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Nonexistence of Solutions to a Hyperbolic Equation with a Time Fractional Damping

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Abstract. We consider the nonlinear hyperbolic equation

$$u_{tt} - \Delta u + D_+^\alpha u = h(t,x)\,|u|^p$$

posed in $Q := (0,\infty) \times \mathbb{R}^N$, where $D_+^\alpha u$, $0 < \alpha < 1$ is a time fractional derivative, with given initial position and velocity $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$. We find the Fujita's exponent which separates in terms of p, α and N, the case of global existence from the one of nonexistence of global solutions. Then, we establish sufficient conditions on $u_1(x)$ and $h(x,t)$ assuring non-existence of local solutions.

Keywords. Fractional damping, non-existence, nonlinear hyperbolic equations

Mathematics Subject Classification (2000). 35L20, 35L70, 26A33, 34A12

1. Introduction

In this paper, we consider the equation

$$u_{tt} - \Delta u + D_+^\alpha u = h(t,x)\,|u|^p \tag{WE}$$

posed in $Q := (0,+\infty) \times \mathbb{R}^N$, subject to the initial conditions

$$u(0,x) = u_0(x) \quad \text{and} \quad u(0,x) = u_1(x), \tag{1}$$

where $\Delta = \partial_1^2 + ...\partial_N^2$ is the Laplacian in the space variable x and D_+^α for $0 < \alpha < 1$ is the time fractional derivative defined by

$$\left(D_+^\alpha f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\sigma)}{(t-\sigma)^\alpha} d\sigma.$$

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This fractional derivative is said to be *left-handed.* The *right-handed* fractional derivative is defined by

$$\left(D_{-}^{\alpha}f\right)(t)=\frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t}^{\infty}\frac{f(\sigma)}{(\sigma-t)^{\alpha}}d\sigma$$

(see [12] and [13] for more on fractional integrals and derivatives). In equation (WE), the term D_{+}^{α} represents an attenuation of fractional type (see [8, 14, 15, 16]). Before we discuss our results in detail, let us briefly dwell on some literature related to the equation (WE).

In the case of purely fractional derivative time modelling, Seredynska and Hanyga [14] considered the nonlinear equation

$$D^{2}u+\gamma D^{1+\eta}u+F(u)=0,$$

where $D^{1+\eta}$ with $0<\eta<1$ represents the $(1+\eta)$-order fractional derivative in the sense of Caputo [12], and γ is the thermo-viscous coefficient. This equation serves as a model for the anomalous attenuation. Our equation can be viewed as an infinite dimensional version of the equation above.

In one of our previous papers, some conditions were obtained for the occurrence of blowing-up of solutions to (WE), with $h(t,x)\equiv 1$, on a bounded domain. More precisely, in [8], it is proved that the solution is unbounded and grows up exponentially in the L^{p}-norm for sufficiently large initial data. This paper has been followed by two others by Tatar [15, 16]. In [16], the set of initial data has been considerably enlarged using a different argument based on a new Lyapunov type functional. Then a blow up in finite time has been proved using an argument similar to the one used in [8] but combined with a technique due to Georgiev and Todorova [4] together with a suitably chosen functional.

Here, in the first part of the paper, we relax completely the conditions on the data and prove a result of different flavor in the sense that a critical exponent is found which separates the case of blow-up from the case of global existence; the decisive point is then made according to the size of data in some functional space. The method of proof we follow here has been already used in [7] (see also [6]) to not only give a short proof of an important result in [17] but also to answer positively an open question raised there concerning the equation (WE) with a linear damping of the form u_t (rather than a time fractional damping). This method of proof appeared first in the book of J. L. Lions [9] for the heat equation with polynomial nonlinearity and then in the paper of Baras and Pierre [2] (see also [3]). It remained dormant till the series of very interesting papers by Qi S. Zhang [18, 19] followed by a sizeable number of articles by Mitidieri, Pohozaev, Kurta, Tesei, Laptev, Veron, Guedda and Kirane collected in [10]. The method is rather simple and consists in a judicious choice of the test function in the weak formulation of equation (WE) accompanied with a scaled variables argument.

