0(t) - \Psi[u_\Gamma(t, \cdot)]. \tag{2.19}$$

3. An equivalent differentiated problem

Let us define the linear differential operators $A = \operatorname{div}(a(\cdot)\nabla)$ and $B = \Delta$, and the following Banach spaces endowed with usual norms:

$$X = C(\overline{\Omega}), \qquad X_2 = D(A) = \left\{ w \in \bigcap_{p>3} W^{2,p}(\Omega) : Aw \in X, \ w|_{\Gamma} = 0 \right\}$$

$$Y = C([0, l]), \quad Y_1 = C^1([0, l]), \quad Y_2 = C^2([0, l]).$$

In this and next section we will transform problem (2.11) – (2.15) into a form which is suitable for an abstract formulation. To this end we have to impose certain basic assumptions on the data $a, \lambda, \mu, u_\Gamma, \widehat{f}, \widehat{u}_0, \widehat{g}_1$ and $\widehat{g}_0$, where $\widehat{f}, \widehat{u}_0, \widehat{g}_1$ and $\widehat{g}_0$ are defined via f, u_0, u_Γ, g_1 and g_0 by means of the formulas (2.16) – (2.19). More exactly, let us assume, for some $\beta \in (0, 1)$, that

$$a \in C^2([0, l]), \ a(y) \geq a_0 > 0, \ \lambda \in C^2(\overline{\Omega}), \ \lambda D_y \rho \in C(\Gamma_l) \tag{3.1}$$

$$\mu \in W^{1,1}(\Omega), \ u_\Gamma \in C^1([0, T]; C^2(\overline{\Omega})), \ \widehat{u}_0 \in X_2 \tag{3.2}$$

$$A\widehat{u}_0 + \widehat{f}(0, \cdot) \in X_2, \ \widehat{f} \in C^{1+\beta}([0, T]; X) \tag{3.3}$$

$$\widehat{g}_1 \in C^{2+\beta}([0, T]; Y) \cap C^{1+\beta}([0, T]; Y_2), \ \widehat{g}_0 \in C^{2+\beta}([0, T]), \tag{3.4}$$

where Γ_l denotes the lateral surface of Ω. Further, let us introduce the following functions depending on the data:

$$v_0(x) = A\widehat{u}_0(x) + \widehat{f}(0, x), \ \psi_1(x) = \Delta u_0(x), \ \psi_2(x) = D_y u_0(x) \tag{3.5}$$

$$b(t, x) = \Delta D_t u_\Gamma(t, x), \ g(t, x) = D_t \widehat{f}(t, x) \tag{3.6}$$

$$q_1(t, y) = D_y \widehat{g}_1(t, y) + \Phi[D_y u_\Gamma(t, \cdot)](y) \tag{3.7}$$

$$q_2(t, y) = D_y^2 \widehat{g}_1(t, y) + \Phi[\Delta u_\Gamma(t, \cdot)](y) \tag{3.8}$$

$$f_1(t, y) = D_t \widehat{g}_1(t, y) - D_y(a(y) D_y \widehat{g}_1(t, y)) - \int_{\Omega(y)} \lambda(x) \widehat{f}(t, x) \, dx', \tag{3.9}$$

and let us define the linear operators Q_1 and Q_2 from $C^1(\overline{\Omega})$ to $C([0, l])$ by

$$(Q_1 w)(y) = -\int_{\Omega(y)} D_y \lambda(x) w(x) \, dx' \tag{3.10}$$

$$(Q_2 w)(y) = \sum_{j=1}^{2} \left[\int_{\Gamma(y)} \lambda(x) n_j D_{x_j} w(x) d\sigma(x') - \int_{\Omega(y)} D_{x_j} \lambda(x) D_{x_j} w(x) dx' \right]$$

$$- \int_{\Gamma(y)} \lambda(x) \overline{\rho}(x) D_y w(x) \, d\sigma(x') \tag{3.11}$$

$$- \int_{\Omega(y)} \left[2 D_y \lambda(x) D_y w(x) + D_y^2 \lambda(x) w(x) \right] dx',$$

where $n = (n_1, n_2, n_3)$ is the outer normal on Γ, $d\sigma(x')$ is the Lebesgue surface measure on $\Gamma(y)$ and

$$\bar{\rho}(x) = \rho\left(\tfrac{x'}{|x'|}y\right)D_y\rho\left(\tfrac{x'}{|x'|}, y\right)\left\{\rho\left(\tfrac{x'}{|x'|}, y\right)^2 + \rho\left(\tfrac{x'}{|x'|}, y\right)^{-2}\right.$$
$$\left. \times \left[-x_2 D_{x_1}\rho\left(\tfrac{x'}{|x'|}, y\right) + x_1 D_{x_2}\rho\left(\tfrac{x'}{|x'|}, y\right)\right]^2\right\}^{-\frac{1}{2}}. \tag{3.12}$$

Remark. In the case of a cylinder we have $\bar{\rho} \equiv 0$, so that the line integral over $\Gamma(y)$ in (2.11) vanishes for any $y \in [0, l]$. Consequently, we can say that function $\bar{\rho}$ *measures the deviation from a cylinder* of our barrelled domain Ω.

Proposition 3.1. *Assume that the assumptions* (3.1) − (3.4) *and the following consistency conditions hold:*

$$\widehat{g}_0(0) = \Psi[\widehat{u}_0]\,,\ \widehat{g}_1(0, y) = \Phi[\widehat{u}_0](y)\,, \quad y \in [0, l] \tag{3.13}$$
$$\widehat{g}_1(t, 0) = \widehat{g}_1(t, l) = 0\,, \qquad\qquad\qquad t \in [0, T] \tag{3.14}$$

$$f_1(0, y) - a(y)(Q_2\widehat{u}_0)(y) - a'(y)(Q_1\widehat{u}_0)(y) = 0\,, \quad y \in [0, l] \tag{3.15}$$

$$\widehat{g}_0'(0) - \int_\Gamma a(y)\mu(x)D_n\widehat{u}_0(x)\,d\sigma + \int_\Omega a(y)\nabla\mu(x)\cdot\nabla\widehat{u}_0(x)dx = \Psi[\widehat{f}(0, \cdot)], \tag{3.16}$$

where D_n denotes the normal derivative on Γ. Then the following assertions are valid:

(i) *If $(\widehat{u}, h) \in \left\{C^{2+\beta}([0, T]; X) \cap C^{1+\beta}([0, T]; X_2)\right\} \times C^\beta([0, T]; Y_1)$ solves the inverse problem* (2.11) − (2.15), *then (v, h), with $v = D_t\widehat{u}$, solves the following problem:*

$$D_t v(t, x) - Av(t, x)$$
$$= \int_0^t h(t - s, y)\big(Bv(s, x) + b(s, x)\big)ds$$
$$+ \int_0^t D_y h(t - s, y)\big(D_y v(s, x) + D_y D_t u_\Gamma(s, x)\big)ds + h(t, y)\psi_1(x) \tag{3.17}$$
$$+ D_y h(t, y)\psi_2(x) + g(t, x)\,, \quad t \in [0, T],\ x = (x', y) \in \Omega$$

$$v(0, x) = v_0(x)\,, \quad x \in \Omega \tag{3.18}$$

$$\left[(Q_1\widehat{u}_0)(y) + q_1(0, y)\right]D_y h(t, y) + \left[(Q_2\widehat{u}_0)(y) + q_2(0, y)\right]h(t, y)$$
$$= -\int_0^t D_y h(t - s, y)\left[(Q_1 v(s, \cdot))(y) + D_s q_1(s, y)\right]ds$$
$$- \int_0^t h(t - s, y)\left[(Q_2 v(s, \cdot))(y) + D_s q_2(s, y)\right]ds + D_t f_1(t, y) \tag{3.19}$$
$$- a(y)(Q_2 v(t, \cdot))(y) - a'(y)(Q_1 v(t, \cdot))(y)\,, \quad y \in [0, l],\ t \in [0, T]\,,$$

$$\int_\Gamma h(t,y)\mu(x)D_n u_0(x)\,d\sigma - \int_\Omega h(t,y)\nabla\mu(x)\cdot\nabla u_0(x)dx$$

$$= -\int_0^t \left[\int_\Gamma h(t-s,y)\mu(x)D_n\big(v(s,x)+D_s u_\Gamma(s,x)\big)\,d\sigma\right.$$

$$\left. -\int_\Omega h(t-s,y)\nabla\mu(x)\cdot\nabla\big(v(s,x)+D_s u_\Gamma(s,x)\big)\,dx\right]ds \qquad (3.20)$$

$$-\int_\Gamma a(y)\mu(x)D_n v(t,x)\,d\sigma + \int_\Omega a(y)\nabla\mu(x)\cdot\nabla v(t,x)\,dx$$

$$+ \widehat{g}_0''(t) - \Psi[D_t\widehat{f}(t,\cdot)], \quad t\in[0,T]$$

where σ is the Lebesgue surface measure on Γ.

(ii) *Conversely, if $(v,h)\in\{C^{1+\beta}([0,T];X)\cap C^{\beta}([0,T];X_2)\}\times C^{\beta}([0,T];Y_1)$ solves problem (3.17) – (3.20), then $(\widehat{u},h)$ with $\widehat{u}(t,x)=\widehat{u}_0(x)+\int_0^t v(s,x)ds$ solves the identification problem (2.11) – (2.15).*

Proof. (i): Differentiating the parabolic equation (2.11) with respect to t and using the definitions (3.5) and (3.6), we derive (3.17). Setting $t=0$ in (2.11), we get the initial condition (3.18).

In order to derive equation (3.19) we first recall the identity (cf. (2.1))

$$\int_{\Omega(y)} z(x)dx = \int_0^{2\pi}d\theta\int_0^{\rho(e^{i\theta},y)} z(re^{i\theta},y)r dr.$$

Then we note that from such an identity, due to the homogeneous boundary condition (2.13), the relation $\Gamma(y)\subset\Gamma$ and the condition (2.14), we easily deduce the equations

$$\int_{\Omega(y)} D_y(\lambda(x)\widehat{u}(t,x))\,dx' = D_y\int_{\Omega(y)}\lambda(x)\widehat{u}(t,x)\,dx' = D_y\widehat{g}_1(t,y). \qquad (3.21)$$

This implies

$$D_y\left[\int_{\Omega(y)} a(y)D_y(\lambda(x)\widehat{u}(t,x))\,dx'\right] = D_y(a(y)D_y\widehat{g}_1(t,y)). \qquad (3.22)$$

Let us now differentiate the integral $\int_{\Omega(y)} a(y)D_y(\lambda(x)\widehat{u}(t,x))\,dx'$ with respect to y, taking advantage of definition (2.2). Moreover, the definition (3.12) of $\overline{\rho}$ and the homogeneous boundary condition for $\widehat{u}$ easily imply

$$D_y\left[\int_{\Omega(y)} a(y)D_y(\lambda(x)\widehat{u}(t,x))\,dx'\right]$$

$$= \int_{\Omega(y)} D_y(a(y)D_y(\lambda(x)\widehat{u}(t,x)))dx' + \int_{\Gamma(y)}\overline{\rho}(x)a(y)D_y(\lambda(x)\widehat{u}(t,x))d\sigma(y) \quad (3.23)$$

$$= \int_{\Omega(y)} D_y(a(y)D_y(\lambda(x)\widehat{u}(t,x)))dx' + \int_{\Gamma(y)}\overline{\rho}(x)a(y)\lambda(x)D_y\widehat{u}(t,x)d\sigma(y).$$

