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A Gauss–Bonnet Formula for Metrics with Varying Signature

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Abstract. A Gauss–Bonnet formula for compact orientable connected Riemannian or Lorentzian 2-manifolds is well-known. We investigate singular metrics on 2-manifolds with varying signature. Such metrics are necessarily degenerate at some points of M where most of the usual definitions for geometric quantities break down. We prove that under some additional assumptions there is a Gauss–Bonnet formula for compact orientable connected 2-manifolds with a singular metric. Some examples are given.

Keywords. Gauss–Bonnet formula, singular metric, pseudo-geodesic, generic metric

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1. Introduction

The Gauss–Bonnet theorem is one of the most important results in differential geometry. The so-called global Gauss–Bonnet formula for a metric g on a compact orientable connected surface M,

$$\int_M K\,dA = 2\pi\chi(M), \tag{1}$$

connects the integral of the intrinsic Gaussian curvature K with the Euler characteristic $\chi(M)$ which is topological invariant. The history of such a formula began with Gauss in [6] in a local version for geodesic triangles (where on the left-hand side of (1) is additionally the sum of the three exterior angles of the triangle, for details see Section 5). Later, Avez in [1] and Chern in [4] independently obtained a global Gauss–Bonnet formula (1) for a compact orientable connected semi-Riemannian manifold (for higher dimension, $K\,dA$ is substituted by an expression in the curvature form). In all of these cases the signature of the metric is constant.

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The question is whether the Gauss–Bonnet formula (1) still holds, if the metric g varies the signature on a connected 2-manifold. These metrics necessarily degenerate at some points of the manifold. In order to make this more precise, we define a *singular metric* $g = \langle \cdot, \cdot \rangle$ on a 2-manifold M as a symmetric 2-tensor on the tangent bundle TM. The *singular locus* $S(g)$ *of* g is defined to be the points of M where g is degenerate (i.e., $S(g) := \{p \in M \mid \text{rank } g < 2\}$).

In 1984, Pelletier in [9] found under some assumptions a global Gauss–Bonnet formula for particular singular metrics on a 2-manifold. These singular metrics degenerate only at distinct simply closed curves. Further assumptions are made concerning the behaviour of the metric in the neighbourhood of the singular locus.

The aim of this paper is to consider singular metrics where the singular locus is the union of simply closed curves which only meet in pairs and transversally. The main result of the present paper is the following theorem where the assumptions will be explained in the sequel of this paper.

Theorem A. *Let* (M, g) *be a compact orientable connected generic* 2*-manifold without boundary. If the singular locus* $S(g) \neq \emptyset$ *is pseudo-geodesic and pseudo-orthogonal, then the Gauss–Bonnet formula*

$$\int_M \overline{K} dA = 2\pi\chi(M)$$

holds, where $\overline{K} := \lambda K$ *is the Gaussian curvature-with-sign where* $\lambda(p)$ *is* -1 *if the signature of* g *is* $(0, 2)$ *at* p *and* 1 *otherwise.*

Theorem A is stated and proved as Theorem 2 in Section 6. In Section 2 we introduce generic 2-manifolds which provides a sufficiently smooth transition between parts of different signature. The pseudo-geodesics are an extension of the concept of a geodesic to the singular case and will be discussed in Section 3. In Section 4 we extend the concept of orthogonality to the singular locus. An overview of local Gauss–Bonnet formulas is given in Section 5. Finally, in Section 6 our main results are stated and proved. Futhermore, some examples of generic 2-manifolds are given and counterexamples where the assertion of Theorem A does not hold.

2. Generic singular metrics

Let us assume in the whole paper that M is a 2-manifold without boundary. We define a *singular metric* $g = \langle \cdot, \cdot \rangle$ on M as a symmetric 2-tensor on the tangent bundle TM. The *singular locus* $S(g)$ *of* g is defined to be the points of M where g is degenerate (i.e., $S(g) := \{p \in M \mid \text{rank } g < 2\}$). We call (M, g) a *singular* 2*-manifold* if M is a 2-manifold and g is a singular metric on M and we denote by $\mathcal{N}(p) := \{X \in T_pM \mid \langle X, Y\rangle = 0 \text{ for all } Y \in T_pM\}$ the *nullspace of g in p*. In this paper we assume sufficient regularity for the metric g. The signature of such a singular metric can vary on the 2-manifold M. Generally, the behaviour of a singular metric is vicious, for example, if g is a singular metric with $S(g) = M$, no quantity of a Riemannian resp. Lorentzian manifold (like Gaussian curvature, geodesic curvature, etc.) exists in the usually sense. For obtaining a Gauss–Bonnet formula (1), the singular locus of g has necessarily to be of measure zero (with respect to some Riemannian metric on M), otherwise we can change the Euler characteristic on the right-hand side of (1) by gluing a loop onto the singular locus without change the integral on the left-hand side. Another problem is the integrability of the Gaussian curvature as an improper integral which depends hardly on the transition between two parts of different signature. Therefore, we study singular metrics having a sufficient smooth transition between the parts of different signature.

Definition 1. Let (M, g) be a singular 2-manifold. Then (M, g) is called *generic* if the following conditions hold:

G_1: The singular locus $S(g)$ is a union of simply closed smooth curves $S_0, \dots, S_m$ $(m \geq 0)$ which only can meet in pairs and transversally. The set of all intersection points, denoted by $I(g)$, of curves S_i and S_j $(i \neq j)$ are called the *intersection points* and the curves S_i are called the *singular curves*.

G_2: The metric g induces on $S(g) - I(g)$ a regular metric (i.e., $\dim \mathcal{N}(p) = 1$ for all $p \in S(g) - I(g)$).

G_3: For all $p \in S(g) - I(g)$ and for all vector field V with $0 \neq V_p \in \mathcal{N}(p)$ it holds $V\langle V, V\rangle_{|p} \neq 0$.

G_4: For all intersection points $p \in I(g)$ the metric g vanishes (i.e., $g_{|_p} = 0$).

G_5: For each simply closed curve γ in $S(g)$ and for all intersection points $p \in I(g)$ on γ it holds $(\dot\gamma\langle\dot\gamma, \dot\gamma\rangle)_p \neq 0$.

Notice that the conditions $G_2 - G_5$ are parameter independent. In local coordinates around an intersection point they state that the determinant function $\det g_{ij}$ of g is a Morse function. A real function f is called a *Morse function* if all critical points p of f (i.e., $\operatorname{grad} f_{|_p} = 0$) are non-degenerate (i.e., the Hessian of f at p has maximal rank).

Pelletier has considered in [9] another type of generic metrics. His generic metrics have no intersection points in the singular locus and the rank of the metric is 1 on the singular locus. These singular metrics are much more special and it is not obvious how to remove the intersection points. However, the situation can be handled for a generic metric in our sense with intersection points.

