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Conditions for Correct Solvability of a Simplest Singular Boundary Value Problem of General Form. I

N. A. Chernyavskaya and L. A. Shuster

Abstract. We consider the singular boundary value problem

$$-r(x)y'(x) + q(x)y(x) = f(x), \quad x \in R$$
$$\lim_{|x|\to\infty} y(x) = 0,$$

where $f \in L_p(\mathbb{R})$, $p \in [1, \infty]$ ($L_\infty(\mathbb{R}) := C(\mathbb{R})$), r is a continuous positive function on $\mathbb{R}$, $0 \le q \in L_1^{\text{loc}}$. A solution of this problem is, by definition, any absolutely continuous function y satisfying the limit condition and almost everywhere the differential equation. This problem is called correctly solvable in a given space $L_p(\mathbb{R})$ if for any function $f \in L_p(\mathbb{R})$ it has a unique solution $y \in L_p(\mathbb{R})$ and if the following inequality holds with an absolute constant $c_p \in (0, \infty)$:

$$\|y\|_{L_p(\mathbb{R})} \le c_p \|f\|_{L_p(\mathbb{R})}, \quad f \in L_p(\mathbb{R}).$$

We find minimal requirements for r and q under which the above problem is correctly solvable in $L_p(\mathbb{R})$.

Keywords. First order linear differential equation, correct solvability

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1. Introduction

We consider the singular boundary value problem

$$-r(x)y'(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R} \tag{1.1}$$
$$\lim_{|x|\to\infty} y(x) = 0. \tag{1.2}$$

N. A. Chernyavskaya: Department of Mathematics and Computer Science, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva, 84105, Israel; nina@math.bgu.ac.il

L. A. Shuster: Department of Mathematics and Computer Science, Bar-Ilan University, 52900 Ramat Gan, Israel; shustel@macs.biu.ac.il

Here and throughout the sequel $f \in L_p(\mathbb{R})$, $p \in [1,\infty]$ ($L_\infty(\mathbb{R}) := C(\mathbb{R})$) and

$$0 < r \in C^{\mathrm{loc}}(\mathbb{R}), \quad 0 \le q \in L_1^{\mathrm{loc}}(\mathbb{R}). \tag{1.3}$$

(In (1.3), we use the symbol $C^{\mathrm{loc}}(\mathbb{R})$ to denote the set of functions defined and continuous for $x \in \mathbb{R}$.) Throughout the paper, we assume that the above conventions are satisfied. We also define a solution of problem (1.1)–(1.2) as any absolutely continuous function y satisfying (1.2) and satisfying (1.1) almost everywhere on $\mathbb{R}$. The main result of the work is a criterion for correct solvability of (1.1)–(1.2) in $L_p(\mathbb{R})$. Note that we call problem (1.1)–(1.2) correctly solvable in a given space $L_p(\mathbb{R})$ (see [6, Chapter III, §6, no. 2]), if the following conditions hold:

I) For every function $f \in L_p(\mathbb{R})$, there exists a unique solution $y \in L_p(\mathbb{R})$ of(1.1)–(1.2);

II) The solution $y \in L_p(\mathbb{R})$ of(1.1)–(1.2) satisfies the following inequality with an absolute constant $c_p \in (0,\infty)$:

$$\|y\|_p \le c_p \|f\|_p, \quad \forall\ f \in L_p(\mathbb{R}). \tag{1.4}$$

We now discuss the main feature of problem(1.1)–(1.2). Its solution y, if it exists, has the form (see Lemma 4.1 in Section 4):

$$y(x) = (Gf)(x) \overset{\mathrm{def}}{=} \int_x^\infty \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(s)}{r(s)}\,ds\right) f(t)\,dt, \quad x \in \mathbb{R}. \tag{1.5}$$

Thus, in order to study (1.1)–(1.2) one has to find minimal requirements to r and q under which the integral operator $G : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is bounded, and, in addition, all the functions from its image vanish on $\pm\infty$. We emphasize that conditions for an integral operator of the form (1.5) to be bounded in $L_p(\mathbb{R})$ may be found by means of Hardy-type inequalities (see [11] and Theorems 3.1 and 3.2 in Section 3). Therefore, it is necessary to strengthen them by a criterion for the validity of (1.2). Note that such a restriction of the initial question does not make it less meaningful. This a priori statement is based on our papers [3], [4] where it is shown that the study of boundary properties of solutions of singular differential equations is a problem separate from the problem of correct solvability of a differential equation in a given space, and thus it requires special analysis.

Let us now briefly describe our work. The requirements I)–II) for $p \in (1,\infty)$, $p = 1$ and $p = \infty$ are studied separately since the known statements on estimating the norm of the operator (1.5) can be divided into exactly three cases (see Section 3). Thus in Section 2, we give three groups of theorems, each of which contains a general (unconditional) criterion for correct solvability of (1.1)–(1.2): Theorems 2.1, 2.5 and 2.8 and a particular criterion of Theorems 2.4, 2.7 and 2.9, which can be applied under a certain a priori assumption on r and q (see Section 2, (2.9)). Theorems 2.1, 2.5 and 2.8 contain conditions expressed in a non-local form, and it might be hard to check them in concrete cases. The statements of Theorems 2.4, 2.7 and 2.9 contain only conditions expressed in a local form. It is much easier to check them in concrete cases since one can use standard tools of local analysis (see Section 8).