Comparing (3.22) and (3.23) we derive the formula

$$
\int_{\Omega(y)} D_y\big(a(y)D_y(\lambda(x)\widehat{u}(t,x))\big)\, dx'
$$
$$
= D_y\big(a(y)D_y\widehat{g}_1(t,y)\big) - \int_{\Gamma(y)} \overline{\rho}(x)a(y)\lambda(x)D_y\widehat{u}(t,x)\, d\sigma(y)\,. \tag{3.24}
$$

Next let us compute

$$
\Phi[\mathrm{div}(a(\cdot)\nabla\widehat{u}(t,\cdot))](y)
$$
$$
= \int_{\Omega(y)} \lambda(x)\mathrm{div}(a(y)\nabla\widehat{u}(t,x))\, dx'
$$
$$
= a(y)\sum_{j=1}^{2}\left[\int_{\Gamma(y)} \lambda(x)n_j D_{x_j}\widehat{u}(t,x)d\sigma(x') - \int_{\Omega(y)} D_{x_j}\lambda(x)D_{x_j}\widehat{u}(t,x)\, dx'\right] \tag{3.25}
$$
$$
+ \int_{\Omega(y)}\big[D_y(a(y)D_y(\lambda(x)\widehat{u}(t,x))) - 2a(y)D_y\lambda(x)D_y\widehat{u}(t,x)
$$
$$
+ \widehat{u}(t,x)D_y(a(y)D_y\lambda(x))\big]dx'.
$$

Using here (3.24) and the definitions (3.10) and (3.11) of operators Q_1 and Q_2, we derive

$$
\Phi[\mathrm{div}(a\nabla\widehat{u}(t,\cdot))] = D_y(aD_y\widehat{g}_1(t,\cdot)) + aQ_2\widehat{u}(t,\cdot) + a'Q_1\widehat{u}(t,\cdot)\,. \tag{3.26}
$$

Analogously we derive the relation

$$
\Phi[\mathrm{div}(h(t-s,\cdot)\nabla\widehat{u}(s,\cdot))](y) = D_y(h(t-s,y)D_y\widehat{g}_1(s,y))
$$
$$
+ h(t-s,y)(Q_2\widehat{u}(s,\cdot))(y)
$$
$$
+ D_y h(t-s,y)(Q_1\widehat{u}(s,\cdot))(y)\,. \tag{3.27}
$$

Moreover, due to the condition (2.14) we have

$$
\Phi[D_t\widehat{u}(t,\cdot)](y) = D_t\widehat{g}_1(t,y)\,. \tag{3.28}
$$

Let us now apply operator Φ to both sides in equation (2.11). In view of (3.26) – (3.28) and definitions (3.7) – (3.9) of q_1, q_2 and f_1, we obtain

$$
a(y)(Q_2\widehat{u}(t,\cdot))(y) + a'(y)(Q_1\widehat{u}(t,\cdot))(y)
$$
$$
+ \int_0^t D_y h(t-s,y)\left[(Q_1\widehat{u}(s,\cdot))(y) + q_1(s,y)\right]ds \tag{3.29}
$$
$$
+ \int_0^t h(t-s,y)\left[(Q_2\widehat{u}(s,\cdot))(y) + q_2(s,y)\right]ds = f_1(t,y)\,.
$$

Differentiating this equation with respect to t, we derive equation (3.19).

In order to complete the proof of (i) we have to derive (3.20). To this end we apply functional Ψ to the parabolic equation (2.11) and observe that

$$\Psi[D_t\widehat{u}(t,\cdot)] \;=\; \widehat{g}_0'(t)\,. \tag{3.30}$$

Hence, we get

$$
\begin{aligned}
&\int_\Gamma a(y)\mu(x)D_n\widehat{u}(t,x)\,d\sigma - \int_\Omega a(y)\nabla\mu(x)\cdot\nabla\widehat{u}(t,x)\,dx \\
&+\int_0^t\left[\int_\Gamma h(t-s,y)\mu(x)D_n(\widehat{u}(s,x)+u_\Gamma(s,x))\,d\sigma\right.\\
&\left.-\int_\Omega h(t-s,y)\nabla\mu(x)\cdot\nabla(\widehat{u}(s,x)+u_\Gamma(s,x))\,dx\right]ds = \widehat{g}_0'(t) - \Psi[\widehat{f}(t,\cdot)].
\end{aligned}
\tag{3.31}
$$

Differentiating this relation we get (3.20).

(ii): Integrating both sides of equation (3.17) over $(0,t)$ and taking advantage of definitions (3.5), (3.6) and of the initial condition (3.18) (cf. (3.5)), we obtain equation (2.11) for $\widehat{u}$.

Further, integrating both sides of equation (3.20) over $(0,t)$ and taking the consistency condition (3.16) into account, we obtain relation (3.31). This, in view of the equation (2.11) for $\widehat{u}$ and the definition of Ψ, can be transformed into the relation $D_t\{\Psi[\widehat{u}(t,\cdot)]-\widehat{g}_0(t)\} = 0$. Due to the first consistency condition in (3.13) we obtain equation (2.15).

It remains to show (2.14). To do this, we first integrate (3.19) over $(0,t)$. From the consistency condition (3.15) we easily deduce (3.29). Arguments similar to those used at the beginning of the proof yield the relations $aQ_2\widehat{u}(t,\cdot) + a'Q_1\widehat{u}(t,\cdot) = \Phi[\mathrm{div}(a\nabla\widehat{u}(t,\cdot))] - D_y(aD_y\Phi[\widehat{u}(t,\cdot)])$ and

$$
\begin{aligned}
&h(t-s,y)(Q_2\widehat{u}(s,\cdot))(y) \;+\; D_yh(t-s,y)(Q_1\widehat{u}(s,\cdot))(y)\\
&\quad = \Phi[\mathrm{div}(h(t-s,\cdot)\nabla\widehat{u}(s,\cdot))](y) \;-\; D_y(h(t-s,y)D_y\Phi[\widehat{u}(s,\cdot)](y))\,.
\end{aligned}
$$

Using these relations in (3.29) and definitions (3.7) – (3.9) as well as equation (2.11) for $\widehat{u}$, we derive the following equation for $z(t,y) = \Phi[\widehat{u}(t,\cdot)](y)-\widehat{g}_1(t,y)$:

$$
D_tz(t,y) - D_y(a(y)D_yz(t,y)) - \int_0^t D_y(h(t-s,y)D_yz(s,y))ds = 0 \tag{3.32}
$$

for $t \in [0,T]$, $y \in [0,l]$. Due to the consistency conditions (3.13), (3.14) and the homogeneous boundary condition for $\widehat{u}$, the solution z to equation (3.32) satisfies homogeneous initial and boundary conditions. Hence, $z \equiv 0$, which implies (2.14). Proposition 3.1 is fully proved. $\square$

4. Diagonalization of the differentiated problem

The purpose of this section is to transform the subsystem (3.19), (3.20) into a fixed-point form for the pair (m, n) defined by

$$m(t) = h(t, 0), \quad n(t, y) = D_y h(t, y), \qquad t \in [0, T], \ y \in [0, l].$$ (4.1)

In order to perform such a transformation we need the following assumptions (cf. relations (3.7), (3.10) (3.21) and (2.9), (2.17)):

$$\left| (Q_1 \widehat{u}_0)(y) + q_1(0, y) \right| = \left| \int_{\Omega(y)} \lambda(x) D_y u_0(x) \, dx' \right| \geq \nu > 0, \quad y \in [0, l]$$ (4.2)

and

$$d_0 := \int_\Gamma \mu(x) D_n u_0(x) e^{-\int_0^y \kappa(z) dz} \, d\sigma - \int_\Omega \nabla \mu(x) \cdot \nabla u_0(x) e^{-\int_0^y \kappa(z) dz} dx \neq 0$$ (4.3)

with q_1, q_2, Q_1, Q_2 defined by (3.7), (3.8), (3.10), (3.11) and

$$\kappa(y) = \big((Q_2 \widehat{u}_0)(y) + q_2(0, y) \big) \big((Q_1 \widehat{u}_0)(y) + q_1(0, y) \big)^{-1}.$$ (4.4)

Remark. In (4.2) we implicitly make use of assumption (2.3). Indeed, if (2.3) did not hold, we should deduce $m_2(\Omega(0))m_2(\Omega(l)) = 0$ since Ω is a domain. Consequently, from definitions (3.7) and (3.10) it would follow $Q_1 u_0(jl) = q_1(0, jl)$ for some $j \in \{0, 1\}$, contradicting (4.2).

Remark. Because of assumption (3.1), the kernel λ cannot be of the form $\lambda(x) = \lambda_1(x') \lambda_2(y)$. Indeed, in this case we would have $\lambda_2 D_y \rho \in C([0, l])$ which would imply $\lambda_2(y) \to 0$ as $y \to kl$, $k = 0, 1$. Therefore, we would deduce $\int_{\Omega(y)} \lambda(x) D_y u_0(x) \, dx' \to 0$ as $y \to kl$, $k = 0, 1$ contradicting (4.2).

From (4.1) we deduce the decomposition $h(t, y) = m(t) + En(t, y)$ with

$$Ew(y) = \int_0^y w(z) dz.$$ (4.5)

Since assumption (4.2) holds, equation (3.19) writes as

$$n(t, y) + \kappa(y) En(t, y)$$
$$= \kappa(y) m(t) + \int_0^t \Big\{ R_1[n(t - s, \cdot), v(s, \cdot)](y)$$
$$+ R_1[n(t - s, \cdot), r(s, \cdot)](y) + m(t - s)[(P_1 v(s, \cdot))(y) + p_1(s, y)] \Big\} ds$$
$$+ (Q_3 v(t, \cdot))(y) + f_2(t, y), \quad y \in [0, l], \ t \in [0, T],$$ (4.6)

where κ is defined by (4.4),

$$R_1[w_1, w_2](y) = -[(Q_1\widehat{u}_0)(y) + q_1(0, y)]^{-1}$$
$$\times \{w_1(y)(Q_1 w_2)(y) + E w_1(y)(Q_2 w_2)(y)\} \tag{4.7}$$

and

$$r(t, y) = (D_t q_1(t, y), D_t q_2(t, y)) \tag{4.8}$$

$$(P_1 w)(y) = -[(Q_1\widehat{u}_0)(y) + q_1(0, y)]^{-1} (Q_2 w)(y) \tag{4.9}$$

$$p_1(t, y) = -[(Q_1\widehat{u}_0)(y) + q_1(0, y)]^{-1} D_t q_2(t, y) \tag{4.10}$$

$$(Q_3 w)(y) = -[(Q_1\widehat{u}_0)(y) + q_1(0, y)]^{-1} \{a(y)(Q_2 w)(y) + a'(y)(Q_1 w)(y)\} \tag{4.11}$$

$$f_2(t, y) = [(Q_1\widehat{u}_0)(y) + q_1(0, y)]^{-1} D_t f_1(t, y). \tag{4.12}$$