The theorems we will present here are concerned with the non-existence of solutions. In case of the existence of a local solution then our results would mean that this solution must blow up in finite.

In the second part of the paper, we establish a sufficient condition on $h(t,x)$ and the initial data assuring non-existence of solutions for any time. Necessary conditions are also established for the existence of global solutions. To this end, we will adapt a method used in Baras and Kersner [1], originally established for parabolic problems. In [1], the following problem has been considered:

$$\begin{cases} u_t - \Delta u = h(x)u^p, & x \in \mathbb{R}^N,\ t > 0 \\ u(0,x) = u_0(x) \geq 0. \end{cases} \tag{PE}$$

It was shown that no local weak nonnegative solution to (PE) exists if the initial data satisfies

$$\lim_{|x|\to\infty} u_0^{p-1} h(x) = +\infty,$$

and any possible local weak nonnegative solution blows-up at a finite time if

$$\lim_{|x|\to\infty} u_0^{p-1} h(x)\,|x|^2 = +\infty.$$

Our plan for the rest of the paper is as follows: In the next section we prove a first result on non-existence of solutions after some time T_*. Section 3 contains the statements and proofs of other results on non-existence of local and global solutions for the same problem but with a space dependent potential.

2. Non-existence of global solutions

The function $h(t,x)$ is assumed to be nonnegative and satisfying $h(tR^2, xR) = R^\rho h(t,x)$ for some ρ positive and R large. Let us make clear first what we mean by a solution to problem (WE). Q_T here will denote the set $Q_T := (0,T) \times \mathbb{R}^N$ and $L^p_{loc}(Q_T,\, h\,dt\,dx)$ will denote the space of all functions $v : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ such that $\int_K |v|^p\, h(t,x)\,dt\,dx < \infty$ for any compact K in $\mathbb{R}^+ \times \mathbb{R}^N$.

Definition 2.1. The continuous function $u \in L^1_{loc}(Q_T)$ is a *local weak solution* of the problem (WE) subject to the initial data (1) on $(0,T)$ $(0 < T < +\infty)$ if $u \in L^p_{loc}(Q_T,\, h\,dt\,dx)$ and is such that

$$\int_Q h(t,x)\,|u|^p\,\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi_0(x) = \int_Q \left(u\varphi_{tt} - u\Delta\varphi + uD^\alpha_-\varphi\right) \tag{2}$$

holds for any $\varphi \in C^2_0(Q_T)$, $\varphi \geq 0$, and satisfying $\varphi = 0$, $\varphi_t = 0$ at $t = T$ and $\varphi_t = 0$ at $t = 0$. Here we have set $\varphi(0,x) =: \varphi_0(x)$.

Remark 2.2. We have the formula (integration by parts) (see [13, p. 46]) $\int_a^b f(x)(D_+^\alpha g)(x)\,dx = \int_a^b g(x)(D_-^\alpha f)(x)\,dx$. In our case we extend u by 0 for $t \le 0$.

Our first result reads

Theorem 2.3. *Assume that* $\int_{\mathbb{R}^N} u_1(x) > 0$, *and* $1 < p \le 1 + \frac{2\alpha+\rho}{2+N-2\alpha}$. *Then, problem* (WE)–(1) *does not admit global non trivial solutions in time.*

Proof. The proof is by contradiction. So, we assume that the solution is global. Let $\varphi_0 \in C_0^2(\mathbf{R})$, $\varphi_0 \ge 0$, φ_0 decreasing and such that

$$\varphi_0(y) = \begin{cases} 1, & \text{if } |y| \le 1 \\ 0, & \text{if } |y| \ge 2. \end{cases}$$