In addition, following a suggestion of S. Luckhaus, we complement Theorems 2.1 – 2.9 by Theorem 1.1 (see below). This theorem presents an example of the relationship between r, q and the parameter $p \in [1, \infty]$, which guarantees the correct solvability of the problem(1.1)–(1.2) in $L_p(\mathbb{R})$.

Theorem 1.1. *Suppose that the following conditions hold:*

1) *The functions $r(x)$ and $q(x)$ are positive and continuous for $x \in \mathbb{R}$.*
2) *There exist $a \geq 1$ and $b > 0$ and an interval (α, β) such that*

$$\frac{1}{a} \leq \frac{r(t)}{r(x)}, \ \frac{q(t)}{q(x)} \leq a \quad \text{for} \quad |t-x| \leq b\,\frac{r(x)}{q(x)}, \qquad x \notin (\alpha, \beta);$$

and, moreover, $3a^2 \exp\left(-\frac{b}{a^2}\right) \leq 1$.

Then problem(1.1)–(1.2) *is correctly solvable in* $L_p(\mathbb{R})$, $p \in [1, \infty]$, *if and only if the conditions from the following table are satisfied:*

Space	$L_1(\mathbb{R})$	$L_p(\mathbb{R})$, $1 < p < \infty$	$C(\mathbb{R})$
Conditions for correct solvability	$r_0 > 0$ $q_0 > 0$	$\sigma_{p'} > 0$ $q_0 > 0$	$q(x) \to \infty$ *as* $\lvert x\rvert \to \infty$

Here $p' = p(p-1)^{-1}$ *for* $p \in (1, \infty)$ *and*

$$r_0 = \inf_{x \in \mathbb{R}} r(x), \quad q_0 = \inf_{x \in \mathbb{R}} q(x), \quad \sigma_{p'} = \inf_{x \in \mathbb{R}} r(x)^{\frac{1}{p}} q(x)^{\frac{1}{p'}}.$$

Remark. This article is the first of two parts. In this part, we state general criteria and give an example of their application. In the second part, we shall state the theorems which supplement the first part. Some statements in this part are given without proofs. (Proofs can be found in [5]). Finally, note that this work is a development of [1], in which the problem (1.1)–(1.2) with $r(x) \equiv 1$ was studied. Proofs of the statements of [1], which can be obtained from the results of this article, are available in the preprint [5].

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2. Main results

Below we give the main results of the first part of this work. Here and throughout the sequel, the symbols $c, c(\cdot), c_1, c_2, \dots$ denote absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations. Let us introduce an auxiliary function d. We temporarily assume that in addition to (1.3), we have also

$$S_1 = \infty, \quad S_1 \stackrel{\text{def}}{=} \int_{-\infty}^{0} \frac{q(t)}{r(t)}\, dt. \tag{2.1}$$

Then for every $x \in \mathbb{R}$, we have a uniquely determined function d where

$$d(x) = \inf_{d>0}\{d : \Phi(x,d) = 2\}, \quad \Phi(x,d) = \int_{x-d}^{x+d} \frac{q(t)}{r(t)}\, dt. \tag{2.2}$$

The functions of type (2.2) were introduced by M. Otelbaev (see [10, 12]).

Theorem 2.1 (§4). *Let $p \in (1,\infty)$, $p' = p(p-1)^{-1}$. Problem* (1.1)–(1.2) *is correctly solvable in $L_p(\mathbb{R})$ if and only if the following conditions hold together:*

1) $M_p < \infty$. *Here* $M_p = \sup_{x\in\mathbb{R}} M_p(x)$, *where* (2.3)

$$\begin{aligned} M_p(x) = {} & \left[\int_{-\infty}^{x} \exp\left(-p\int_t^x \frac{q(\xi)}{r(\xi)}d\xi\right) dt\right]^{\frac{1}{p}} \\ & \cdot \left[\int_{x}^{\infty} \frac{1}{r(t)^{p'}} \exp\left(-p'\int_x^t \frac{q(\xi)}{r(\xi)}d\xi\right) dt\right]^{\frac{1}{p'}} \end{aligned} \tag{2.4}$$

2) $S_1 = \infty$ *(see* (2.1)*)* (2.5)

3) $A_{p'} < \infty$. *Here* $A_{p'} = \sup_{x\in\mathbb{R}} A_{p'}(x)$, *where* (2.6)

$$A_{p'}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}}, \quad x \in \mathbb{R}. \tag{2.7}$$

Corollary 2.1.1 ([5]). *Let $p \in (1,\infty)$, and suppose that problem*(1.1)–(1.2) *is correctly solvable in $L_p(\mathbb{R})$. Then its solution y satisfies the inequality*

$$\left\|\left(\frac{q}{r}\right)^{\frac{1}{p}} y\right\|_p \le c(p)\|f\|_p, \quad \forall\, f(x) \in L_p(\mathbb{R}).$$

Remark 2.2. Theorem 2.1 is "not convenient" for application to concrete problems (1.1)–(1.2) since the condition (2.3) is non-local, and therefore may be hard to check. At the same time, under an additional assumption (to (1.3)),

$$d_0 < \infty, \quad d_0 = \sup_{x\in\mathbb{R}} d(x), \tag{2.8}$$

conditions (2.3) and (2.5) hold automatically(see Theorem 2.4 below), and the checking of 3) can be made by local tools (see (2.7), (2.8). In the following lemma we obtain the properties of r and q guaranteeing (2.8).