To solve equation (4.6) with respect to the left-hand side we note that, for any $\eta \in C([0, l])$, the unique solution to the integral equation $w(y) + \kappa(y) E w(y) = \eta(y)$, $y \in [0, l]$ is

$$w(y) = L_1\eta(y), \quad L_1 = I - \kappa(\cdot)L, \tag{4.13}$$

I being the identity operator and

$$L\eta(y) = \int_0^y e^{-\int_z^y \kappa(\tau)d\tau} \eta(z) dz. \tag{4.14}$$

Moreover, w satisfies the relation $E w(y) = L\eta(y)$, $y \in [0, l]$. Consequently, from (4.6) we get

$$n(t, y) = -L_1\kappa(y)m(t)$$
$$+ \int_0^t \Big\{ R_2[n(t - s, \cdot), v(s, \cdot)](y) + R_2[n(t - s, \cdot), r(s, \cdot)](y)$$
$$+ m(t - s)[(P_2 v(s, \cdot))(y) + p_2(s, y)] \Big\} ds$$
$$+ (Q_4 v(t, \cdot))(y) + f_3(t, y), \quad y \in [0, l], \ t \in [0, T], \tag{4.15}$$

and

$$E n(t, y) = -L\kappa(y)m(t)$$
$$+ \int_0^t \Big\{ R_3[n(t - s, \cdot), v(s, \cdot)](y) + R_3[n(t - s, \cdot), r(s, \cdot)](y)$$
$$+ m(t - s)[(P_3 v(s, \cdot))(y) + p_3(s, y)] \Big\} ds$$
$$+ (Q_5 v(t, \cdot))(y) + f_4(t, y), \quad y \in [0, l], \ t \in [0, T],$$
$$\tag{4.16}$$

where

$$R_2 = L_1 R_1, \quad P_2 = L_1 P_1, \quad p_2 = L_1 p_1, \quad Q_4 = L_1 Q_3, \quad f_3 = L_1 f_2 \left.\right\}$$
$$R_3 = LR_1, \quad P_3 = LP_1, \quad p_3 = Lp_1, \quad Q_5 = LQ_3, \quad f_4 = Lf_2 \,. \left.\right\} \tag{4.17}$$

Next let us deal with equation (3.20). Since $h = m + En$, we can rewrite (3.20) in the form

$$\overline{L}(m(t) + En(t, \cdot))$$
$$= \int_0^t \Big\{ R_1^0[n(t - s, \cdot), v(s, \cdot)] + R_1^0[n(t - s, \cdot), w_\Gamma(s, \cdot)] \tag{4.18}$$
$$+ m(t - s)[P_1^0 v(s, \cdot) + p_1^0(s)] \Big\} ds + Q_1^0 v(t, \cdot) + f_1^0(t), \quad t \in [0, T],$$

where

$$\overline{L}w = \int_\Gamma \mu(x) D_n u_0(x) w(y) \, d\sigma - \int_\Omega \nabla \mu(x) \cdot \nabla u_0(x) w(y) dx \tag{4.19}$$

$$R_1^0[w_1, w_2] = -\int_\Gamma E w_1(y) \mu(x) D_n w_2(x) \, d\sigma \tag{4.20}$$

$$w_\Gamma(t, x) = D_t u_\Gamma(t, x) \tag{4.21}$$

$$P_1^0 w = -\int_\Gamma \mu(x) D_n w(x) \, d\sigma + \int_\Omega \nabla \mu(x) \cdot \nabla w(x) dx \tag{4.22}$$

$$p_1^0(t) = -\int_\Gamma \mu(x) D_n D_t u_\Gamma(t, x) \, d\sigma + \int_\Omega \nabla \mu(x) \cdot \nabla D_t u_\Gamma(t, x) dx \tag{4.23}$$

$$Q_1^0 w = -\int_\Gamma a(y) \mu(x) D_n w(x) \, d\sigma + \int_\Omega a(y) \nabla \mu(x) \cdot \nabla w(x) dx \tag{4.24}$$

$$f_1^0(t) = \widehat{g}_0''(t) - \Psi[D_t \widehat{f}(t, \cdot)]. \tag{4.25}$$

Our next step consists in replacing the expression for En from (4.16) into the left-hand side of (4.18) and express m in terms of integrals containing n, m, v. To this end we use assumption (4.3). We note that

$$d_0 = \overline{L}(1 - L\kappa(\cdot)). \tag{4.26}$$

Hence, inserting (4.16) into the left-hand side of (4.18) and taking relation (4.26) for d_0 into account, we arrive at the following equation, where $t \in [0, T]$:

$$m(t) = \int_0^t \Big\{ R^0[n(t - s, \cdot), v(s, \cdot)] + R^0[n(t - s, \cdot), r(s, \cdot)]$$
$$+ m(t - s)[P^0 v(s, \cdot) + p^0(s)] \Big\} ds + Q^0 v(t, \cdot) + f^0(t), \tag{4.27}$$

where

$$R^0[w_1, w_2] = d_0^{-1} \left\{ R_1^0[w_1, w_2] - \overline{L} R_3[w_1, w_2](\cdot) \right\} \qquad (4.28)$$

$$P^0 w = d_0^{-1} \left\{ P_1^0 w - \overline{L}(P_3 w)(\cdot) \right\} \qquad (4.29)$$

$$p^0(t) = d_0^{-1} \left\{ p_1^0(t) - \overline{L} p_3(t, \cdot) \right\} \qquad (4.30)$$

$$Q^0 w = d_0^{-1} \left\{ Q_1^0 w - \overline{L}(Q_5 w)(\cdot) \right\} \qquad (4.31)$$

$$f^0(t) = d_0^{-1} \left\{ f_1^0(t) - \overline{L} f_4(t, \cdot) \right\}. \qquad (4.32)$$

Finally, substituting m from (4.27) into the first addend in the right-hand side of (4.15), we derive the equation for n:

$$\begin{aligned}
n(t, y) = \int_0^t \Big\{ &R^1[n(t - s, \cdot), v(s, \cdot)](y) + (\overline{R}^1 n(t - s, \cdot))(s, y) \\
&+ m(t - s) \left[(P^1 v(s, \cdot))(y) + p^1(s, y) \right] \Big\} ds \\
&+ (Q^1 v(t, \cdot))(y) + f^1(t, y), \quad t \in [0, T], \ y \in [0, l],
\end{aligned} \qquad (4.33)$$

where

$$R^1[w_1, w_2](y) = R_2[w_1, w_2](y) - R^0[w_1, w_2] L_1 \kappa(y) \qquad (4.34)$$

$$(P^1[w])(y) = (P_2[w])(y) - P^0 w L_1 \kappa(y) \qquad (4.35)$$

$$p^1(t, y) = p_2(t, y) - p^0(t) L_1 \kappa(y) \qquad (4.36)$$

$$(Q^1[w])(y) = (Q_4[w])(y) - Q^0 w L_1 \kappa(y) \qquad (4.37)$$

$$f^1(t, y) = f_3(t, y) - f^0(t) L_1 \kappa(y). \qquad (4.38)$$

The parabolic equation (3.17) can be rewritten in the form

$$\begin{aligned}
D_t v(t, x) - &A v(t, x) \\
= \int_0^t \Big\{ &R[n(t - s, \cdot), v(s, \cdot)](x) + (\overline{R} n(t - s, \cdot))(s, x) \\
&+ m(t - s) \left[(B v(s, \cdot))(x) + b(s, x) \right] \Big\} ds \\
&+ m(t) \psi_1(x) + (S n(t, \cdot))(x) + g(t, x), \quad t \in [0, T], \ x \in \Omega,
\end{aligned} \qquad (4.39)$$

where

$$R[w_1, w_2](x) = E w_1(y) B w_2(x) + w_1(y) D_y w_2(x) \qquad (4.40)$$

$$(\overline{R} w)(t, x) = E w(y) b(t, x) + w(y) D_y D_t u_\Gamma(t, x) \qquad (4.41)$$

$$(S w)(x) = E w(y) \psi_1(x) + w(y) \psi_2(x). \qquad (4.42)$$

Summing up, we have proved the following proposition, where the spaces X, X_2 and Y are defined at the beginning of Section 3.

Proposition 4.1. *Let assumptions* (3.1) − (3.4) *and* (4.2), (4.3) *be fulfilled. Then the following assertions hold:*

(i) *If* $(v, h) \in \{C^{1+\beta}([0, T]; X) \cap C^{\beta}([0, T]; X_2)\} \times C^{\beta}([0, T]; Y_1)$ *solves problem* (3.17) − (3.20), *then* (v, m, n) *with* $m(t) = h(t, 0)$ *and* $n(t, y) = D_y h(t, y)$ *solves problem* (4.39), (3.18), (4.27), (4.33).

(ii) *Conversely, if* $(v, m, n) \in \{C^{1+\beta}([0, T]; X) \cap C^{\beta}([0, T]; X_2)\} \times C^{\beta}[0, T] \times C^{\beta}([0, T]; Y)$ *solves problem* (4.39), (3.18), (4.27), (4.33), *then* (v, h) *with* $h(t, y) = m(t) + \int_0^y n(t, z)dz$ *solves problem* (3.17) − (3.20).

Let us introduce the following intermediate space between X and X_2: $X_1 = C_0^{1+2\varepsilon}(\overline{\Omega}) = \{w \in C^{1+2\varepsilon}(\overline{\Omega}) : w|_{\Gamma} = 0\}$ where ε is some number from the interval $[0, \frac{1}{2})$. According to the definitions of operators and functions introduced in (4.20) − (4.25), (4.28) − (4.32), (4.40) − (4.42) and (3.5), (3.6), we immediately deduce the following proposition.

Proposition 4.2. *Let assumptions of Proposition* 4.1 *be fulfilled. Then:*

$$R, \ R^1, \ R^0 \ are \ continuous \ bilinear \ operators \ such \ that$$
$$R : Y \times X_2 \to X, \ R^1 : Y \times X_1 \to Y, \ R^0 : Y \times X_1 \to \mathbb{R} \tag{4.43}$$

$$B \in \mathcal{L}(X_2; X), \ P^1, Q^1 \in \mathcal{L}(X_1; Y)$$
$$P^0, Q^0 \in \mathcal{L}(X_1; \mathbb{R}), \ S \in \mathcal{L}(Y; X) \tag{4.44}$$

$$b \in C([0, T]; X), \ p^0 \in C[0, T], \ p^1 \in C([0, T]; Y)$$
$$g \in C^{\beta}([0, T]; X), \ f^0 \in C^{\beta}[0, T], \ f^1 \in C^{\beta}([0, T]; Y) \tag{4.45}$$

$$v_0 \in X_2 \ \text{and} \ \psi_1 \in X. \tag{4.46}$$

5. Abstract formulation of the identification problem

Let X and A be a Banach space and a linear closed operator in X, respectively. We define $X_2 = D(A)$ and endow it with the graph-norm $\|w\|_{X_2} = \|w\|_X + \|Aw\|_X$. We assume that

(i) there exists a $\theta \in (\frac{\pi}{2}, \pi)$ such that the resolvent set of A contains 0 and the open sector $\Sigma_{\theta} = \{\xi \in \mathbb{C} \setminus \{0\} : |\arg \xi| < \theta\}$

(ii) there exists an $M > 0$ such that $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M|\lambda|^{-1}$ for any $\lambda \in \Sigma_{\theta}$. $\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\}$ (5.1)

Then (see [16]) operator A generates an analytic semigroup in X, $\{e^{tA}\}_{t \geq 0}$, $e^{0A} = I$, possibly discontinuous at $t = 0$.