We choose $\varphi(t,x) = \varphi_0^\lambda(\xi)$, where $\xi = R^{-4}(t^2 + |x|^4)$, R is a positive real number, λ is any real greater than p, and such that

$$\int_{\operatorname{supp}\Delta\varphi} |\Delta\varphi|^q (h\varphi)^{1-q} + \int_{\operatorname{supp}\varphi_{tt}} |\varphi_{tt}|^q (h\varphi)^{1-q} + \int_{\operatorname{supp} D_-^\alpha\varphi} \left|D_-^\alpha\varphi\right|^q (h\varphi)^{1-q} < \infty$$

with $p + q = pq$. Here and in the whole paper supp will stand for support. We clearly have $\varphi_t(0,x) = 0$. This function φ will be taken as a test function in (2). First, let us write

$$\int_Q u\varphi_{tt} = \int_Q u(h\varphi)^{\frac{1}{p}}(h\varphi)^{-\frac{1}{p}}\varphi_{tt}.$$

As φ is of compact support, using Hölder's inequality, we obtain

$$\int_Q u\varphi_{tt} \le \left(\int_{\operatorname{supp}\varphi} |u|^p h\varphi\right)^{\frac{1}{p}} \left(\int_{\operatorname{supp}\varphi_{tt}} (h\varphi)^{-\frac{q}{p}} |\varphi_{tt}|^q\right)^{\frac{1}{q}}. \tag{3}$$

We can appeal to the ε-Young inequality to get

$$\int_Q u\varphi_{tt} \le \varepsilon \int_{\operatorname{supp}\varphi} |u|^p h\varphi + C_\varepsilon \int_{\operatorname{supp}\varphi_{tt}} (h\varphi)^{-\frac{q}{p}} |\varphi_{tt}|^q \tag{4}$$

for some $\varepsilon > 0$. Likewise, we have the estimates

$$\int_Q u\Delta\varphi \le \left(\int_{\operatorname{supp}\varphi} |u|^p h\varphi\right)^{\frac{1}{p}} \left(\int_{\operatorname{supp}\Delta\varphi} (h\varphi)^{-\frac{q}{p}} |\Delta\varphi|^q\right)^{\frac{1}{q}} \tag{5}$$

and

$$\int_Q u\Delta\varphi \le \varepsilon \int_{\operatorname{supp}\varphi} |u|^p h\varphi + C_\varepsilon \int_{\operatorname{supp}\Delta\varphi} (h\varphi)^{-\frac{q}{p}} |\Delta\varphi|^q. \tag{6}$$

The same is true for the third term in the right hand side of (2)

$$\int_Q uD^\alpha_-\varphi \le \left(\int_{\operatorname{supp}\varphi} |u|^p\, h\varphi\right)^{\frac{1}{p}} \left(\int_{\operatorname{supp} D^\alpha_-\varphi} (h\varphi)^{-\frac{q}{p}} \left|D^\alpha_-\varphi\right|^q\right)^{\frac{1}{q}} \tag{7}$$

and

$$\int_Q uD^\alpha_-\varphi \le \varepsilon \int_{\operatorname{supp}\varphi} |u|^p\, h\varphi + C_\varepsilon \int_{\operatorname{supp} D^\alpha_-\varphi} (h\varphi)^{-\frac{q}{p}} \left|D^\alpha_-\varphi\right|^q. \tag{8}$$

Summing up, (4), (6) and (8), with ε small enough, we infer that

$$\int_Q |u|^p\, h\varphi + \int_{\mathbb{R}^N} u_1(x)\varphi_0(x) \le C \int_{\operatorname{supp}\varphi} (h\varphi)^{-\frac{q}{p}} \left(|\varphi_{tt}|^q + |\Delta\varphi|^q + \left|D^\alpha_-\varphi\right|^q\right) \tag{9}$$

for some positive constant C. From now on the constant C will denote a generic positive constant. At this stage, we introduce the scaled variables $t = \tau R^2$, $x = yR$ and set $\Omega := \{(\tau, y) \in \mathbb{R}^+ \times \mathbb{R}^N; \tau^2 + |y|^4 \le 2\}$. Therefore, writing $\varphi(t,x) = \varphi(\tau R^2, yR) =: \chi(\tau, y)$, we have