Lemma 1. *Let (M, g) be a generic 2-manifold, then the following holds:*

(i) *The distribution of the signature of g around a connected piece of $S(g) - I(g)$ is either*

$$(0,2)\ \left|^{(-)}\right.\ (1,1) \qquad \text{or} \qquad (2,0)\ \left|^{(+)}\right.\ (1,1)$$

where $(+)$ (resp. $(-)$) means that the segment is spacelike (resp. timelike).

(ii) *The distribution of the signature of g around an intersection point is the following (up to rotations around the intersection point)*

$$\begin{array}{c|c} (1,1) & {}_{(+)}(2,0) \\ \hline {}^{(-)} & {}^{(+)} \\ (0,2) & {}^{(-)}(1,1) \end{array}$$

Proof. (i) From G_2 it follows that the connected piece of $S(g) - I(g)$ is either spacelike $(+)$ or timelike $(-)$. Therefore, both adjacent components of $M - S(g)$ have a spacelike resp. timelike direction. Furthermore, G_3 makes sure that we have always a change of the signature.

(ii) From G_2 and G_5 it follows that a singular curve S_i of $S(g)$ changes the type at an intersection point (from spacelike $(+)$ to timelike $(-)$ resp. vice versa). By (i) the described distribution is the only possibility. $\square$

3. Pseudo-geodesics in the singular locus

For a singular 2-manifold (M, g) the Levi–Civita connection ∇ is well defined only outside the singular locus. In order to obtain a kind of a connection in every point of M, we define the *Levi–Civita dual connection* $\square_X Y(Z)$ (cf. [7] and [9]) by the right-hand side of the Koszul formula

$$\begin{aligned} \square_X Y(Z) := \frac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle \Big) \end{aligned} \tag{2}$$

for all vector fields X, Y, Z on M (notice that $\square_X Y(Z)$ is defined everywhere on M). Outside the singular locus the Levi–Civita dual connection is nothing but

$$\square_X Y(Z) = \langle \nabla_X Y, Z\rangle \tag{3}$$

for all vector fields X, Y, Z. Therefore, in this setting the equation $\nabla_{\dot\gamma}\dot\gamma = \alpha\dot\gamma$ of an ordinary geodesic γ turns into the condition

$$\Box_X X(Z) = 0, \tag{4}$$

whenever X is a vector field tangential to γ and Z is a vector field orthogonal to γ where X and Z do not vanish on γ. Notice that (4) does not depend on the choise of X and Z (i.e., if (4) holds for one then also for all such vector fields). Furthermore, (4) is well defined even in the singular locus.

Definition 2. Let (M, g) be a generic 2-manifold. We call a curve γ a *pseudo-geodesic* if $\Box_X X(Z) = 0$ holds whenever X is a vector field tangential to γ and Z is a vector field orthogonal to γ where X and Z do not vanish on γ. The singular locus $S(g)$ is called *pseudo-geodesic* if $S(g) - I(g)$ consists only of pseudo-geodesics.

Let (M, g) be a generic 2-manifold and let S_i be a singular curve of $S(g)$. In local coordinates around a point $p \in S_i - I(g)$, the equality (4) leads to the following. By $G_2 - G_5$, we can always choose a parametrization

$$\phi_1 : (-1,1)^2 \longrightarrow U_p \tag{5}$$

in a neighbourhood U_p of $p = \phi_1(0,0)$ with coordinates (x, y) such that the conditions

$$\phi_1((-1,1)\times\{0\}) \subset S_i \quad \text{and} \quad \phi_1((-1,1)^2) \cap S(g) = (-1,1)\times\{0\}$$

hold and g has the expression $\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$ with

$$\begin{aligned} g_{11} &\neq 0 \quad \text{on } (-1,1)^2 && (6)\\ g_{22} &= 0 \quad \text{for } y = 0 && (7)\\ g_{22} &\neq 0 \quad \text{for } y \neq 0 && (8)\\ \tfrac{\partial}{\partial y} g_{22} &\neq 0 \quad \text{for } y = 0. && (9) \end{aligned}$$

Writing $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$, in these coordinates the Levi–Civita connection ∇ is outside the singular locus (i.e., $y \neq 0$) nothing but

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{2} \Gamma_{ij}^k \partial_k$$

with the Christoffel symbols $\Gamma_{ij}^k = \sum_m \Gamma_{ij,m} g^{mk}$ where $g^{mk} := (g_{rs})^{-1}$. Writing

$$(\widehat{g}^{mk}) := (\det g_{rs}) \cdot (g^{mk}) \quad \text{and} \quad \widehat{\Gamma}_{ij}^k := \sum_m \Gamma_{ij,m}\widehat{g}^{mk},$$

we obtain a pseudo-connection $\widehat{\nabla}$ defined on $(-1,1)^2$ with

$$\nabla_{\partial_i}\partial_j = \frac{1}{\det\, g_{ij}} \underbrace{\sum_{k=1}^{2} \widehat{\Gamma}_{ij}^k \partial_k}_{=:\widehat{\nabla}_{\partial_i}\partial_j} = (\det\, g_{ij})^{-1} \cdot \widehat{\nabla}_{\partial_i}\partial_j. \tag{10}$$

The following lemma tells us how to extend the Levi–Civita connection ∇ to $(-1,1)^2$ in this parametrization. It follows from the rule of Bernoulli–l'Hospital.

Lemma 2. *Let $f, h : (-1,1)^2 \to \mathbb{R}$ be two smooth functions. If $\{(x,y) \mid f(x,y) = 0\} = \{(x,y) \mid h(x,y) = 0\} = (-1,1) \times \{0\}$ and $\frac{\partial}{\partial y} h(x,0) \neq 0$ for all x, then $F := \frac{f}{h}$ is extendible to $(-1,1)^2$ with $F(x,0) = \frac{\partial f}{\partial y} / \frac{\partial h}{\partial y} \mid_{(x,0)}$.*

Choosing $h := \det\, g_{ij}$ and $f := \widehat{\Gamma}_{ij}^k$ and considering the fact that from (6) and (9) follows

$$\tfrac{\partial}{\partial y} h = \tfrac{\partial}{\partial y}(\det\, g_{ij}) = \tfrac{\partial}{\partial y} g_{11} \cdot g_{22} + \tfrac{\partial}{\partial y} g_{22} \cdot g_{11} = \tfrac{\partial}{\partial y} g_{22} \cdot g_{11} \neq 0$$

for $y = 0$, all assumptions of Lemma 2 except the following are satisfied. The leftover assumption is $f = \widehat{\Gamma}_{ij}^k = 0$ for $y = 0$. By determining

$$\widehat{\nabla}_{\partial_1}\partial_1 = \frac{1}{2}\left(\tfrac{\partial}{\partial x} g_{11} \cdot g_{22}\, \partial_1 - \tfrac{\partial}{\partial y} g_{11} \cdot g_{11}\, \partial_2\right) \tag{11}$$

$$\widehat{\nabla}_{\partial_2}\partial_1 = \widehat{\nabla}_{\partial_1}\partial_2 = \frac{1}{2}\left(\tfrac{\partial}{\partial y} g_{11} \cdot g_{22}\, \partial_1 + \tfrac{\partial}{\partial x} g_{22} \cdot g_{11}\, \partial_2\right) \tag{12}$$