We will make use of the following interpolation spaces $D_A(\alpha, \infty)$ related to operator A:

$$D_A(\alpha, \infty) = \{w \in X \ : \ \eta \to \|\eta^{1-\alpha} A e^{\eta A} w\| \in L^{\infty}(0, 1)\},$$

where $0 \leq \alpha \leq 1$. In particular, $D_A(0, \infty) = X$ and $D_A(1, \infty) = X_2$. The spaces $D_A(\alpha, \infty)$ for $0 < \alpha < 1$ are endowed with the norms $\|w\|_{D_A(\alpha,\infty)} = \|w\|_X + \sup_{0<\eta<1} \left\| \eta^{1-\alpha} A e^{\eta A} w \right\|_X$.

Let β and ε be two given real numbers such that

$$0 < \beta < \tfrac{1}{2}, \quad 0 < \varepsilon < \tfrac{1}{2} - \beta, \tag{5.2}$$

and let

$$X_1 = D_A\left(\tfrac{1}{2} + \varepsilon, \infty\right). \tag{5.3}$$

Moreover, assume that we are given a Banach space Y, the continuous bilinear operators

$$R : Y \times X_2 \to X, \quad R^1 : Y \times X_1 \to Y, \quad R^0 : Y \times X_1 \to \mathbb{R}, \tag{5.4}$$

the linear operators

$$B \in \mathcal{L}(X_2; X), \ P^1, Q^1 \in \mathcal{L}(X_1; Y), \ P^0, Q^0 \in \mathcal{L}(X_1; \mathbb{R}), \ S \in \mathcal{L}(Y; X), \tag{5.5}$$

the functions

$$\left. \begin{array}{lll} b \in C([0,T]; X), & p^0 \in C([0,T]), & p^1 \in C([0,T]; Y) \\ g \in C^\beta([0,T]; X), & f^0 \in C^\beta[0,T], & f^1 \in C^\beta([0,T]; Y) \end{array} \right\} \tag{5.6}$$

and the elements

$$v_0 \in X_2, \quad \psi_1 \in X. \tag{5.7}$$

We can now formulate the following abstract problem: *find* $v : [0,T] \to X_2$, $m : [0,T] \to \mathbb{R}$ and $n : [0,T] \to Y$ satisfying the equations

$$v'(t) - Av(t) = \int_0^t \Big\{ R[n(t-s), v(s)] + (\overline{R}n(t-s))(s) \tag{5.8}$$
$$+ m(t-s)[Bv(s) + b(s)] \Big\} ds + m(t)\psi_1 + Sn(t) + g(t)$$

$$v(0) = v_0 \tag{5.9}$$

$$m(t) = \int_0^t \Big\{ R^0[n(t-s), v(s)] + (\overline{R}^0 n(t-s))(s) \tag{5.10}$$
$$+ m(t-s)[P^0 v(s) + p^0(s)] \Big\} ds + Q^0 v(t) + f^0(t)$$

$$n(t) = \int_0^t \Big\{ R^1[n(t-s), v(s)] + (\overline{R}^1 n(t-s))(s) \tag{5.11}$$
$$+ m(t-s)[P^1 v(s) + p^1(s)] \Big\} ds + Q^1 v(t) + f^1(t).$$

We note that from (5.10), (5.11), in view of the initial condition (5.9), we get the following explicit formulae for $m(0)$ and $n(0)$:

$$m(0) = Q^0 v_0 + f^0(0), \quad n(0) = Q^1 v_0 + f^1(0). \tag{5.12}$$

Our aim is to transform system (5.8) – (5.11) into a fixed-point form. To this end we need the following well-known theorem concerning parabolic equations [18].

Theorem 5.1. *If $\phi \in C^\beta([0, T]; X)$, $v_0 \in X_2$ and $Av_0 + f(0) \in D_A(\beta, \infty)$ then the Cauchy problem $v'(t) - Av(t) = \phi(t)$, $t \in [0, T]$, $v(0) = v_0$ admits a unique solution v in $C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; X_2)$ represented by the formula $v(t) = \int_0^t e^{(t-s)A}\phi(s)ds + e^{tA}v_0$. Moreover, if $\phi(0) = 0$, $e^{tA} * \phi$ satisfies the estimate*

$$\|e^{tA} * \phi\|_{C^\beta([0,T];X_2)} \leq C\|\phi\|_{C^\beta([0,T];X)}, \tag{5.13}$$

the positive constant C being independent of ϕ.

In order to apply Theorem 5.1 to the Cauchy problem (5.8), (5.9) we have to introduce the further assumption

$$Av_0 + g(0) + (Q^0 v_0 + f^0(0))\psi_1 + S(Q^1 v_0 + f^1(0)) \in D_A(\beta, \infty). \tag{5.14}$$

The relations (5.12) and (5.14) imply

$$Av_0 + g(0) + m(0)\psi_1 + Sn(0) \in D_A(\beta, \infty). \tag{5.15}$$

Now we observe that, due to Theorem 5.1 and property (5.14), the Cauchy problem (5.8), (5.9) for $(v, m, n) \in \{C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; X_2)\} \times C^\beta([0, T]) \times C^\beta([0, T]; Y)$ is equivalent to the operator equation

$$\begin{aligned} v(t) = &\int_0^t e^{(t-s)A}K[v, m, n](s)ds + \int_0^t e^{(t-s)A}\{m(s)\psi_1 + Sn(s)\}ds \\ &+ \int_0^t e^{(t-s)A}g(s)ds + e^{tA}v_0, \quad t \in [0, T], \end{aligned} \tag{5.16}$$

where

$$K[v, m, n](t) = \int_0^t \left\{R[n(t-s), v(s)] + (\overline{R}n(t-s))(s) + m(t-s)[Bv(s) + b(s)]\right\}ds.$$

Next we substitute the right-hand side in (5.16) for v into the terms $Q^0 v$ and $Q^1 v$ of the equations (5.10) and (5.11), respectively. We obtain the fixed-point equations

$$m(t) = N^0[v, m, n](t) + m_1(t), \quad t \in [0, T], \tag{5.17}$$

$$n(t) = N^1[v, m, n](t) + n_1(t), \quad t \in [0, T], \tag{5.18}$$

where

$$N^0[v, m, n](t) = K^0[v, m, n](t) + Q^0\left[\int_0^t e^{(t-s)A} K[v, m, n](s)ds\right.$$
$$\left. + \int_0^t e^{(t-s)A}\left\{m(s)\psi_1 + Sn(s)\right\}ds\right] \tag{5.19}$$

$$K^0[v, m, n](t) = \int_0^t \left\{R^0[n(t-s), v(s)] + (\overline{R}^0 n(t-s))(s)\right.$$
$$\left. + m(t-s)\left[P^0 v(s) + p^0(s)\right]\right\}ds \tag{5.20}$$

$$m_1(t) = Q^0\left[\int_0^t e^{(t-s)A} g(s)ds + e^{tA}v_0\right] + f^0(t) \tag{5.21}$$

$$N^1[v, m, n](t) = K^1[v, m, n](t) + Q^1\left[\int_0^t e^{(t-s)A} K[v, m, n](s)ds\right.$$
$$\left. + \int_0^t e^{(t-s)A}\left\{m(s)\psi_1 + Sn(s)\right\}ds\right] \tag{5.22}$$

$$K^1[v, m, n](t) = \int_0^t \left\{R^1[n(t-s), v(s)] + (\overline{R}^1 n(t-s))(s)\right.$$
$$\left. + m(t-s)\left[P^1 v(s) + p^1(s)\right]\right\}ds \tag{5.23}$$

$$n_1(t) = Q^1\left[\int_0^t e^{(t-s)A} g(s)ds + e^{tA}v_0\right] + f^1(t). \tag{5.24}$$

Finally, we substitute the right-hand sides in (5.17) and (5.18) into the term $\int_0^t e^{(t-s)A}\left\{m(s)\psi_1 + Sn(s)\right\}ds$ in equation (5.16). We derive the relation

$$v(t) = N[v, m, n](t) + v_1(t), \quad t \in [0, T], \tag{5.25}$$

where

$$N[v, m, n](t) = \int_0^t e^{(t-s)A} K[v, m, n](s)\,ds$$
$$+ \int_0^t e^{(t-s)A}\left[N^0[v, m, n](s)\psi_1 + SN^1[v, m, n](s)\right]ds, \tag{5.26}$$

and

$$v_1(t) = \int_0^t e^{(t-s)A}\left[m_1(s)\psi_1 + Sn_1(s)\right]ds + \int_0^t e^{(t-s)A} g(s)ds + e^{tA}v_0. \tag{5.27}$$

Conversely, it is an easy task to show that any solution (v, m, n) to (5.17), (5.18), (5.25), with the stated regularity, solves (5.8) – (5.11). Consequently, we have proved the following proposition.

Proposition 5.2. *Let the assumptions* (5.1) – (5.7) *and* (5.14) *be fulfilled. Then system* (5.8) – (5.11) *for* $(v, m, n) \in \left\{C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; X_2)\right\} \times C^\beta([0, T]) \times C^\beta([0, T]; Y)$ *is equivalent to the system* (5.17), (5.18), (5.25).

6. Preliminary lemmas

Let us introduce some notation. With any Banach space Z we associate the function Banach space $C^\beta([0,T];Z)$, $0<\beta<1$, normed by $\|w\|_{\beta,0,Z} = \|w\|_{0,0,Z} + [w]_{\beta,0,Z}$, $w \in C^\beta([0,T];Z)$ where $\|w\|_{0,0,Z} = \max_{0\le t\le T}\|w(t)\|_Z$ and $[w]_{\beta,0,Z} = \sup_{0\le\tau<t\le T}(t-\tau)^{-\beta}\|w(t)-w(\tau)\|_Z$. Moreover, we introduce in $C^\beta([0,T];Z)$ the following weighted norm, equivalent to the previous one and depending on a non-negative parameter γ: $\|w\|_{\beta,\gamma,Z} = \|e^{-\gamma\cdot}w\|_{\beta,0,Z} = \|w\|_{0,\gamma,Z} + [w]_{\beta,\gamma,Z}$ where $\|w\|_{0,\gamma,Z} = \|e^{-\gamma\cdot}w\|_{0,0,Z}$, $[w]_{\beta,\gamma,Z} = [e^{-\gamma\cdot}w]_{\beta,0,Z}$.

   In the sequel we will denote by C any non-negative constant, which may vary from line to line.