$$\int_\Omega (h\varphi)^{-\frac{q}{p}} |\varphi_{tt}|^q = R^{2+N-4q-\frac{q\rho}{p}} \int_\Omega h^{-\frac{q}{p}} |\chi_{\tau\tau}|^q \chi^{-\frac{q}{p}}$$

$$\int_{\operatorname{supp}\Delta\varphi} (h\varphi)^{-\frac{q}{p}} |\Delta\varphi|^q = R^{2+N-4q-\frac{q\rho}{p}} \int_\Omega h^{-\frac{q}{p}} |\Delta\chi|^q \chi^{-\frac{q}{p}}$$

and

$$\int_{\operatorname{supp} D^\alpha_-\varphi} (h\varphi)^{-\frac{q}{p}} \left|D^\alpha_-\varphi\right|^q = R^{2+N-2\alpha q-\frac{q\rho}{p}} \int_\Omega h^{-\frac{q}{p}} \left|D^\alpha_-\chi\right|^q \chi^{-\frac{q}{p}}.$$

So, we have

$$\int |u|^p\, h\varphi + \int u_1(x)\varphi_0(x) \le C\left\{R^{2+N-4q-\frac{q\rho}{p}} + R^{2+N-2\alpha q-\frac{q\rho}{p}}\right\}. \tag{10}$$

Observe that we have chosen φ_0 in such a way to have $|\chi_{\tau\tau}|^q$ and $|\Delta\chi|^q$ at the same magnitude in R. Now we impose the condition

$$1 < p \le 1 + \frac{2\alpha + \rho}{2 + N - 2\alpha} =: p_\alpha.$$

In the estimate (10), we have to distinguish two cases:

`Either` $p < p_\alpha$: In this case, passing to the limit as $R \to \infty$ in (10) we obtain

$$\lim_{R\to\infty} \left\{\int |u|^p\, h\varphi + \int u_1(x)\varphi_0(x)\right\} = \int h\, |u|^p + \int u_1(x) \le 0.$$

This contradicts the requirement $\int u_1(x) > 0$.

Or $p = p_\alpha$: In this case, we obtain from (10) $\int h\,|u|^p\,\varphi + \int u_1\varphi_0 \leq C$ and therefore $\int h\,|u|^p\,\varphi \leq C$. Letting $R \to \infty$, we obtain $\int h\,|u|^p \leq C$. So

$$\lim_{R\to\infty}\int_{C_R} |u|^p\, h\varphi = 0, \tag{11}$$

where $C_R := \left\{(t,x) : R^4 \leq t^2 + |x|^4 \leq 2R^4\right\}$.

Using (2) and the estimates (3), (5) and (7), we may write

$$\begin{aligned}\int |u|^p\, h\varphi + \int u_1(x)\varphi_0(x) \leq \left(\int_{C_R} |u|^p\, h\varphi\right)^{\frac{1}{p}} \Bigg\{ \left(\int (h\varphi)^{-\frac{q}{p}}\, |\varphi_{tt}|^q\right)^{\frac{1}{q}} \\ + \left(\int (h\varphi)^{-\frac{q}{p}}\, |\Delta\varphi|^q\right)^{\frac{1}{q}} + \left(\int (h\varphi)^{-\frac{q}{p}}\, \left|D_-^\alpha\varphi\right|^q\right)^{\frac{1}{q}} \Bigg\}.\end{aligned} \tag{12}$$

Passing to the limit as $R \to \infty$ in (12) and taking into account (11), we obtain

$$\lim_{R\to\infty}\left\{\int |u|^p\, h\varphi + \int u_1(x)\varphi_0(x)\right\} = 0.$$

This is again in contradiction with $\int u_1 > 0$. The proof is complete. □

Remark 2.4. Observe that in the limiting case when $\alpha \to 1$, the critical exponent is $p_{cwd} = 1 + \frac{2+\rho}{N}$. This is in agreement with the one found in [17] and [7].