$$\widehat{\nabla}_{\partial_2}\partial_2 = \frac{1}{2}\left(-\tfrac{\partial}{\partial x} g_{22} \cdot g_{22}\, \partial_1 + \tfrac{\partial}{\partial y} g_{22} \cdot g_{11}\, \partial_2\right), \tag{13}$$

it turns out that in the singular locus (i.e., $y = 0$) we have

$$\widehat{\nabla}_{\partial_1}\partial_1 = -\frac{1}{2}\left(\tfrac{\partial}{\partial y} g_{11} \cdot g_{11}\, \partial_2\right) \tag{14}$$

$$\widehat{\nabla}_{\partial_2}\partial_1 = \widehat{\nabla}_{\partial_1}\partial_2 = 0 \tag{15}$$

$$\widehat{\nabla}_{\partial_2}\partial_2 = \frac{1}{2}\left(\tfrac{\partial}{\partial y} g_{22} \cdot g_{11}\, \partial_2\right) \neq 0 \tag{16}$$

On the other hand, by (2) the pseudo-geodesic condition of the singular locus is equivalent to

$$\Box_{\partial_1}\partial_1(\partial_2) = -\frac{1}{2}\tfrac{\partial}{\partial y} g_{11} = 0 \tag{17}$$

for $y = 0$. Combining Lemma 2 and (14)–(17) we obtain the following proposition.

Proposition 1. *Let* (M, g) *be a generic* 2*-manifold then the following conditions are equivalent:*

(i) $S(g)$ *is pseudo-geodesic.*

(ii) *In the parametrization* (5), $\nabla_{\partial_i}\partial_1$ $(i = 1, 2)$ *is extendible to* $(-1, 1)^2$.

(iii) *In the parametrization* (5), $\frac{\partial}{\partial y} g_{11} = 0$ *for* $y = 0$.

Proof. (ii) $\Leftrightarrow$ (iii): As $g_{11} \neq 0$ on $(-1, 1)^2$, this follows from (14)–(15), (17) and Lemma 2. (i) $\Leftrightarrow$ (iii): This follows directly from (17). $\square$

Parameter independently, Proposition 1 (ii) states that the Levi–Civita connection is local extendible to

$$\nabla' : \mathfrak{X}(M) \times \mathfrak{X}^{\top}(M) \to \mathfrak{X}(M)$$

where $\mathfrak{X}(M)$ is the set of all vector fields on M and $\mathfrak{X}^{\top}(M)$ is the set of all vector fields on M which are tangential to S_i. By Proposition 1 (iii) we have a simple method to decide whether the singular locus is pseudo-geodesic or not. Furthermore, Proposition 1 (iii) is also helpful for construction of generic metrics with a pseudo-geodesic singular locus. In [10] the condition (ii) in Proposition 1 is called *auto-parallel*. For further and general propositions we refer to [7] and [10].

4. Pseudo-orthogonality of the singular locus

Let (M, g) be a generic 2-manifold. As the singular metric g is degenerate in $S(g)$, we are not able to measure angles in the usual way. In order to talk about orthogonality of two singular curves S_i and S_j at an intersection point $p \in S_i \cap S_j$ with respect to g, we make the following considerations.

Let S_i be a singular curve of $S(g)$. As the rank of g is equal to 1 on $S_i - I(g)$ there exists a non-vanishing vector field N^i on $S_i - I(g)$ with $N^i_p \in \mathcal{N}(p)$ for all $p \in S_i - I(g)$. By G_2, N^i is not tangential to S_i. If N^i is extendible in the sense that there exists a non-vanishing vector field $\overline{N}^i$ on S_i with $\overline{N}^i \in \mathbb{R}N^i$ on $S_i - I(g)$ then the extension $\overline{N}^i$ at the intersection point can play the role of an orthogonal direction of S_i. More precisely, we can introduce the following definition.

Definition 3. Let (M, g) be a generic 2-manifold. We call the singular locus *pseudo-orthogonal* if for every intersection point $p = S_i \cap S_j$ $(i \neq j)$ of two singular curves there exist around p non-vanishing vector fields N^i on S_i and N^j on S_j satisfying the following conditions

(i) N^i resp. N^j lies in the nullspace on S_i resp. S_j.

(ii) It hold $N^i_p \in T_pS_j$ and $N^j_p \in T_pS_i$.

In local coordinates around an intersection point p of two singular curves S_i and S_j ($i \neq j$), we can express the conditions in Definition 3 in the following way. There is always a parametrization

$$\phi_2(x,y) : (-1,1)^2 \to U_p \subset M \tag{18}$$

in a neighbourhood U_p of an intersection point $p = \phi_2(0,0)$ with

$$\phi_2\big((-1,1)\times\{0\}\big) = S_i \cap U_p, \quad \phi_2\big(\{0\}\times(-1,1)\big) = S_j \cap U_p$$

and $S_i \cup S_j \supset S(g) \cap U_p$.

As $g_{11}(x,0) = 0$ only if $x = 0$, and $g_{22}(0,y) = 0$ only if $y = 0$, the vector fields N^i and N^j in Definition 3 can be chosen as

$$N^i_{(x,0)} := \Big(-\frac{g_{12}}{g_{11}}, 1\Big)_{|(x,0)} \quad \text{and} \quad N^j_{(0,y)} := \Big(1, -\frac{g_{12}}{g_{22}}\Big)_{|(0,y)},$$

with $x, y \in (-1,1)\backslash\{0\}$. By G_5 and the rule of Bernoulli–l'Hospital , for a generic metric these two vector fields are always extendible to $x, y \in (-1,1)$ with

$$N^i_{(0,0)} := \left(-\frac{\frac{\partial}{\partial x}g_{12}}{\frac{\partial}{\partial x}g_{11}}_{|(0,0)}, 1\right) \quad \text{and} \quad N^j_{(0,0)} = \left(1, -\frac{\frac{\partial}{\partial y}g_{12}}{\frac{\partial}{\partial y}g_{22}}_{|(0,0)}\right). \tag{19}$$

In the sense of Definition 3, the singular locus $S(g)$ is pseudo-orthogonal if and only if the extension of N^i in $x = 0$ resp. N^j in $y = 0$ is

$$N^i_{(0,0)} = (0,1) \quad \text{resp.} \quad N^j_{(0,0)} = (1,0). \tag{20}$$

This leads to the following proposition.

Proposition 2. *Let (M,g) be a generic 2-manifold. Then the following conditions are equivalent:*

(i) *The singular locus is pseudo-orthogonal.*

(ii) *At all intersection points $p \in I(g)$ we have $\frac{\partial}{\partial x}g_{12} = \frac{\partial}{\partial y}g_{12} = 0$ in the parametrization* (18).

(iii) *There exists a parametrization* (18) *which is orthogonal in the sense that*

$$(g_{ij}) = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

with consequently $g_{11} \cdot g_{22} = 0$ if and only if $x \cdot y = 0$.