Lemma 6.1. *Let $0 \le \beta < 1$ and let Z, Z_1, Z_2 be three Banach spaces. Moreover, let M be a continuous bilinear operator from $Z_1 \times Z_2$ to $C([0,T];Z)$. Then for any $w_1 \in C^\beta([0,T];Z_1)$, $w_2 \in C^\beta([0,T];Z_2)$ the following estimates hold:*

$$\left\|\int_0^\cdot M[w_1(\cdot - s), w_2(s)]ds\right\|_{\beta,\gamma,Z} \le C \begin{cases} (T + T^{1-\beta})\|w_1\|_{\beta,\gamma,Z_1}\|w_2\|_{0,\gamma,Z_2} \\ \gamma^{\beta-1}(1 + \gamma^{-\beta})\|w_1\|_{\beta,\gamma,Z_1}\|w_2\|_{0,0,Z_2}. \end{cases} \tag{6.1}$$

Proof. Using the simple relations $\int_0^t ds \le T$ and $\int_0^t e^{-\gamma s}ds \le \gamma^{-1}$, $0 \le t \le T$, we easily obtain the estimate

$$e^{-\gamma t}\int_0^t \|M[w_1(t-s), w_2(s)]\|_Z \, ds$$
$$\le C\int_0^t e^{-\gamma(t-s)}\|w_1(t-s)\|_{Z_1}e^{-\gamma s}\|w_2(s)\|_{Z_2} \, ds$$
$$\le C\|w_1\|_{0,\gamma,Z_1}\int_0^t e^{-\gamma s}\|w_2(s)\|_{Z_2} \, ds \tag{6.2}$$
$$\le C\begin{cases} T\|w_1\|_{0,\gamma,Z_1}\|w_2\|_{0,\gamma,Z_2} \\ \gamma^{-1}\|w_1\|_{0,\gamma,Z_1}\|w_2\|_{0,0,Z_2}. \end{cases}$$

Further, using the following relations, where $0 \le \tau < t \le T$ and $0 < \beta < 1$,

$$(t-\tau)^{-\beta}\int_\tau^t ds \le T^{1-\beta}$$
$$(t-\tau)^{-\beta}\int_\tau^t e^{-\gamma s}ds \le (t-\tau)^{-\beta}\left(\int_\tau^t ds\right)^\beta\left(\int_\tau^{+\infty} e^{-\gamma(1-\beta)^{-1}s}ds\right)^{1-\beta} \le c\gamma^{\beta-1}, \tag{6.3}$$

we easily get the estimate

$$(t-\tau)^{-\beta}\left\|e^{-\gamma t}\int_0^t M[w_1(t-s), w_2] \, ds - e^{-\gamma\tau}\int_0^\tau M[w_1(\tau-s), w_2(s)] \, ds\right\|_Z$$
$$\le C\|w_1\|_{\beta,\gamma,Z_1}\left\{(t-\tau)^{-\beta}\int_\tau^t e^{-\gamma s}\|w_2(s)\|_{Z_2} \, ds + \int_0^\tau e^{-\gamma s}\|w_2(s)\|_{Z_2} \, ds\right\}.$$

This, in turn, implies

$$\left[\int_0^{\cdot} M[w_1(\cdot - s), w_2(s)]ds\right]_{\beta,\gamma,Z} \leq C \begin{cases} (T + T^{1-\beta}) \|w_1\|_{\beta,\gamma,Z_1} \|w_2\|_{0,\gamma,Z_2} \\ \gamma^{\beta-1}(1 + \gamma^{-\beta}) \|w_1\|_{\beta,\gamma,Z_1} \|w_2\|_{0,0,Z_2}. \end{cases} \quad (6.4)$$

The inequalities (6.2) and (6.4) yield (6.1). $\qquad\qquad\square$

Lemma 6.2. *Let Z be a Banach space and let $w \in C^{\alpha}([0,T]; Z)$, $0 \leq \alpha < 1$. Then*

$$\|w\|_{\alpha,\gamma,Z} \leq C(1 + \gamma^{\alpha})\|w\|_{\alpha,0,Z}. \quad (6.5)$$

If, in addition $w(0) = 0$ and $\alpha' \in [0, \alpha]$, then for any positive γ

$$\|w\|_{\alpha',\gamma,Z} \leq C\gamma^{\alpha'-\alpha}(1 + \gamma^{-\alpha'})[w]_{\alpha,0,Z}. \quad (6.6)$$

Proof. Note that

$$\|w\|_{\alpha,\gamma,Z} \leq \max_{0 \leq t \leq T} \left\|e^{-\gamma t}w(t)\right\|_Z + \sup_{0 \leq \tau < t \leq T} (t-\tau)^{-\alpha} \left\|e^{-\gamma t}[w(t) - w(\tau)]\right\|_Z$$
$$+ \sup_{0 \leq \tau < t \leq T} (t-\tau)^{-\alpha} \left\|\left[e^{-\gamma t} - e^{-\gamma\tau}\right]w(\tau)\right\|_Z. \quad (6.7)$$

Since $0 < e^{-\gamma t} \leq 1$ and $(t-\tau)^{-\alpha}|e^{-\gamma t} - e^{-\gamma\tau}| = \gamma(t-\tau)^{-\alpha}\int_\tau^t e^{-\gamma s}ds \leq C\gamma^\alpha$, from (6.7) we immediately obtain $\|w\|_{\alpha,\gamma,Z} \leq \|w\|_{0,0,Z} + [w]_{\alpha,0,Z} + C\gamma^\alpha\|w\|_{0,0,Z} \leq C(1 + \gamma^\alpha)\|w\|_{\alpha,0,Z}$. This proves (6.5).

Let now $w(0) = 0$. Then replacing α with α' in (6.7), we derive

$$\|w\|_{\alpha',\gamma,Z} \leq \left\{\gamma^{-\alpha} \sup_{0 \leq \zeta < +\infty} \zeta^\alpha e^{-\zeta} + \gamma^{\alpha'-\alpha}\left[\sup_{0 \leq \xi < \zeta < +\infty} (\zeta - \xi)^{\alpha-\alpha'} e^{-\zeta}\right.\right.$$
$$\left.\left.+ \sup_{0 \leq \xi < \zeta < +\infty} (\zeta - \xi)^{-\alpha'}\xi^\alpha\left(e^{-\xi} - e^{-\zeta}\right)\right]\right\}[w]_{\alpha,0,Z}. \quad (6.8)$$

Here we have denoted $\gamma\tau$ and γt by ξ and ζ, respectively. Since $\zeta^\alpha e^{-\zeta} \leq C$, $(\zeta - \xi)^{\alpha-\alpha'} e^{-\zeta} \leq \zeta^{\alpha-\alpha'} e^{-\zeta} \leq C$ and $(\zeta - \xi)^{-\alpha'}\xi^\alpha(e^{-\xi} - e^{-\zeta}) = \xi^\alpha e^{-\xi}(\zeta - \xi)^{-\alpha'}(1 - e^{-(\zeta-\xi)} \leq C$, from (6.8) we derive (6.6). $\qquad\square$

Lemma 6.3. *Let $0 \leq \alpha < 1$, $0 \leq \beta < 1 - \alpha$ and let A satisfy property (5.1). Then, for any $f \in C([0,T]; X)$ and $\gamma \in [1, +\infty)$,*

$$\left\|\int_0^{\cdot} e^{(\cdot-s)A}f(s)ds\right\|_{\beta,\gamma,D_A(\alpha,\infty)} \leq C\left[\gamma^{-1+\alpha} + (\gamma - 1)^{-1+\alpha+\beta}\right]\|f\|_{0,\gamma,X}, \quad (6.9)$$

the positive constant C being independent of γ.

Proof. First we recall that from the first part of the proof of Proposition 2.2.7 in [16], with β being replaced with α, we get the estimate

$$\|t^{k+\alpha}A^k e^{tA}\|_{\mathcal{L}(X;D_A(\alpha,\infty))} \leq C_k(\alpha) \quad \forall t \in (0,+\infty), \ k \in \mathbb{N}, \tag{6.10}$$

since the semigroup $\{e^{tA}\}_{t\geq 0}$ is *uniformly bounded*. Hence, we deduce the estimates, for all $t \in (0,+\infty)$,

$$\|t^\alpha e^{t(A-\gamma I)}\|_{\mathcal{L}(X;D_A(\alpha,\infty))} \leq C_0(\alpha)e^{-\gamma t} \tag{6.11}$$

$$\|t^{1+\alpha}(A-\gamma I)e^{t(A-\gamma I)}\|_{\mathcal{L}(X;D_A(\alpha,\infty))} \leq C_1(\alpha)e^{-\gamma t} + \gamma t C_0(\alpha)e^{-\gamma t}$$
$$\leq C(\alpha)e^{-(\gamma-1)t} \quad \forall \gamma \in [1,+\infty) \,. \tag{6.12}$$

From the previous inequalities we easily deduce the following estimates, where $0 \leq \tau < t \leq T$:

$$\left\|\int_0^t e^{(t-s)(A-\gamma I)}e^{-\gamma s}f(s)ds\right\|_{D_A(\alpha,\infty)} \leq \|f\|_{0,\gamma,X}\int_0^t \|e^{s(A-\gamma I)}\|_{\mathcal{L}(X;D_A(\alpha,\infty))}ds$$
$$\leq C\|f\|_{0,\gamma,X}\int_0^{+\infty} s^{-\alpha}e^{-(\gamma-1)s}ds \tag{6.13}$$
$$= C(\gamma-1)^{-1+\alpha}\|f\|_{0,\gamma,X}\,,$$

and

$$\left\|\int_0^t e^{(t-s)(A-\gamma I)}e^{-\gamma s}f(s)ds - \int_0^\tau e^{(\tau-s)(A-\gamma I)}e^{-\gamma s}f(s)ds\right\|_{D_A(\alpha,\infty)}$$
$$\leq \|f\|_{0,\gamma,X}\left\{\int_\tau^t \|e^{(t-s)(A-\gamma I)}\|_{\mathcal{L}(X;D_A(\alpha,\infty))}ds\right.$$
$$\left. + \int_0^\tau \|e^{(t-s)(A-\gamma I)} - e^{(\tau-s)(A-\gamma I)}\|_{\mathcal{L}(X;D_A(\alpha,\infty))}ds\right\} \tag{6.14}$$
$$\leq C(\gamma-1)^{-1+\alpha+\beta}(t-\tau)^\beta\|f\|_{0,\gamma,X}\,.$$

We have thus proved that $e^{tA}*f \in C^\beta([0,T];D_A(\alpha,\infty))$ for any $f \in C^\beta([0,T];X)$ and estimate (6.8) holds. $\qquad\square$

Lemma 6.4. *Let* $0 \leq \beta < 1$ *and* A *satisfy* (5.1). *Moreover, let* $f \in C^\beta([0,T];X)$ *and* $f(0) = 0$. *Then*

$$\left\|\int_0^\cdot e^{(\cdot-s)A}f(s)ds\right\|_{\beta,\gamma,X_2} \leq C\left(1+\gamma^{-1}\right)[f]_{\beta,\gamma,X}\,. \tag{6.15}$$

Proof. We have

$$\left\|\int_0^\cdot e^{(\cdot-s)A}f(s)ds\right\|_{\beta,\gamma,X_2} \leq \left\|\int_0^\cdot e^{(\cdot-s)A}f(s)ds\right\|_{\beta,\gamma,X} + \left\|A\int_0^\cdot e^{(\cdot-s)A}f(s)ds\right\|_{\beta,\gamma,X}. \tag{6.16}$$

For the first addend in the right-hand side of (6.16) we derive the estimate

$$
\begin{aligned}
\left\| \int_0^{\cdot} e^{(\cdot-s)A} f(s) ds \right\|_{\beta,\gamma,X} & \\
\leq \max_{0 \leq t \leq T} & \left\| \int_0^t e^{-\gamma(t-s)} e^{(t-s)A} e^{-\gamma s} f(s) ds \right\|_X \\
+ \sup_{0 \leq \tau < t \leq T} & (t-\tau)^{-\beta} \Bigg\{ \left\| \int_\tau^t e^{-\gamma s} e^{sA} e^{-\gamma(t-s)} f(t-s) ds \right\|_X \\
+ & \left\| \int_0^\tau e^{-\gamma s} e^{sA} \left[e^{-\gamma(t-s)} f(t-s) - e^{-\gamma(\tau-s)} f(\tau-s) \right] ds \right\|_X \Bigg\}.
\end{aligned}
\tag{6.17}
$$