Remark 2.5. Notice that the previous argument works perfectly as well for the case $1 \leq \alpha < 2$. In this case we use the definitions (see [14, p. 37])

$$\left(D_+^\alpha f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{f(\sigma)}{(t-\sigma)^{\alpha-n+1}}\, d\sigma, \qquad n = [\alpha]+1$$

and

$$\left(D_-^\alpha f\right)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_t^\infty \frac{f(\sigma)}{(\sigma-t)^{\alpha-n+1}}\, d\sigma, \quad n = [\alpha]+1.$$

3. Necessary conditions for local and global solutions

In this section we assume that $\inf_{t\in\mathbb{R}^+} h(t,x) > 0$.

Remark 3.1. From the formula (see [13, p. 36])

$$\left(D_-^\alpha f\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(T)}{(T-t)^\alpha} - \int_t^T \frac{f'(\sigma)}{(\sigma-t)^\alpha} d\sigma\right],$$

(for absolutely continuous functions) it is clear that if $f(T)=0$, then the right-handed fractional derivative reduces to

$$\left(D_{-}^{\alpha} f\right)(t)=\frac{-1}{\Gamma(1-\alpha)} \int_{t}^{T} \frac{f^{\prime}(\sigma)}{(\sigma-t)^{\alpha}} d\sigma.$$

This is to be compared with the fractional derivative in the sense of Caputo.

Our first results in this section are the following

Theorem 3.2. *Let u be a local solution to* (WE)–(1) *where $T<+\infty$ and $p>1$. Then, there exist constants γ and L such that*

$$\liminf_{|x|\to\infty} u_1(x) h^{\frac{p}{p-1}}(t,x) \leq \frac{1}{q}\left(\frac{4}{p}\right)^{\frac{q}{p}}\left(\frac{\gamma^q}{T^{2q-1}}+L T^{(1-\alpha)q}\right).$$

Proof. By the definition of a weak solution, for any $\varphi \in C_0^\infty(Q_T)$, $\varphi \geq 0$ such that $\operatorname{supp}\varphi \subset \left\{x \in \mathbb{R}^N : |x|>R_0>0\right\}$, we have

$$\int_{\mathbb{R}^N} u_1(x)\varphi_0(x)+\int_{Q_T} h(t,x)|u|^p \varphi \leq \int_{Q_T}\left(|u|\left|\varphi_{tt}\right|+|u||\Delta\varphi|+|u|\left|D_{-}^{\alpha}\varphi\right|\right). \quad (13)$$

Using the ε-Young inequality $ab \leq \varepsilon a^p+C_\varepsilon b^q$ (with $C_\varepsilon=\frac{1}{q}(p\varepsilon)^{-\frac{q}{p}}$) we can estimate all three terms in the right hand side of (13). Indeed, writing $|u|\left|\varphi_{tt}\right|=|u|(\varphi h)^{\frac{1}{p}}(\varphi h)^{-\frac{1}{p}}\left|\varphi_{tt}\right|$, we find for $\varepsilon>0$

$$\int_{Q_T}|u|\left|\varphi_{tt}\right| \leq \varepsilon \int_{Q_T}|u|^p h\varphi+C_\varepsilon \int_{Q_T}\left|\varphi_{tt}\right|^q(\varphi h)^{-\frac{q}{p}}, \quad (14)$$

where q is always the conjugate exponent of p. Likewise, we obtain for the other two terms

$$\int_{Q_T}|u||\Delta\varphi| \leq \varepsilon \int_{Q_T}|u|^p h\varphi+C_\varepsilon \int_{Q_T}|\Delta\varphi|^q(\varphi h)^{-\frac{q}{p}} \quad (15)$$

and

$$\int_{Q_T}|u|\left|D_{-}^{\alpha}\varphi\right| \leq \varepsilon \int_{Q_T}|u|^p h\varphi+C_\varepsilon \int_{Q_T}\left|D_{-}^{\alpha}\varphi\right|^q(\varphi h)^{-\frac{q}{p}}. \quad (16)$$