Proof. (i) $\Leftrightarrow$ (ii): This follows directly from (19) and (20). (iii) $\Rightarrow$ (ii) is obvious. (i) $\Rightarrow$ (iii): By (20), in the parametrization (18), we can always find orthogonal vector fields $V^1, V^2 \neq 0$ on $(-1,1)^2$ with

$$V^1_{(x,0)} = N^i_{(x,0)},\ V^1_{(0,y)} \in TS_j \quad \text{and} \quad V^2_{(0,y)} = N^j_{(0,y)},\ V^2_{(x,0)} \in TS_i.$$

We obtain the desired parametrization (orthogonal coordinates) by reparametrizing so that the derivative of the coordinate lines point into the directions of V^1 resp. V^2 and so that the x-axis and y-axis are preserved. □

5. Local Gauss–Bonnet formulas and the topological structure of Lorentzian parts

Let (M,g) be a compact orientable connected generic 2-manifold then $M' := M - S(g)$ is a union of connected orientable open 2-manifolds $\{M_1,\dots,M_n\}$, each with constant signature (2,0), (1,1) or (0,2). If the metric g is Riemannian on M_i (i.e., with signature (2,0)), then there is the ordinary (exterior) angle, the ordinary geodesic curvature κ_g and the ordinary Gaussian curvature K. Furthermore, there holds the well-known local Gauss–Bonnet formula for a compact oriented 2-manifold $D \subset M_i$ with a piecewise smooth boundary Γ

$$\int_D K d\mathrm{A} + \int_\Gamma \kappa_g d\mathrm{s} + \sum_i \alpha_i = 2\pi\chi(D), \tag{21}$$

where the α_i's are the exterior angles at the non-smooth points of Γ and $\chi(D)$ is the Euler characteristic of D.

In the case where g has the signature (0,2), the local Gauss–Bonnet formula (21) holds for $-g$. By taking account of that $K_g = -K_{-g}$ and $\kappa_g = -\kappa_{-g}$ (where K_{-g} (resp. κ_{-g}) is the Gaussian curvature (resp. geodesic curvature) with respect to $-g$), formula (21) turns into

$$\int_D -K_g d\mathrm{A} + \int_\Gamma -\kappa_g ds + \sum_i \alpha_i = 2\pi\chi(D), \tag{22}$$

where α_i are the exterior angles in the Riemannian sense. Notice that the sign in the first integral is the reason of the Gaussian curvature-with-sign in Theorem A.

In the Lorentzian case (i.e., the metric g has signature (1,1)), there are different local Gauss–Bonnet formulas. Birman and Nomizu (cf. [2]) appear to be the first to consider a Lorentzian version of the classical local Gauss–Bonnet theorem. They assumed that the boundary consists only of timelike segments. Dzan (cf. [5]) proved a local Gauss–Bonnet formula for regions with either timelike or spacelike piecewise smooth boundary (using an imaginary 'geodesic curvature' and a special kind of angle). Later, LAW (cf. [8]) extended this to a local Gauss–Bonnet formula for regions with piecewise smooth non-null (i.e., timelike or spacelike) boundary. In this paper we will use the local Gauss–Bonnet formula from LAW. Before we introduce this local Gauss–Bonnet formula we have to define the complex exterior angle of two non-lightlike vectors in the tangent plane which LAW (and DZAN) used.

Let h be a Lorentzian metric

$$h = dx_1^2 - dx_2^2,$$

on $\mathbb{R}^2$ and $\{\mathbf{t}, \mathbf{x}\}$ a basis defined by $\mathbf{t} := (1,0)^\mathsf{T}$ and $\mathbf{x} := (0,-1)^\mathsf{T}$. With respect to this choice of $\{\mathbf{t}, \mathbf{x}\}$, the two null directions of g divide the tangent space $\mathbb{R}^2$ into quadrants, each containing one component of $S^{1,1} := \{u \mid h(u,u) = \pm 1\}$. If $u, v \in S^{1,1}$ are lying in the same quadrant, then there is a unique defined number α with $u = L(\alpha)v$ where

$$L(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

and $\cosh|\alpha| = |h(u,v)|$. If $u, v \in S^{1,1}$ do not lie in the same quadrant, we have to rotate them. Writing

$$C_+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad C_- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

there is a unique number α and a unique number $n \in \{0,1,2,3\}$ such that

$$u = L(\alpha)C_+^n v = L(\alpha)C_-^{4-n} v. \tag{23}$$

Definition 4 ([8, Definition 3.7]). Let u and v be non-null unit vectors. With the notations above, the *oriented angle* from v to u in the positive resp. negative sense is defined by $(v,u)_+ := \alpha + n(i\frac{\pi}{2})$ resp. $(v,u)_- := \alpha + (4-n)(-i\frac{\pi}{2})$.

Notice that if u and v are orthogonal, then the oriented angle is purely imaginary. The imaginary part is then 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ in the positive sense. In this paper we only consider orthogonal unit vectors, so we refer for further considerations to [5] and [8].

Theorem 1 ([8, Theorem 5.1]). *Let g be a Lorentzian metric on a domain D with a piecewise smooth boundary Γ consisting of a finite number of non-null segments, then*

$$\int_D K dA + \int_\Gamma \kappa_g ds + \sum_i \beta_i = \pm 2\pi i \tag{24}$$

holds, where β_i denotes the complex oriented exterior angle at the non-smooth points of Γ.

We only take an interest in the real part of (24). Therefore, we do not have to regard the orientation of the exterior angles. If the boundary is orthogonal at its non-smooth points then the exterior angle is purely imaginary. The real part of the Gauss–Bonnet formula (24) turns into

$$\int_D K dA + \int_\Gamma \kappa_g ds = 0. \tag{25}$$

Notice that in the real part of (24) resp. (25), the Euler characteristic $\chi(D)$ is missing resp. zero, so that it is necessary to study the topological structure of the connected components of $M - S(g)$ where g is Lorentzian (cf. Propositon 3).

In order to obtain a global Gauss–Bonnet formula for generic 2-manifolds using these local Gauss–Bonnet formulas we need to approximate the singular locus. For the simply closed singular curves S_i of the singular locus we have tubular neighbourhoods $\Psi_i : S_i \times (-\varepsilon, \varepsilon) \to M$, which we can always find (for example by taking a Riemannian metric on M). Notice that ε is fixed for every curve S_i. Next to an intersection point $p \in S_i \cap S_j$ $(i \neq j)$ the intersection of the two tubular neighbourhoods Ψ_i and Ψ_j are not empty. Let the singular locus $S(g)$ be pseudo-orthogonal then we obtain from Proposition 2(iii) in a neighbourhood of any $p \in I(g)$ orthogonal coordinates. By taking the coordinate lines as segments of leaves we can always construct a tubular neighbourhood Ψ_i of every S_i with the property that two leaves $\Psi_i^t := \Psi_i(S_i \times \{t\})$ and $\Psi_j^t := \Psi_j(S_j \times \{t\})$ $(i \neq j)$ can only meet orthogonally. Furthermore, if we choose $\varepsilon > 0$ small enough these tubular neighbourhoods Ψ_i have non-lightlike leaves outside the singular locus. This is possible because pieces of the curves S_i between two intersection points are non-lightlike (cf. G_2). In this sense we call a set $\Psi = \{\Psi_0, \ldots, \Psi_m\}$ of tubular neighbourhoods *proper* if the conditions above are satisfied. This leads to the following proposition.