Due to $\|e^{tA}\|_{\mathcal{L}(X)} \leq C$ for $0 \leq t \leq T$ and the relations, for $\tau \leq s \leq t$,

$$
\left\| e^{-\gamma(t-s)} f(t-s) \right\|_X \leq (t-s)^\beta [f]_{\beta,\gamma,X} \leq (t-\tau)^\beta [f]_{\beta,\gamma,X}
$$
$$
\left\| e^{-\gamma(t-s)} f(t-s) - e^{-\gamma(\tau-s)} f(\tau-s) \right\|_X \leq (t-\tau)^\beta [f]_{\beta,\gamma,X}
$$

following from $f \in C^\beta([0,T];X)$ and $f(0) = 0$, from (6.17) we obtain

$$
\begin{aligned}
\left\| \int_0^{\cdot} e^{(\cdot-s)A} f(s) ds \right\|_{\beta,\gamma,X} \leq C \Bigg\{ & T^\beta \max_{0 \leq t \leq T} \int_0^t e^{-\gamma(t-s)} ds \\
+ \sup_{0 \leq \tau < t \leq T} & \left(T^\beta \int_\tau^t e^{-\gamma s} ds + \int_0^\tau e^{-\gamma s} ds \right) \Bigg\} [f]_{\beta,\gamma,X} \\
\leq & \ C \gamma^{-1} [f]_{\beta,\gamma,X}.
\end{aligned}
\tag{6.18}
$$

In order to estimate the second addend in (6.16) we will make use of the following relation (see [5, Theorem 4.1 (ii)]):

$$
\left\| A \int_0^{\cdot} e^{(\cdot-s)A} f(s) ds \right\|_{\beta,0,X} \leq \overline{C}(\beta,\theta,T) M [f]_{\beta,0,X},
\tag{6.19}
$$

where θ and M are the constants in (5.1) and $\overline{C}$ is a positive function. Let us denote $A_\gamma = A - \gamma I$. It is easy to check that if A satisfies (5.1) with the parameters θ and M, then the operator A_γ for $\gamma \geq 0$ satisfies (5.1) with the parameters $\theta_1 = \theta$ and $M_1 = M[\cos(\theta - \frac{\pi}{2})]^{-1}$. Hence, using (6.19) we obtain

$$
\begin{aligned}
\left\| (A - \gamma I) \int_0^{\cdot} e^{(\cdot-s)A} f(s) ds \right\|_{\beta,\gamma,X} &= \left\| A_\gamma \int_0^{\cdot} e^{(\cdot-s)A_\gamma} e^{-\gamma s} f(s) ds \right\|_{\beta,0,X} \\
&\leq \overline{C}(\beta,\theta_1,T) M_1 [e^{-\gamma \cdot} f]_{\beta,0,X} \\
&= C [f]_{\beta,\gamma,X}.
\end{aligned}
\tag{6.20}
$$

Combining (6.20) with (6.19) we have

$$\left\| A \int_0^{\cdot} e^{(\cdot-s)A} f(s) ds \right\|_{\beta,\gamma,X} \le C \left[f \right]_{\beta,\gamma,X} . \tag{6.21}$$

Finally, from (6.16), (6.19) and (6.21) we derive the estimate (6.15). $\qquad\square$

7. Estimates of basic operators

We start by proving the following lemma.

Lemma 7.1. *Let assumptions* (5.1) $-$ (5.3), (5.5) $-$ (5.7) *and* (5.14) *hold. Then the functions* m_1, n_1 *and* v_1 *defined by* (5.21), (5.24) *and* (5.27), *respectively, satisfy* $m_1 \in C^{\beta}[0,T]$, $n_1 \in C^{\beta}([0,T];Y)$ *and* $v_1 \in C^{1+\beta}([0,T];X) \cap C^{\beta}([0,T];X_2)$.

Proof. First we prove $m_1 \in C^{\beta}([0,T])$, $n_1 \in C^{\beta}([0,T];Y)$. Since, by assumption, $f^0 \in C^{\beta}([0,T])$, $Q^0 \in \mathcal{L}(X_1,\mathbb{R})$ and $f^1 \in C^{\beta}([0,T];Y)$, $Q^1 \in \mathcal{L}(X_1,Y)$, we have to show that

$$\int_0^{\cdot} e^{(\cdot-s)A} g(s) ds + e^{\cdot A} v_0 \in C^{\beta}([0,T];X_1) . \tag{7.1}$$

Since $g \in C^{\beta}([0,T];X) \subset C([0,T];X)$, Lemma 6.3 with $\alpha = \frac{1}{2} + \varepsilon$ implies $\int_0^{\cdot} e^{(\cdot-s)A} g(s) ds \in C^{\beta}\big([0,T]; D_A\big(\frac{1}{2} + \varepsilon, \infty\big)\big) = C^{\beta}([0,T];X_1)$. To show (7.1) it remains to prove the relation

$$e^{\cdot A} v_0 \in C^{\beta}([0,T];X_1) . \tag{7.2}$$

In view of the assumption $v_0 \in X_2$, since $0 < \beta + \varepsilon < \frac{1}{2}$, we obtain

$$
\begin{aligned}
\left[e^{\cdot A} v_0 \right]_{\beta,0,X_1} &= \sup_{0 \le \tau < t \le T} (t-\tau)^{-\beta} \sup_{0 < \eta < 1} \left\| \eta^{1-\frac{1}{2}-\varepsilon} A e^{\eta A} \left(e^{tA} - e^{\tau A} \right) v_0 \right\|_X \\
&= \sup_{0 \le \tau < t \le T} (t-\tau)^{-\beta} \sup_{0 < \eta < 1} \left\| \eta^{\frac{1}{2}-\varepsilon} \int_{\tau}^{t} A e^{(\eta+z)A} A v_0 \, dz \right\|_X \\
&\le C \|A v_0\|_X \sup_{0 \le \tau < t \le T} (t-\tau)^{-\beta} \sup_{0 < \eta < 1} \eta^{\frac{1}{2}-\varepsilon} \int_{\tau}^{t} (\eta+z)^{-1} dz \\
&\le C \sup_{0 \le \tau < t \le T} (t-\tau)^{-\beta} \int_{\tau}^{t} z^{-\frac{1}{2}-\varepsilon} dz \\
&\le C(t-\tau)^{\frac{1}{2}-\beta-\varepsilon} \\
&\le C T^{\frac{1}{2}-\beta-\varepsilon} .
\end{aligned}
\tag{7.3}
$$

So, estimate (7.3) yields (7.2). Therefore, $m_1 \in C^{\beta}([0,T])$, $n_1 \in C^{\beta}([0,T];Y)$.

Next let us prove the assertion $v_1 \in C^{1+\beta}([0,T];X) \cap C^{\beta}([0,T];X_2)$. Due to the assumptions $S \in \mathcal{L}(Y;X)$ and $g \in C^{\beta}([0,T];X)$ and the proved inclusions $m_1 \in C^{\beta}([0,T])$ and $n_1 \in C^{\beta}([0,T];Y)$, the relation

$$m_1 \psi_1 S n_1 + g \in C^{\beta}([0,T];X) \tag{7.4}$$

holds. Further, relation (5.15) is implied by the assumption (5.14) and formulas (5.12). Due to (5.14), (7.4) and Lemma 5.1 the function v_1 is a unique solution in $C^{1+\beta}([0,T];X) \cap C^{\beta}([0,T];X_2)$ to the Cauchy problem $v_1'(t) - A v_1(t) = m_1(t)\psi_1 + S n_1(t) + g(t)$, $v(0) = v_0$. $\qquad \square$

Let us now introduce the Banach spaces

$$\mathcal{U}^{\beta,\gamma} = \left\{ C^{1+\beta}([0,T];X) \cap C^{\beta}([0,T];X_2) \right\} \times C^{\beta}([0,T]) \times C^{\beta}([0,T];Y) \tag{7.5}$$

depending on the pair of parameters $(\beta,\gamma) \in (0,1) \times (0,+\infty)$. We endow $\mathcal{U}^{\beta,\gamma}$ with the weighted norm $\|U\|_{\beta,\gamma} = \|v\|_{\beta,\gamma,X_2} + \|m\|_{\beta,\gamma,\mathbb{R}} + \|n\|_{\beta,\gamma,Y}$, $U = (v,m,n)$. Moreover, let Λ stand for the space of non-negative functions $\omega(\gamma)$ continuous on $(1,\infty)$ and satisfying the condition $\omega(\gamma) \to 0$ as $\gamma \to +\infty$.

Our next task consists in estimating the operators K, K^0 and K^1 defined by (5.17), (5.20) and (5.23), respectively.

Lemma 7.2. *Let assumptions* (5.1) $-$ (5.7), (5.14) *hold. Then, for any triplet* $U = (v,m,n)$, $U = (v_1,m_1,n_1)$ *and* $U = (\tilde{v},\tilde{m},\tilde{n})$ *in* $\mathcal{U}^{\beta,\gamma}$, *the following estimates hold for any* $\gamma > 1$ *and some* $\omega_1, \omega_2 \in \Lambda$:

$$\|K[v,m,n]\|_{\beta,\gamma,X} + \|K^0[v,m,n]\|_{\beta,\gamma,\mathbb{R}} + \|K^1[v,m,n]\|_{\beta,\gamma,Y}$$
$$\leq C \|U - U_1\|_{\beta,\gamma}^2 + \omega_1(\gamma)\left(\|U - U_1\|_{\beta,\gamma} + 1\right), \tag{7.6}$$

$$\|K[v,m,n] - K[\tilde{v},\tilde{m},\tilde{n}]\|_{\beta,\gamma,X} + \|K^0[v,m,n] - K^0[\tilde{v},\tilde{m},\tilde{n}]\|_{\beta,\gamma,\mathbb{R}}$$
$$+ \|K^1[v,m,n] - K^1[\tilde{v},\tilde{m},\tilde{n}]\|_{\beta,\gamma,Y} \tag{7.7}$$
$$\leq C\left\{ \|U - U_1\|_{\beta,\gamma} + \|\tilde{U} - U_1\|_{\beta,\gamma} + \omega_2(\gamma) \right\} \|U - \tilde{U}\|_{\beta,\gamma}.$$

Proof. Note that

$$\int_0^t R[n(t-s), v(s)]ds = \int_0^t \Big\{ R[(n-n_1)(t-s), (v-v_1)(s)]$$
$$+ R[(v-v_1)(t-s), n_1(s)] + R[(n-n_1)(t-s), v_1(s)]$$
$$+ R[n_1(t-s), v_1(s)] \Big\}ds,$$

where $\widehat{R}[w_1, w_2] = R[w_2, w_1]$. Observing that R is a continuous bilinear operator from $Y \times X_2$ to X, from Lemma 6.1, estimate (6.5) in Lemma 6.2 with $\alpha = \beta$,

and the inequality $\beta < \frac{1}{2}$ we obtain the following estimates for some $\omega_3 \in \Lambda$:

$$
\left\| \int_0^{\cdot} R[n(\cdot - s), v(s)] ds \right\|_{\beta,\gamma,X}
$$

$$
\leq C \Big\{ \|n - n_1\|_{\beta,\gamma,Y} \|v - v_1\|_{0,\gamma,X_2}
$$

$$
+ \gamma^{\beta-1}(1 + \gamma^{-\beta}) \big[\|v - v_1\|_{\beta,\gamma,X_2} \|n_1\|_{0,0,Y} \tag{7.8}
$$

$$
+ \|n - n_1\|_{\beta,\gamma,Y} \|v_1\|_{0,0,X_2} + (1 + \gamma^\beta) \|n_1\|_{\beta,0,Y} \|v_1\|_{0,0,X_2} \big] \Big\}
$$

$$
\leq C \|U - U_1\|^2_{\beta,\gamma} + \omega_3(\gamma) \left(\|U - U_1\|_{\beta,\gamma} + 1 \right) .
$$

Likewise, from (7.8) with $v = r$, we deduce the following estimate for some $\omega_4 \in \Lambda$:

$$
\left\| \int_0^{\cdot} R[n(\cdot - s), r(s)] \right\|_{\beta,\gamma,X} \leq \omega_4(\gamma) \left(\|U - U_1\|_{\beta,\gamma} + 1 \right) . \tag{7.9}
$$

Analogously, for some $\omega_5 \in \Lambda$, we derive the estimate

$$
\left\| \int_0^{\cdot} m(\cdot - s)(Bv(s) + b(s)) ds \right\|_{\beta,\gamma,X} \leq C \|U - U_1\|^2_{\beta,\gamma} + \omega_5(\gamma) \left(\|U - U_1\|_{\beta,\gamma} + 1 \right).
$$

Taking advantage of this relation and (7.8) – (7.9) in (5.17), we obtain the assertion (7.6) for K.