Taking $\varepsilon=\frac{1}{4}$, we deduce from (14)–(16) and (13) that

$$J:=\int_{\mathbb{R}^N} u_1(x)\varphi_0(x) \leq C_{1/4}\int_{Q_T}\left(\left|\varphi_{tt}\right|^q+|\Delta\varphi|^q+\left|D_{-}^{\alpha}\varphi\right|^q\right)(\varphi h)^{-\frac{q}{p}}, \quad (17)$$

with $C_{1/4}=\frac{1}{q}\left(\frac{4}{p}\right)^{\frac{q}{p}}$. At this stage, we make the choice

$$\varphi(t,x):=\Phi\left(\frac{x}{R}\right)\left(1-\frac{t^2}{T^2}\right)^{2q}$$

where $\Phi\in C_0^\infty(Q_T)$, $\Phi\geq 0$, $\operatorname{supp}\Phi\subset\{x\in\mathbb{R}^N:1<|x|<2\}$ and $|\Delta\Phi|\leq k\Phi$. It is clear that the requirements previously set for φ are satisfied ($\varphi(T,x)\equiv\varphi_t(T,x)\equiv\varphi_t(0,x)\equiv 0$). Next, we estimate the three terms in the right hand side of (17). Let us make the change of variables $t=\tau T$ and put $\gamma=q(q-1)$. Using this and the assumptions on φ, we find,

$$\int_{Q_T}|\varphi_{tt}|^q(\varphi h)^{-\frac{q}{p}}\leq\gamma^q T^{1-2q}\int_{Q_1}h^{1-q}\Phi, \tag{18}$$

and

$$\int_{Q_T}|\Delta\varphi|^q(\varphi h)^{-\frac{q}{p}}\leq k^q R^{-2q}T\int_{Q_1}h^{1-q}\Phi. \tag{19}$$

For the third term, it is easy to see that

$$\int_{Q_T}\left|D_-^\alpha\varphi\right|^q(\varphi h)^{-\frac{q}{p}}=\int_{Q_T}h^{1-q}\left(1-\frac{t^2}{T^2}\right)^{2q(1-q)}\Phi\left|D_-^\alpha\left(1-\frac{t^2}{T^2}\right)^{2q}\right|^q. \tag{20}$$

Now we compute the right-handed fractional derivative

$$\Gamma(1-\alpha)D_-^\alpha\left(1-\frac{t^2}{T^2}\right)^{2q}=-\frac{d}{dt}\int_t^T\frac{\left(1-\frac{\sigma^2}{T^2}\right)^{2q}}{(\sigma-t)^\alpha}d\sigma=-T^{-4q}\frac{d}{dt}\int_t^T\frac{\left(T^2-\sigma^2\right)^{2q}}{(\sigma-t)^\alpha}d\sigma.$$

Using the Euler's change of variable $y=\frac{\sigma-t}{T-t}$ we see that $1-y=\frac{T-\sigma}{T-t}$ and $1-y^2=\frac{T^2-\sigma^2}{(T-t)^2}-2t\frac{1-y}{T-t}$. Therefore

$$I:=\int_t^T\frac{\left(T^2-\sigma^2\right)^{2q}}{(\sigma-t)^\alpha}d\sigma=\int_0^1\left[(1-y^2)+2t\frac{1-y}{T-t}\right]^{2q}(T-t)^{4q-\alpha+1}y^{-\alpha}dy$$

or

$$I=(T-t)^{4q-\alpha+1}\int_0^1 y^{-\alpha}(1-y)^{2q}\left[(1+y)+\frac{2t}{T-t}\right]^{2q}dy.$$