Proposition 3. *Let (M, g) be a orientable generic 2-manifold with a pseudo-orthogonal singular locus $S(g) \neq \emptyset$ and let M_i be a Lorentzian connected component of $M - S(g)$. Furthermore, let $\Psi = \{\Psi_0, \ldots, \Psi_m\}$ be a proper set of tubular neighbourhoods then $M_{it} := M_i - \bigcup_{k=0}^m \Psi_k(S_k \times (-t, t))$ is either a topological disc with a boundary consisting of 4 segments which are orthogonal at the edges or a topological cylinder with a boundary consisting of two simply closed curves.*

Proof. Let us introduce the following notations:

$e_1, \ldots, e_n$: the non-smooth connected components of ∂M_{it}

$f_1, \ldots, f_k$: the smooth connected components of ∂M_{it}

l : the number of loops inside M_{it}

Notice that the edges of e_j are timelike and spacelike in alternating mane (cf. Lemma 1). As g is Lorentzian on M_i there exists a non-vanishing timelike vector

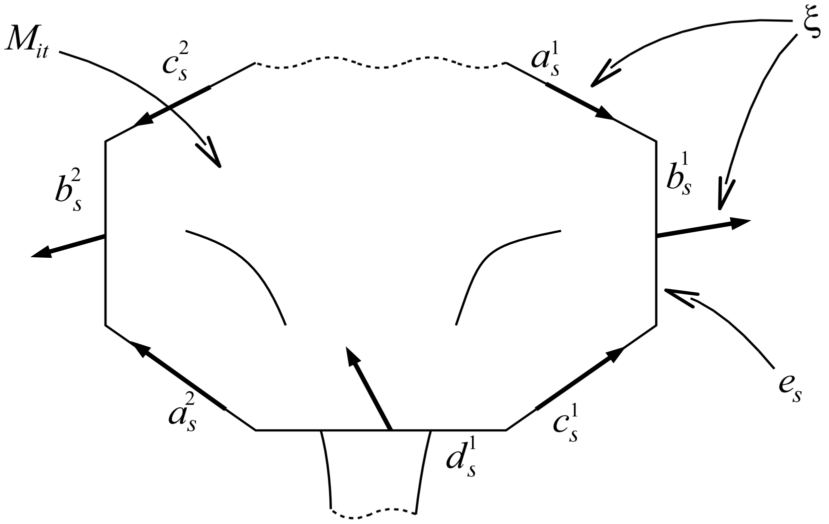

Figure 1: The connected component e_i of ∂M_{it}.

field ξ on M_i resp. M_{it}. We can assume that ξ is tangential to the edges of e_s which are timelike and also tangential to f_s where f_s is timelike. By Lemma 1 and the existence of ξ, it follows that the non-smooth connected components e_s of ∂M_{it} has the form

$$e_s = a_s^1\, b_s^1\, c_s^1\, d_s^1\, a_s^2\, b_s^2\, c_s^2\, d_s^2 \ldots\, a_s^{k_s}\, b_s^{k_s}\, c_s^{k_s}\, d_s^{k_s},$$

where a_s^j and c_s^j are timelike and b_s^j and d_s^j are spacelike so that the behaviour of ξ is given in Figure 1. Notice that the rotation of ξ along e_s follows from the fact that Ψ is proper and Lemma 1 (ii).

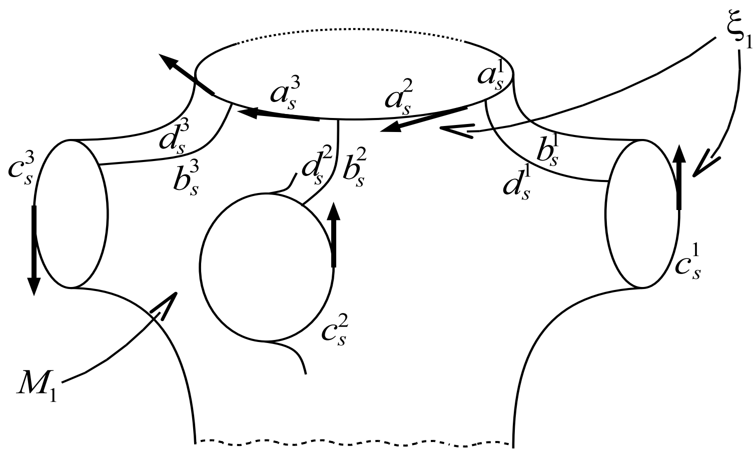

Figure 2: The 2-manifold M_1.

By gluing b_s^j and d_s^j together for every s and j like given in Figure 2 so that the direction of ξ is preserved, we obtain a orientable 2-manifold M_1 with smooth boundary consisting of simply closed smooth curves $(1 \leq s \leq n)$

$$a_s^1\, a_s^2 \ldots a_s^{k_s},\ c_s^1,\ \ldots,\ c_s^{k_s},\ f_1,\ f_2,\ \ldots,\ f_k.$$

As the direction of ξ was preserved by the gluing, on M_1 there exists a non-vanishing vector field ξ_1 so that ξ_1 is tangential to $a_s^1\, a_s^2 \ldots a_s^{k_s},\ c_s^1,\ \ldots,\ c_s^{k_s}$, $f_1,\ f_2,\ \ldots,\ f_k$ if they are timelike. The number H of connected components of ∂M_1 is

$$H := k + \sum_{s=1}^{n} (k_s + 1).$$

Let $(\overline{M}_1, \xi_1)$ and $(\overline{M}_2, -\xi_1)$ be two copies of M_1 endowed with the vector field ξ_1 resp. $-\xi_1$. We can now glue the same connected components of $\partial\overline{M}_1$ resp. $\partial\overline{M}_2$ together (cf. Figure 3) and we obtain a oriented compact 2-manifold M_2 without boundary. The number of loops in M_2 is

$$2l + H - 1 = 2l + k + \sum_{s=1}^{n} (k_s + 1) - 1.$$

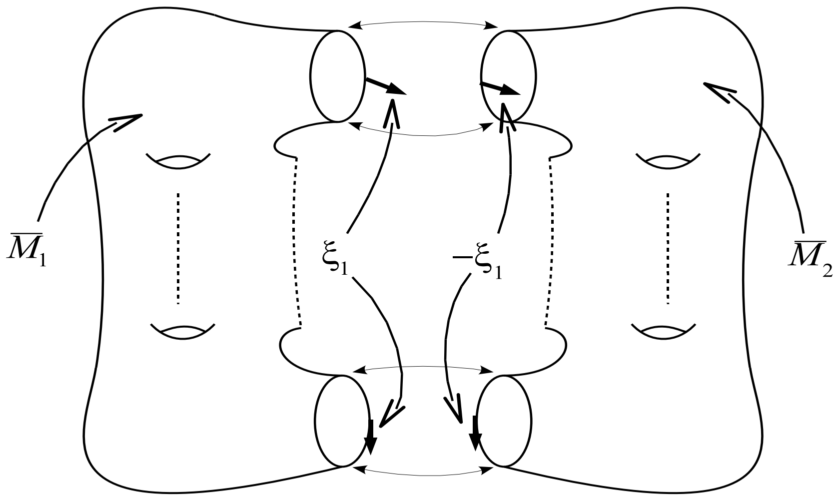

Figure 3: The 2-manifold M_2.