Consider now the identity

$$
\int_0^t \left\{ R[n(t - s), v(s)] - R[\widetilde{n}(t - s), \widetilde{v}(s)] \right\} ds
$$

$$
= \int_0^t \Big\{ R[(n - \widetilde{n})(t - s), (v - v_1)(s) + R[(n - \widetilde{n})(t - s), v_1(s)]
$$

$$
+ R[(\widetilde{n} - n_1)(t - s), (v - \widetilde{v})(s) + R[(v - \widetilde{v}_1)(t - s), n_1(s)] \Big\} ds ,
$$

where $\widehat{R}[w_1, w_2] = R[w_2, w_1]$, again. Using Lemma 6.1, we obtain

$$
\left\| \int_0^{\cdot} \left\{ R[n(\cdot - s), v(s)] - R[\widetilde{n}(\cdot - s), \widetilde{v}(s)] \right\} ds \right\|_{\beta,\gamma,X}
$$

$$
\leq C \Big\{ \|n - \widetilde{n}\|_{\beta,\gamma,Y} \|v - v_1\|_{0,\gamma,X_2}
$$

$$
+ \gamma^{\beta-1}(1 + \gamma^{-\beta}) \|n - \widetilde{n}\|_{\beta,\gamma,Y} \|v_1\|_{0,0,X_2}
$$

$$
+ \|\widetilde{n} - n_1\|_{\beta,\gamma,Y} \|v - \widetilde{v}\|_{0,\gamma,X_2} \tag{7.10}
$$

$$
+ \gamma^{\beta-1}(1 + \gamma^{-\beta}) \|v - \widetilde{v}\|_{\beta,\gamma,X_2} \|n_1\|_{0,0,Y} \Big\}
$$

$$
\leq C \left\{ \|U - U_1\|_{\beta,\gamma} + \|\widetilde{U} - U_1\|_{\beta,\gamma} + \omega_6(\gamma) \right\} \|U - \widetilde{U}\|_{\beta,\gamma}
$$

for some $\omega_6 \in \Lambda$. Proceeding as above, we easily derive

$$
\begin{aligned}
\Bigg\| \int_0^{\cdot} & \Big\{ R[n(t-s), r(s)] + m(\cdot - s)\,[Bv(s) + b(s)] \\
& - R[\widetilde{n}(t-s), r(s)] - \widetilde{m}(\cdot - s)\,[B\widetilde{v}(s) + b(s)] \Big\}\, ds \Bigg\|_{\beta, \gamma, X} \\
& \leq C \Big\{ \|U - U_1\|_{\beta, \gamma} + \|\widetilde{U} - U_1\|_{\beta, \gamma} + \omega_7(\gamma) \Big\} \|U - \widetilde{U}\|_{\beta, \gamma}
\end{aligned}
\tag{7.11}
$$

with some $\omega_7 \in \Lambda$. Estimates (7.10), (7.11), in view of (5.17), yield (7.7) for K. The assertions (7.6), (7.7) for K^0 and K^1 are proved in a similar manner. $\quad\square$

Finally, we derive estimates for the operators N^0, N^1 and N defined by (5.19), (5.19) and (5.26), respectively.

Lemma 7.3. *Let assumptions* (5.1) – (5.7), (5.14) *hold. Then for any triplet* $U = (v, m, n)$, $U = (v_1, m_1, n_1)$ *and* $U = (\widetilde{v}, \widetilde{m}, \widetilde{n})$ *in* $\mathcal{U}^{\beta, \gamma}$ *the following estimates hold for any* $\gamma > 1$ *and some* $\omega_8, \omega_9 \in \Lambda$:

$$
\begin{aligned}
\|N\,[v, m, n]\|_{\beta, \gamma, X_2} & + \|N^0[v, m, n]\|_{\beta, \gamma, \mathbb{R}} + \|N^1[v, m, n]\|_{\beta, \gamma, Y} \\
& \leq C\|U - U_1\|_{\beta, \gamma}^2 + \omega_8(\gamma)\,(\|U - U_1\|_{\beta, \gamma} + 1)
\end{aligned}
\tag{7.12}
$$

$$
\begin{aligned}
\|N\,[v, m, n] & - N[\widetilde{v}, \widetilde{m}, \widetilde{n}]\|_{\beta, \gamma, X_2} + \|N^0[v, m, n] - N^0[\widetilde{v}, \widetilde{m}, \widetilde{n}]\|_{\beta, \gamma, \mathbb{R}} \\
& + \|N^1[v, m, n] - N^1[\widetilde{v}, \widetilde{m}, \widetilde{n}]\|_{\beta, \gamma, Y} \\
& \leq C \Big\{ \|U - U_1\|_{\beta, \gamma} + \|\widetilde{U} - U_1\|_{\beta, \gamma} + \omega_9(\gamma) \Big\} \|U - \widetilde{U}\|_{\beta, \gamma}.
\end{aligned}
\tag{7.13}
$$

Proof. First we deal with operator N^0 defined by (5.19). Observing that $Q^0 \in \mathcal{L}(X_1, \mathbb{R})$, $S \in \mathcal{L}(Y, X)$, $X_1 = D_A(\frac{1}{2} + \varepsilon, \infty)$ and $\varepsilon < \frac{1}{2} - \beta$, by virtue of Lemma 6.3 with $\alpha = \frac{1}{2} + \varepsilon$, Lemma 6.2 with $\alpha = 0$ and Lemma 7.2, we obtain the following estimates for any $\gamma > 0$ and some $\omega_{10}, \omega_{11} \in \Lambda$:

$$
\begin{aligned}
\|N^0[v, m, n]\|_{\beta, \gamma, \mathbb{R}} \leq & \ \|K^0[v, m, n]\|_{\beta, \gamma, \mathbb{R}} \\
& + \|Q^0\|_{\mathcal{L}(X_1, \mathbb{R})} C\,\gamma^{\varepsilon + \beta - \frac{1}{2}} \Big(1 + \gamma^{-\frac{1}{2} - \varepsilon} + \gamma^{-\beta}\Big) \\
& \times \Big\{ \|K[v, m, n]\|_{\beta, \gamma, X} + \big(\|m - m_1\|_{\beta, \gamma, \mathbb{R}} + \|m_1\|_{0, 0, \mathbb{R}}\big)\|\psi_1\|_X \\
& + \|S_2\|_{\mathcal{L}(Y, X)} \big(\|n - n_1\|_{\beta, \gamma, Y} + \|n_1\|_{0, 0, Y}\big) \Big\} \\
& \leq C\|U - U_1\|_{\beta, \gamma}^2 + \omega_{10}(\gamma)\big(\|U - U_1\|_{\beta, \gamma} + 1\big)
\end{aligned}
$$

and

$$\|N^0[v, m, n] - N^0[\widetilde{v}, \widetilde{m}, \widetilde{n}]\|_{\beta, \gamma, \mathbb{R}}$$
$$\leq \|K^0[v, m, n] - K^0[\widetilde{v}, \widetilde{m}, \widetilde{n}]\|_{\beta, \gamma, \mathbb{R}}$$
$$+ \|Q^0\|_{\mathcal{L}(X_1, \mathbb{R})} C \gamma^{\varepsilon + \beta - \frac{1}{2}} \left(1 + \gamma^{-\frac{1}{2} - \varepsilon} + \gamma^{-\beta}\right)$$
$$\times \left\{ \|K[v, m, n] - K[\widetilde{v}, \widetilde{m}, \widetilde{n}]\|_{\beta, \gamma, X} \right.$$
$$\left. + \|m - \widetilde{m}\|_{\beta, \gamma, \mathbb{R}} \|\psi_1\|_X + \|S_2\|_{\mathcal{L}(Y, X)} \|n - \widetilde{n}\|_{\beta, \gamma, Y} \right\}$$
$$\leq C \left\{ \|U - U_1\|_{\beta, \gamma} + \|\widetilde{U} - U_1\|_{\beta, \gamma} + \omega_{11}(\gamma) \right\} \|U - \widetilde{U}\|_{\beta, \gamma}.$$

Hence we have proved the estimates (7.12) and (7.13) for N^0. The estimates (7.12) and (7.13) for N^1 can be proved in a similar manner.

Next let us consider the operator N defined by (5.26). Since $K[v, m, n](0) = 0$, $N^0[v, m, n](0) = 0$ and $N^1[v, m, n](0) = 0$ (see (5.17), (5.19), (5.22)), we can make use of Lemma 6.4 and relations (7.11) for N^0 and N^1 to estimate the operator N. We find the following estimates for any $\gamma \geq 1$ and some $\omega_{12} \in \Lambda$:

$$\|N[v, m, n]\|_{\beta, \gamma, X_2} \leq C \Big\{ \|K[v, m, n]\|_{\beta, \gamma, X} + \|N^0[v, m, n]\|_{\beta, \gamma, \mathbb{R}} \|\psi_1\|_X$$
$$+ \|S_2\|_{\mathcal{L}(Y, X)} \|N^1[v, m, n]\|_{\beta, \gamma, Y} \Big\}$$
$$\leq C \|U - U_1\|_{\beta, \gamma}^2 + \omega_{12}(\gamma) \left(\|U - U_1\|_{\beta, \gamma} + 1\right).$$

Thus we have proved (7.11). Estimate (7.12) for N can be proved similarly by means of Lemma 6.4 using the estimates (7.12) for N^0 and N^1. $\square$

8. Main results

In this section we formulate and prove the main existence and uniqueness results of the paper. First we deal with the abstract identification problem (5.8) – (5.11).

Theorem 8.1. *Let the assumptions* (5.1) – (5.7), (5.14) *hold. Then the abstract problem* (5.8) – (5.11) *has a solution* (v, m, n) *in* $\mathcal{U}^{\beta, 0}$. *The solution is unique in* $\mathcal{U}^{\beta', 0}$ *for any* $\beta' \in (0, \beta)$.

Proof. By Proposition 5.1 problem (5.8) – (5.11) is equivalent to system (5.17), (5.18), (5.25), which we rewrite in the operator form $U = F(U)$ where, as before, $U = (v, m, n)$ and $F = (F_1, F_2, F_3)$ with $F_1(U) = N[v, m, n] + v_1$, $F_2(U) = N^0[v, m, n] + m_1$, $F_3(U) = N^1[v, m, n] + n_1$.