By the binomial formula we may write

$$I=\sum\nolimits_{l=0}^{2q}2^{2q-l}C_l^{2q}t^{2q-l}(T-t)^{2q-\alpha+l+1}\int_0^1 y^{-\alpha}(1-y)^{2q}(1+y)^l dy$$

where $C_l^{2q}=\frac{2q(2q-1)(2q-2)\dots(2q-l+1)}{l!}$. Using the formula

$$\int_0^1(1-\tau)^{u-1}\tau^{v-1}d\tau=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},\quad u,v>0$$

we obtain

$$I = \sum_{l=0}^{2q} 2^{2q-l} C_l^{2q} M_l t^{2q-l} (T-t)^{2q-\alpha+l+1}$$

where $M_l := \sum_{n=0}^{l} C_n^l \frac{\Gamma(2q+1)\Gamma(n-\alpha+1)}{\Gamma(2q-\alpha+n+2)}$. Hence

$$\begin{aligned} &D_-^\alpha \left(1 - \tfrac{t^2}{T^2}\right)^{2q} \\ &\quad = \frac{-T^{-4q}}{\Gamma(1-\alpha)} \sum_{l=0}^{2q} 2^{2q-l} C_l^{2q} M_l t^{2q-l-1} (T-t)^{2q-\alpha+l} \left[(2q-l)T - (4q-\alpha+1)t\right]. \end{aligned}$$

Substituting this expression in (20) we obtain that

$$\begin{aligned} &\int_{Q_T} \left|D_-^\alpha \varphi\right|^q (\varphi h)^{-\frac{q}{p}} \\ &\quad = \frac{T^{1-\alpha q}}{\Gamma(1-\alpha)} \int_{Q_1} h^{1-q} \left(1-\tau^2\right)^{2q(1-q)} \Phi \\ &\quad \times \left| \sum_{l=0}^{2q} 2^{2q-l} C_l^{2q} M_l \tau^{2q-l-1} (1-\tau)^{2q-\alpha+l} \left[(2q-l) - (4q-\alpha+1)\tau\right] \right|^q. \end{aligned}$$

It is not difficult to see that, as $l + 2 - \alpha > 0$, we have the estimation

$$\int_{Q_T} \left|D_-^\alpha \varphi\right|^q (\varphi h)^{-\frac{q}{p}} \le L T^{1-\alpha q} \int_{Q_1} h^{1-q} \Phi \tag{21}$$

with $L := \frac{1}{\Gamma(1-\alpha)} \left[\sum_{l=0}^{2q} 2^{2q-l} C_l^{2q} M_l (6q - l - \alpha + 1)\right]^q$.

Now observing that

$$\inf_{|x|>R} \left(u_1(x) h^{q-1}\right) \int_{\mathbb{R}^N} h^{1-q} \Phi \le \int_{\mathbb{R}^N} u_1(x) \varphi_0(x) = \int_{\mathbb{R}^N} u_1(x) \Phi(x)$$

and gathering the relations (17)–(19) and (21), we infer that

$$\inf_{|x|>R} \left(u_1(x) h^{q-1}\right) \int_{\mathbb{R}^N} h^{1-q} \Phi \le \left[\gamma^q T^{1-2q} + k^q R^{-2q} T + L T^{1-\alpha q}\right] C_{1/4} \int_{Q_1} h^{1-q} \Phi. \tag{22}$$

Taking the supremum with respect to t of both sides of (22) and making use of the assumption $\inf_{t\in\mathbb{R}^+} h(t,x) > 0$, we can divide by $\int_{\mathbb{R}^N} \left(\inf_{t\in\mathbb{R}^+} h\right)^{1-q} \Phi > 0$ (recall that $1 - q < 0$). Then, letting $R \to +\infty$, we obtain

$$\liminf_{|x|\to\infty} \left(u_1(x) h^{q-1}\right) \le \left(\frac{\gamma^q}{T^{2q-1}} + L T^{1-\alpha q}\right) C_{1/4}, \tag{23}$$

which completes the proof. □

We can immediately deduce the following results.