By adapting ξ_1 on $\overline{M}_1$ and $-\xi_1$ on $\overline{M}_2$, we obtain a non-vanishing vector field ξ_3 on M_2. As the torus is the only compact orientable connected 2-manifold admitting a non-vanishing vector field, M_2 is a torus (i.e., there is only one loop). It follows that the number of loops is equal to 1, i.e.,

$$2l + k + \sum_{s=1}^{n} (k_s + 1) - 1 = 1.$$

The only solutions are

$l = 1,\ n = k = 0$	$\Rightarrow$	M_{it} is a torus and $S(g) = \emptyset$
$l = 0,\ n = 0,\ k = 2$	$\Rightarrow$	M_{it} is topological a disc with a boundary consisting of 4 segments which are orthogonal at the edges
$l = k = 0,\ n = k_1 = 1$	$\Rightarrow$	M_{it} is topological a cylinder with a boundary consisting of 2 simply closed

As $S(g) \neq \emptyset$, only the last two cases are possible. □

Notice that byProposition 3 the Gauss–Bonnet formula (25) for the Lorentzian part M_{it} turns into

$$\int_{M_{it}} K dA + \int_{\partial M_{it}} \kappa_g ds = 2\pi\chi(M_{it}) - w_i \frac{\pi}{2} = 0, \tag{26}$$

with the notations and the assumptions of Proposition 3, where w_i is the number of non-smooth points of ∂M_{it}.

6. A global Gauss–Bonnet formula

Let (M, g) be a compact orientable connected generic 2-manifold and let $\Psi = \{\Psi_0, \ldots, \Psi_m\}$ be a proper set of tubular neighbourhoods. If the Gaussian curvature is integrable on M (resp. on $M - S(g)$), then we can use the limits $(t \to 0)$ of the local Gauss–Bonnet formulas in Section 5 applied to $M_t := M - \bigcup_{k=0}^m \Psi_k(S_k \times (-t, t))$ to obtain a global Gauss–Bonnet formula. In this sense the behaviour of the geodesic curvature of the boundary ∂M_t (resp. of the leaves Ψ_i^t of the proper set of tubular neighbourhoods $\{\Psi_0, \ldots, \Psi_m\}$) is important and need a specification. As the set of tubular neighbourhoods is proper, (the real part of) the exterior angle at the non-smooth points of ∂M_t is constant. We can establish the following propositions.

Proposition 4. *Let (M, g) be a compact orientable connected generic 2-manifold with a pseudo-geodesic and pseudo-orthogonal singular locus $S(g) \neq \emptyset$ and let $\Psi = \{\Psi_0, \ldots, \Psi_m\}$ be a proper set of tubular neighbourhoods. Then for every leaf $\Psi_i^t := \Psi_i(S_i \times \{t\})$ of Ψ_i*

$$\lim_{t \to 0} \int_{\Psi_i^t} |\kappa_g| \ d\sigma = 0$$

holds, where κ_g denotes the geodesic curvature of Ψ_i^t.

Proof. Let S_i be a singular curve of $S(g)$. We can assume that the transport of an intersection point $p \in S_i \cap S_j$ $(i \neq j)$ via the tubular neighbourhood Ψ_i is along S_j (i.e., $\Psi_i(p,t) \in S_j$ for all t) and we can assume that $t > 0$.

Let $c : [a,b] \to S_i$ be the segment between two adjoined intersection points (resp. if there is only one intersection point, c is the segment $S_i - I(g)$ and if there isn't any intersection on S_i, c is the segment S_i with an arbitrary point of S_i as start- and endpoint). Writing $c_t(k) := \Psi_i(c(k),t)$ as the transportation of c, then the following holds for $t > 0$:

$$\int_{c_t} |k_g| \, ds = \int_a^b \underbrace{\frac{\left|\Box_{c_t'} c_t'(N)\right|}{\sqrt{|\langle N,N\rangle|}}}_{=:F(k,t)} dk, \tag{27}$$

where $N \neq 0$ is a vector field which is orthogonal to c_t' and $N_p \in \mathcal{N}(p)$ for all $p \in S_i$. As $N\langle N,N\rangle \neq 0$ on $c_0([a,b])$ (cf. G_3), it follows from the rule of Bernoulli–l'Hospital that F is extendible to $\Psi_i(c([a,b]) \times [0,\varepsilon[)$. Furthermore, as S_i is pseudo-geodesic it follows that $\Box_{c_t'} c_t'(N) = 0$ for $t = 0$. This implies that $F(k,0) = 0$. Thus

$$\lim_{t\to 0} \int_{c_t} |k_g| \, ds = \lim_{t\to 0} \int_a^b F(k,t)\, dk = \int_a^b \lim_{t\to 0} F(k,t)\, dk = \int_a^b F(k,0)\, dk = 0. \qquad \Box$$

Proposition 5. *Let (M,g) be a compact orientable generic 2-manifold with a pseudo-orthogonal singular locus $S(g) \neq \emptyset$. Let M_i $(i = 1, \cdots, n)$ be the connected components of $M - S(g)$ and let $\Psi = \{\Psi_0, \ldots, \Psi_m\}$ be a proper set of tube neighbourhoods. Writing $M_{it} := M_i - \bigcup_{k=0}^m \Psi_k(S_k \times (-t,t))$, then the following holds:*

(i) *Let w_i be the number of non-smooth points of ∂M_{it} $(t \neq 0)$, then for $1 \leq j \leq w_i$*

$$\angle_i^j := \lim_{t\to 0} \alpha_i^j = \begin{cases} 0 & : \ \textit{signature } (1,1) \\ \frac{\pi}{2} & : \ \textit{signature } (2,0) \textit{ or } (0,2) \end{cases}$$

holds, where α_i^j denotes the (real part of the) exterior angle of the j-th non-smooth point of ∂M_{it}.

(ii) *In the notation of* (i) *it holds*

$$\sum_{i=1}^n \sum_{j=1}^{w_i} \angle_i^j = \pi \, |I(g)| \, ,$$

where $|I(g)|$ denotes the number of intersection points.