Let us now define the following balls in $\mathcal{U}^{\beta,\gamma}$: $\mathcal{B}(\beta,\gamma,r) = \{U \in \mathcal{U}^{\beta,\gamma} : \|U - U_1\|_{\beta,\gamma} \le r\}$, where $\gamma \ge 1$ and $r > 0$. By the definitions of $F(U)$, U_1, $\mathcal{B}(\beta,\gamma,r)$ and estimate (7.11) of Lemma 7.3 we deduce the estimate

$$\|F(U) - U_1\|_{\beta,\gamma} \le \widehat{C}\, r^2 + \widehat{\omega}(\gamma)\,(r+1) \quad \forall U \in \mathcal{B}(\beta,\gamma,r)\,, \tag{8.1}$$

for some positive constant $\widehat{C}$ and some function $\widehat{\omega} \in \Lambda$.

Choose now $r_1 = \widehat{C}^{-1}$. Since $\widehat{\omega}(\gamma) \to 0$ as $\gamma \to +\infty$, we can find, for any $r < r_1$, a number $\gamma_1(r) \ge 1$ such that $\widehat{\omega}(\gamma) \le \frac{r - \widehat{C}r^2}{r+1}$ for all $\gamma \ge \gamma_1(r)$. According to this definition, the inequality $\widehat{C}\,r^2 + \widehat{\omega}(\gamma)\,(r+1) \le r$ holds for any $r < r_1$ and $\gamma \ge \gamma_1(r)$. Hence, by virtue of (8.1), we can conclude that

$$F(\mathcal{B}(\beta,\gamma,r)) \subseteq \mathcal{B}(\beta,\gamma,r) \quad \text{if } r < r_1 \text{ and } \gamma \ge \gamma_1(r)\,. \tag{8.2}$$

Further, by estimate (7.12) in Lemma 7.3 we get the estimate

$$\|F(U) - F(\widetilde{U})\|_{\beta,\gamma} \le \overline{C}\,\{2r + \overline{\omega}(\gamma)\}\,\|U - \widetilde{U}\|_{\beta,\gamma} \quad \forall U, \widetilde{U} \in \mathcal{B}(\beta,\gamma,r)\,, \tag{8.3}$$

for some positive constant $\overline{C}$ and some function $\overline{\omega} \in \Lambda$.

Define then $r_2 = (2\overline{C})^{-1}$ and choose, for any $r < r_2$, a number $\gamma_2(r) \ge 1$ such that $\overline{\omega}(\gamma) < \frac{1 - 2\overline{C}r}{\overline{C}}$ for all $\gamma \ge \gamma_2(r)$. Thus, the inequality $\overline{C}\,[2r + \overline{\omega}(\gamma)] < 1$ holds for any $r < r_2$ and $\gamma \ge \gamma_2(r)$. Consequently, from (8.3) it follows that

$$F \text{ is a contraction mapping in } \mathcal{B}(\beta,\gamma,r) \quad \text{if } r < r_2 \text{ and } \gamma \ge \gamma_2(r)\,. \tag{8.4}$$

Summing up, (8.2) and (8.4) imply that equation $U = F(U)$ has a unique solution in each ball $\mathcal{B}(\beta,\gamma,r)$ such that $r < r_3$ and $\gamma \ge \gamma_3(r)$, where $r_3 = \min\{r_1, r_2\}$ and $\gamma_3(r) = \max\{\gamma_1(r), \gamma_2(r)\}$. Evidently r_3 and ω_3 depend on β, i.e., $r_3 = r_3[\beta]$ and $\omega(r) = \omega[\beta](r)$. We have proved the existence of the solution to (5.17), (5.18), (5.25) in $\mathcal{U}^{\beta,\gamma}$.

Next we are going to show that the solution of system (5.17), (5.18), (5.25) is unique in the space $\widehat{\mathcal{U}}^{\beta,\gamma} = \bigcup_{\beta' \in (0,\beta)} \mathcal{U}^{\beta',\gamma}$. Suppose that system (5.17), (5.18), (5.25), or equivalently equation $U = F(U)$, has two solutions U and $\widetilde{U}$ in $\widehat{\mathcal{U}}^{\beta,\gamma}$. Consequently, there exists a $\beta' \in (0,\beta)$ such that $U, \widetilde{U} \in \mathcal{U}^{\beta',\gamma}$. Let us now choose some $\beta'' \in (0,\beta')$. Since $(U - U_1)(0) = (\widetilde{U} - U_1)(0) = 0$ (cf. system (5.17), (5.18), (5.27) for U, $\widetilde{U}$ and formulas (5.19), (5.20), (5.22), (5.23) and (5.26) for the operators entering this system), from estimate (6.6) in Lemma 6.2 we obtain the relations

$$\|U - U_1\|_{\beta'',\gamma} \to 0\,, \quad \|\widetilde{U} - U_1\|_{\beta'',\gamma} \to 0 \quad \text{as} \quad \gamma \to +\infty. \tag{8.5}$$

We mention that the assumptions of Theorem 8.1 remain valid if we replace β with β''. Hence the existence part of the proof also remains valid if we substitute

β'' for β. The relations (8.5) imply that there exists $\gamma_0 \geq \gamma_3[\beta''](r_3[\beta'']/2)$ such that $U, \widetilde{U} \in \mathcal{B}(\beta'', \gamma_0, r_3[\beta'']/2)$. As in the ball $\mathcal{B}(\beta'', \gamma_0, r_3[\beta'']/2)$ the uniqueness has already been shown, we conclude that $U = \widetilde{U}$.

In order to complete the proof it remains to show that the first component v of the solution $U = (v, m, n)$ belongs to $C^{1+\beta}([0, T]; X)$. But this easily follows because the right-hand side of the equation (5.25) belongs to $C^{1+\beta}([0, T]; X)$ for any $(v, m, n) \in \mathcal{U}^{\beta, \gamma}$. $\qquad\qquad\square$

Finally, let us return to our explicit identification problem (2.4) – (2.8) for (u, h), which is equivalent to problem (2.11) – (2.15) for $(\widehat{u}, h)$ with $\widehat{u} = u - u_\Gamma$. Due to Propositions 3.1, 4.1 and 4.2, problem (2.11) – (2.15) is a particular case of the abstract problem (5.8) – (5.11), since conditions (5.1) – (5.7) are satisfied. Consequently, applying Theorem 8.1, we obtain the following theorem.

Theorem 8.2. *Let assumptions* (3.1) – (3.4) *with* $0 < \beta < \frac{1}{2}$ *and* (3.13) – (3.16), (4.2), (4.3) *be satisfied. Moreover, let the inclusion* $Av_0 + g(0, \cdot) + (Q^0 v_0 + f^0(0))\psi_1 + S(Q^1 v_0 + f^1(0, \cdot)) \in D_A(\beta, \infty)$ *hold, where the operator* S *is given by* (4.42), *functions* v_0, ψ_1, ψ_2, g *are defined by formulas* (3.5), (3.6) *and* Q^0, f^0, Q^1 *and* f^1 *are defined by* (4.31), (4.32), (4.37) *and* (4.38), *respectively*[1]. *Then problem* (2.4) – (2.8) *has a solution* (u, h) *in the space* $\mathcal{U}_\beta \times C^\beta([0, T]; C^1[0, l])$, *where* $\mathcal{U}_\beta = \{u = u_\Gamma + \widehat{u} : \widehat{u} \in C^{2+\beta}([0, T]; X) \cap C^{1+\beta}([0, T]; X_2)\}$. *The solution is unique in the space* $\bigcup_{\beta' \in (0, \beta)} \mathcal{U}_{\beta'} \times C^{\beta'}([0, T]; C^1[0, l])$.

References

[1]   Colombo, F., Guidetti, D. and Lorenzi, A., Integro-differential identification problems for thermal materials with memory in non-smooth plane domains. Preprint no. 37/2001. Department of Mathematics F. Enriques, Università degli Studi di Milano.

[2]   Colombo, F., Guidetti, D. and Lorenzi, A., Integro-differential identification problems for the heat equation in cylindrical domains. Preprint no. 3/2002. Department of Mathematics F. Enriques, Università degli Studi di Milano.

[3]   Colombo, F. and Lorenzi, A., Identification of time- and space-dependent relaxation kernels for materials with memory related to cylindrical domains I. *J. Math. Anal. Appl.* 213 (1997), 32 – 62.

[4]   Colombo, F. and Lorenzi, A., Identification of time- and space-dependent relaxation kernels for materials with memory related to cylindrical domains II. *J. Math. Anal. Appl.* 213 (1997), 63 – 90.

[5]   Colombo, F. and Lorenzi, A., An identification problem related to parabolic integrodifferential equations with non-commuting spatial operators. *J. Inverse Ill-Posed Probl.* 8 (2000), 505 – 540.

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[1]See also definitions (3.7), (3.8), (3.10), (3.11), (4.3), (4.4), (4.11) – (4.14) and (4.17) for the functions entering these expressions.

[6] Favaron, A. and Lorenzi, A., Parabolic integrodifferential identification problems related to memory kernels I, II. *J. Inverse Ill-Posed Probl.* 9 (2001), 489 – 529, 595 – 614.

[7] Favaron, A. and Lorenzi, A., Parabolic integrodifferential identification problems related to memory kernels with special symmetries. *J. Inverse Ill-Posed Probl.* 11 (2003), 18 – 28.

[8] Grasselli, M., An identification problem for a linear integro-differential equation occuring in heat flow. *Math. Meth. Appl. Sci.* 15 (1992), 167 – 186.

[9] Janno, J. and Wolfersdorf, L. v., Inverse problems for identification of memory kernels in heat flow. *J. Inverse Ill-Posed Probl.* 4 (1996), 39 – 66.

[10] Janno, J. and Wolfersdorf, L. v., On a class of multilinear operator equations. *Z. Anal. Anwendungen* 15 (1996), 935 – 948.

[11] Janno, J. and Wolfersdorf, L. v., Identification of weakly singular memory kernels in heat conduction. *Z. Angew. Math. Mech.* 77 (1997), 243 – 257.

[12] Janno, J. and Wolfersdorf, L. v., Identification of memory kernels in general linear heat flow. *J. Inverse Ill-Posed Probl.* 6 (1998), 141 – 164.

[13] Lorenzi, A. and Sinestrari, E., An inverse problem in the theory of materials with memory. *Nonlinear Anal.* 12 (1988), 1317 – 1335.

[14] Lorenzi, A. and Sinestrari, E., An inverse problem in the theory of materials with memory II. In: *Semigroup Theory and Applications. Ser. Pure Appl. Math.* 161 (1989), 261 – 290.

[15] Lorenzi, A. and Yakhno, V. G., An identification problem related to an isotropic nonhomogeneous stratified viscoelastic body. *J. Inverse Ill-Posed Probl.* 5 (1997), 29 – 53.

[16] Lunardi, A., *Analytic Semigroups and Optimal Regularity in Parabolic Problems.* Basel: Birkhäuser 1995.

[17] Prüss, J., *Evolutionary Integral Equations and Applications.* Basel: Birkhäuser 1993.

[18] Sinestrari, E., On the Cauchy problem in spaces of continuous functions. *J. Math. Anal. Appl.* 107 (1985), 16 – 66.