Corollary 3.3. *Let* $p > 1$*. Assume that*

$$\liminf_{|x|\to\infty} u_1(x)h^{\frac{p}{p-1}}(t,x) = +\infty,$$

then problem (WE)–(1) *has no weak local solution for any* $T > 0$*.*

Corollary 3.4. *Suppose that* $1 < p < \frac{1}{1-\alpha}$ *and* $u_1(x) \geq 0$*. If* (WE)–(1) *admits a global weak solution, then*

$$\liminf_{|x|\to\infty} \left[u_1(x) \Big(\inf_{t\in\mathbb{R}^+} h(t,x) \Big)^{q-1} \right] = 0.$$

Proof. Suppose that (WE)–(1) has a global weak solution and that

$$P := \liminf_{|x|\to\infty} \left[u_1(x) \Big(\inf_{t\in\mathbb{R}^+} h(t,x) \Big)^{q-1} \right] > 0.$$

Then from (23), it appears that

$$T \leq \max \left\{ \left(\frac{\gamma^q + L}{P} C_{1/4} \right)^{\frac{1}{\alpha q - 1}}, \left(\frac{\gamma^q + L}{P} C_{1/4} \right)^{\frac{1}{2q-1}} \right\}.$$

This is a contradiction. □

The next theorem gives another necessary condition for existence of a global weak solution. At the same time it provides (in case $u_1(x) \geq 0$) a sufficient condition for blow up in finite time of any possible local solution.

Theorem 3.5. *Suppose that* $1 < p < \frac{1}{1-\alpha}$ *and* u *is a global weak solution to* (WE)–(1)*. Then, there exists a positive constant* K *such that*

$$\liminf_{|x|\to\infty} \left(u_1(x)h^{q-1} |x|^{2\frac{\alpha q - 1}{\alpha}} \right) \leq K.$$

Proof. As $p < \frac{1}{1-\alpha}$, we have $\alpha q - 1 > 0$ and then for $T > 1$ we may write

$$\gamma^q T^{1-2q} + k^q R^{-2q} T + L T^{1-\alpha q} \leq \frac{\gamma^q + L}{T^{\alpha q - 1}} + k^q R^{-2q} T.$$

From (22) we see that

$$\inf_{|x|>R} \left(u_1(x) h^{q-1} \right) \int_{\mathbb{R}^N} h^{1-q}\Phi \leq \left(\frac{\gamma^q + L}{T^{\alpha q-1}} + k^q R^{-2q} T \right) C_{1/4} \int_{Q_1} h^{1-q}\Phi. \tag{24}$$

Minimizing the left hand side expression in (24) with respect to T, we obtain

$$\inf_{|x|>R} \left(u_1(x)h^{q-1}\right) \int_{\mathbb{R}^N} h^{1-q}\Phi \le \left(k^q + \frac{\gamma^q + L}{K_1}\right) C_{1/4} K_1^{\frac{1}{\alpha q}} R^{-2\frac{\alpha q-1}{\alpha}} \int_{Q_1} h^{1-q}\Phi\,,$$

where $K_1 := \frac{1}{k^q}(\alpha q - 1)(\gamma^q + L)$. Now, using the assumptions on Φ (namely, $R < |x| < 2R$), we see that

$$\begin{aligned} &\inf_{|x|>R} \left(u_1(x)h^{q-1}\,|x|^{2\frac{\alpha q-1}{\alpha}}\right) \int_{\mathbb{R}^N} h^{1-q}\,|x|^{-2\frac{\alpha q-1}{\alpha}}\,\Phi \\ &\qquad \le \left(k^q + \frac{\gamma^q + L}{K_1}\right) C_{1/4} 2^{-2\frac{\alpha q-1}{\alpha}} K_1^{\frac{1}{\alpha q}} \int_{Q_1} h^{1-q}\,|x|^{-2\frac{\alpha q-1}{\alpha}}\,\Phi. \end{aligned} \tag{25}$$

To conclude it suffices to take the sup with respect to t of both sides of (25) and divide by $\int_{\mathbb{R}^N} \left[\inf_{t\in\mathbb{R}^+} h(t,x)\right]^{1-q} |x|^{-2(\alpha q-1)/\alpha}\,\Phi$. □

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