Proof. (i): As ψ is proper, all leaves of the tubular neighbourhoods of Ψ can only meet orthogonally. Thus by the definition of the exterior angles in Section 5, the real part of the exterior angle at a vertex of ∂M_{it} is either $\frac{\pi}{2}$ (if g has the signature (2,0) or (0,2)) or 0 (if g has the signature (1,1))

(ii): We have for every intersection point four non-smooth points of certain ∂M_{it}. We have only to count the angles for every intersection point with the distribution in Lemma 1. Therefore, we obtain $(0+\frac{\pi}{2}+0+\frac{\pi}{2})\,|I(g)|=\pi\,|I(g)|$. $\square$

Before we establish a global Gauss–Bonnet formula, we have to ensure the integrability of the Gaussian curvature K on $M - S(g)$. The following proposition shows us that this is always satisfied.

Proposition 6. *Let (M, g) be a orientable compact generic 2-manifold with a pseudo-geodesic and pseudo-orthogonal singular locus $S(g)$, then the Gaussian curvature K is integrable on M (resp. on $M - S(g)$).*

Proof. In orthogonal coordinates, the integrand of the total curvature is

$$K\,dA = \frac{\langle R(\partial_1,\partial_2)\partial_1,\partial_2\rangle}{g_{11}\cdot g_{22}}\sqrt{\epsilon g_{11}g_{22}}\,dx\,dy, \tag{28}$$

where ϵ is -1 if the signature is (1,1) and 1 otherwise, and R is the Riemannian curvature tensor. As K is integrable on every closed connected subset of $M - S(g)$ we have to show that the Gaussian curvature is integrable around the singular points. First, we consider an intersection point $p \in I(g)$ and take the parametrization (18) in orthogonal coordinate form (cf. Proposition 2 (iii)) so that the distribution of the signature is equal to the figure given in Lemma 1 (ii). As g is generic, we know that $g_{11} = x \cdot \varphi_1(x, y)$ and $g_{22} = y \cdot \varphi_2(x, y)$ with $\varphi_1, \varphi_2 > 0$ (cf. G_3 and G_5). Calculating $R_{1212} := \langle R(\partial_1,\partial_2)\partial_1,\partial_2\rangle$, we obtain

$$\begin{aligned} 4R_{1212} = &-2x\tfrac{\partial^2}{\partial^2 y}\varphi_1 - 2y\tfrac{\partial^2}{\partial^2 x}\varphi_2 + \frac{x\left(\frac{\partial}{\partial y}\varphi_1\right)^2}{\varphi_1} + \frac{\frac{\partial}{\partial x}\varphi_2}{x}y \\ &+ \frac{1}{\varphi_1}\tfrac{\partial}{\partial x}\varphi_1\tfrac{\partial}{\partial x}\varphi_2 + \frac{y\left(\frac{\partial}{\partial x}\varphi_2\right)^2}{\varphi_2} + \frac{\frac{\partial}{\partial y}\varphi_1}{y}x + \frac{1}{\varphi_2}\tfrac{\partial}{\partial y}\varphi_1\tfrac{\partial}{\partial y}\varphi_2. \end{aligned} \tag{29}$$

As the singular locus is pseudo-geodesic it follows that $\frac{\partial}{\partial y}\varphi_1 = 0$ (resp. $\frac{\partial}{\partial x}\varphi_2 = 0$) for $y = 0$ (resp. $x = 0$). By Lemma 2 it follows that $\frac{\partial}{\partial y}\varphi_1/y$ and $\frac{\partial}{\partial x}\varphi_2/x$ are extendible to $(-1, 1)^2$. Therefore, by (29) it follows that R_{1212} is extendible to $(-1, 1)^2$. By (28) the Gaussian curvature is integrable on $[-\frac{1}{2}, \frac{1}{2}]^2$ because

$$\int_{[-\frac{1}{2},\frac{1}{2}]^2} |K|\,dA \le \left(\max_{[-\frac{1}{2},\frac{1}{2}]^2}\frac{|R_{1212}|}{\sqrt{\varphi_1\varphi_2}}\right)\int_{[-\frac{1}{2},\frac{1}{2}]^2}\frac{1}{\sqrt{|xy|}}\,dx\,dy < \infty.$$

The same arguments work for any $p \in S(g) - I(g)$ with parametrization (5). $\square$

We are now able to prove the validity of a global Gauss–Bonnet formula for generic 2-manifolds as follows.

Theorem 2. *Let (M, g) be a compact orientable connected generic 2-manifold without boundary with a pseudo-geodesic and pseudo-orthogonal singular locus $S(g) \neq \emptyset$. Let $M_1, \ldots, M_n$ be the connected components of $M - S(g)$, then the following holds:*

(i) *Let $\Psi = \{\Psi_0, \ldots, \Psi_m\}$ be a proper set of tubular neighbourhoods then for every $i = 1, \ldots, n$ and $t > 0$*

$$\int_{M_i} \lambda_i K dA = 2\pi\chi(M_{it}) - w_i \frac{\pi}{2}$$

holds, where λ_i is -1 if the signature of g is $(0, 2)$ on M_i and 1 otherwise, M_{it} is defined as in Proposition 5 and w_i is the number of non-smooth points of ∂M_{it}. Notice that $\chi(M_{it})$ is constant.

(ii) *The Gauss–Bonnet formula*

$$\int_M \overline{K} dA = 2\pi\chi(M)$$

holds, where $\overline{K} := \lambda_i K$ is the Gaussian curvature-with-sign with λ_i from (i)*.*

Notice that the statement of Theorem 2 generally does not hold if we omit anyone of the assumptions. We give example for these cases (see Example 1 and Example 2).

Proof. (i): If g has the signature (2,0) or (0,2) on M_i, from Proposition 5 it follows that the exterior angle are always $\frac{\pi}{2}$. By (21) and (22) it follows

$$\int_{M_{it}} \lambda_i K d\mathrm{A} + \int_{\partial M_{it}} \lambda_i \kappa_g \, d\sigma = 2\pi\chi(M_{it}) - w_i \frac{\pi}{2}. \tag{30}$$

If g is Lorentzian on M_i, then (30) follows directly from (26). By taking the limit $(t \to 0)$ of (30) and Proposition 4 we obtain the desired equality. Notice that the right-hand side of (30) is constant for all t.

(ii): As all tubular neighbourhoods Ψ_i are strips, it follows that $\chi(\overline{\Psi}_t) = |I(g)|$ with $\overline{\Psi}_t := \bigcup_{k=0}^{m} \Psi_k(S_k \times [-t, t])$ $(t > 0)$. Thus

$$\begin{aligned} \int_M \overline{K} dA &\overset{Prop.\ 6}{=} \sum_{i=0}^{n} \int_{M_i} \overline{K} dA \\ &\overset{(i)}{=} \sum_{i=0}^{n} \left(2\pi\chi(M_{it}) - w_i \frac{\pi}{2}\right) \\ &= 2\pi \left(\sum_{i=0}^{n} \chi(M_{it}) - |I(g)| \right) \\ &= 2\pi\chi(M) \end{aligned}$$

□

In the special case where $S(g)$ does not have any intersection points, Theorem 2 with comparable assumptions is already observed by Pelletier in [9]. The topological structure of the closure of each connected component of $M - S(g)$ where g induces a Lorentzian metric can only be a cylinder.

Theorem 3. *Let (M, g) be a compact orientable connected generic 2-manifold with a pseudo-geodesic singular locus $S(g) \neq \emptyset$ without any intersection points (i.e., $I(g) = \emptyset$). Then the Gauss–Bonnet formula*

$$\int_M \lambda K dA = 2\pi\chi(M)$$

holds, where the factor $\lambda(p)$ is -1 if the signature of g is $(0, 2)$ at p and 1 otherwise.

We now give some examples of generic 2-manifolds. First, we will give a simple example how to construct a generic 2-manifold with a pseudo-geodesic and pseudo-orthogonal singular locus.

Example 1. Let h_1 be the generic metric

$$ds^2 = \sin(k\alpha)d\alpha^2 + \sin(j\beta)d\beta^2,$$

($k, j \neq 0$ fixed) on the torus $T = S^1 \times S^1$. The singular locus $\mathcal{S}(h_1)$ of h_1 is the union of circles where $\sin(k\alpha)\sin(j\beta) = 0$. The singular locus is pseudo-geodesic because $\frac{\partial}{\partial\beta}\sin(k\alpha) = \frac{\partial}{\partial\alpha}\sin(j\beta) = 0$ on $\mathcal{S}(h_1)$ and obvious pseudo-orthogonal.

A second generic metric h_2 is given by

$$ds^2 = -\cos(t)dt^2 + d\alpha^2$$

on the cylinder $C =]-\frac{3}{2}\pi, \frac{3}{2}\pi[\times S^1$. The singular locus $\mathcal{S}(h_2)$ is union of the two circles $\{-\frac{\pi}{2}\} \times S^1$ and $\{\frac{\pi}{2}\} \times S^1$. $\mathcal{S}(h_2)$ is also pseudo-geodesic because $\frac{\partial}{\partial\alpha}\cos(t) = 0$ on $\mathcal{S}(h_2)$.

With these two generic metrics, we can construct a generic metric with a pseudo-geodesic and pseudo-orthogonal singular locus on a compact orientable connected 2-manifold with arbitrary Euler characteristic. First, we take the generic 2-manifold (T, h_1) $(k, j \neq 0)$ and cutting out one disc from a connected component of $T - S(h_1)$ where h_1 is of signature (2,0) and one disc from a connected component of $T - S(h_1)$ where h_1 is of signature (0,2) (cf. Figure 4). Now, taking the generic 2-manifold (C, h_2) and gluing the curves $c_1 = \{-\frac{5}{4}\pi\} \times S^1$ and $c_2 = \{\frac{5}{4}\pi\} \times S^1$ of the cylinder C into the holes of T (in the right way, i.e, so that the orientation of T and C are preserved, cf. Figure 4). The result is a compact orientable connected 2-manifold with a generic metric h. Notice that the singular locus of h is still pseudo-geodesic and pseudo-orthogonal. If we do this repeatedly, we obtain a 2-manifold of arbitrary Euler characteristic.

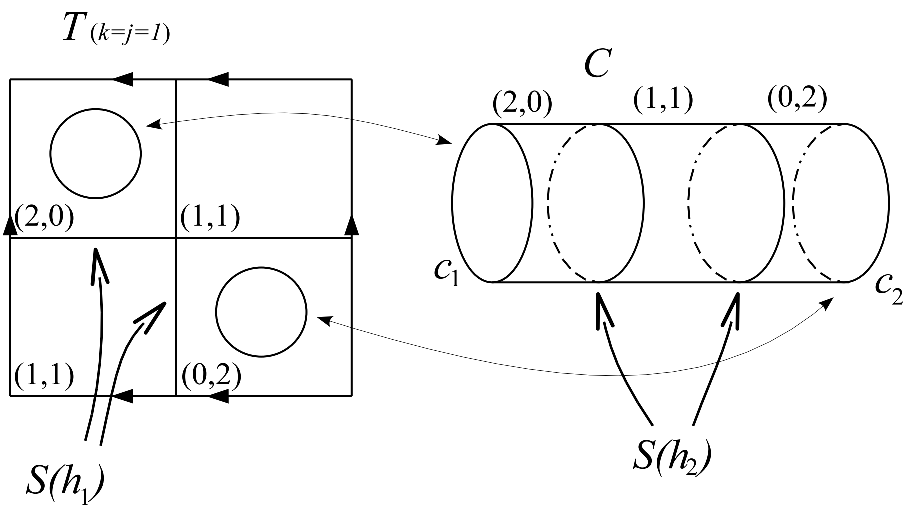

Figure 4: The generic metrics h_1 and h_2.

The following example shows that the Theorem 2 becomes incorrect, if we omit one of the assumptions.

Example 2. Let (C, h_1) be the cylinder $C = (-\frac{\pi}{2}, \frac{3\pi}{2}) \times S^1$ with the singular metric h_1

$$ds^2 = -\sin(t)dt^2 + (2 + \sin(t))d\alpha^2.$$

It holds $\mathcal{S}(h_1) = \{0\} \times S^1 \cup \{\pi\} \times S^1$. This singular metric is generic, but the singular locus isn't pseudo-geodesic because $\frac{\partial}{\partial t}(2 + \sin(t)) = \cos(t) \neq 0$ on $\mathcal{S}(h_1)$. The Gaussian curvature on $C' :=]0, \frac{\pi}{2}] \times S^1$ is

$$K = \frac{\sin(t) - \cos^2(t) + 2}{2\sin^2(t)(2 + \sin(t))^2}.$$

The Gaussian curvature is not integrable on C' because

$$\begin{aligned}
\int_{C'} K\, dA &= \lim_{s \searrow 0} \int_{]s, \frac{\pi}{2}] \times S^1} K\, dA \\
&= \lim_{s \searrow 0} 2\pi \int_s^{\frac{\pi}{2}} K \sqrt{\sin(t)(2 + \sin(t))}\, dt \\
&= \lim_{s \searrow 0} 2\pi \int_s^{\frac{\pi}{2}} \frac{\sin(t) - \cos^2(t) + 2}{2\sin^{\frac{3}{2}}(t)(2 + \sin(t))^{\frac{3}{2}}} dt \\
&= \lim_{s \searrow 0} 2\pi \left. \frac{-\sqrt{2}\sqrt{4 + 2\sin(t)}\cos(t)}{4\sqrt{\sin(t)}(2 + \sin(t))} \right|_s^{\frac{\pi}{2}} = -\infty
\end{aligned}$$

The autor was not able to find an example of a generic 2-manifold with a pseudo-geodesic but not pseudo-orthogonal singular locus so that the Gauss–Bonnet formula in Theorem 2 (ii) does not hold. However, it is obvious that pseudo-orthogonal does not follow from pseudo-geodesic.

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