Lemma 2.3 (§3). *The conditions $S_1 = \infty$ and $d_0 < \infty$ (see* (2.1) *and* (2.8)*) hold together if and only if*

$$\exists\ a \in (0,\infty) : B(a) > 0, \quad B(a) \overset{def}{=} \inf_{x\in\mathbb{R}} \int_{x-a}^{x+a} \frac{q(t)}{r(t)}\,dt. \tag{2.9}$$

In the next statement, conditions for correct solvability of (1.1)–(1.2) in $L_p(\mathbb{R})$ contain only local requirements.

Theorem 2.4 (§4). *Let $p \in (1,\infty)$, and* (2.9) *holds. Then problem* (1.1)*–*(1.2) *is correctly solvable in $L_p(\mathbb{R})$ if and only if $A_{p'} < \infty$ (see* (2.6)*).*

Now we give conditions for correct solvability of(1.1)–(1.2) in $L_1(\mathbb{R})$.

Theorem 2.5 (§5). *Problem*(1.1)*–*(1.2) *is correctly solvable in $L_1(\mathbb{R})$ if and only if the following conditions hold together:*

1) $S_1 = \infty$ *(see* (2.1)*)* (2.10)

2) $r_0 > 0,\ r_0 = \inf_{x\in\mathbb{R}} r(x) > 0$ (2.11)

3) $M_1 < \infty$. *Here* $M_1 = \sup_{x\in\mathbb{R}} M_1(x) < \infty,$ *where* (2.12)

$$M_1(x) = \frac{1}{r(x)} \int_{-\infty}^{x} \exp\left(-\int_t^x \frac{q(\xi)}{r(\xi)}d\xi\right) dt, \quad x \in \mathbb{R}. \tag{2.13}$$

Corollary 2.5.1 ([5]). *Suppose that* (1.1)*–*(1.2) *is correctly solvable in $L_1(\mathbb{R})$. Then its solution y satisfies the following inequality (see* (2.11)*)*

$$\|y'\|_1 + \left\|\frac{q}{r}y\right\|_1 \le \frac{3}{r_0}\|f\|_1, \quad \forall\ f \in L_1(\mathbb{R}). \tag{2.14}$$

Remark 2.6. Inequality (2.14) shows that on the set of solutions of problem (1.1)–(1.2), equation (1.1) has the property of separability in the space $L_\theta^1(\mathbb{R})$ where $\theta(x) = \frac{1}{r(x)}$, $x \in \mathbb{R}$. Note that the problem of separability of singular differential operators was first considered in [7], [8].

In the next theorem, conditions for correct solvability of (1.1)–(1.2) in $L_1(\mathbb{R})$ contain only local requirements.

Theorem 2.7 (§5). *Suppose that condition* (2.9) *holds. Then the problem* (1.1)*–*(1.2) *is correctly solvable in $L_1(\mathbb{R})$ if and only if $r_0 > 0$ (see* (2.11)*).*

In the following theorem, we give conditions for correct solvability of problem (1.1)– (1.2) in $C(\mathbb{R})$.

Theorem 2.8 (§6). *Problem* (1.1)–(1.2) *is correctly solvable in* $C(\mathbb{R})$ *if and only if* $A_0 = 0$, *where* $A_0 = \lim\limits_{|x|\to\infty} A(x)$. *Here*

$$A(x) = \int_x^\infty \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(\xi)}{r(\xi)} d\xi\right) dt, \quad x \in \mathbb{R}. \tag{2.15}$$

Moreover, if $A_0 = 0$, *then* $S_1 = \infty$ *(see* (2.1)*).*

A local form of the condition for correct solvability of (1.1)–(1.2) is given in the following theorem.

Theorem 2.9 (§6). *Under assumption* (2.9), *problem* (1.1)–(1.2) *is correctly solvable in* $C(\mathbb{R})$ *if and only if* $\tilde{A}_0 = 0$, *where* $\tilde{A}_0 = \lim\limits_{|x|\to\infty} \tilde{A}(x)$. *Here*

$$\tilde{A}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)}, \quad x \in \mathbb{R}.$$

In the next theorem, we give conditions for correct unsolvability in $L_p(\mathbb{R})$, $p \in [1,\infty]$ of the problem (1.1)–(1.2).

Theorem 2.10 (§7). *Let* $p \in [1,\infty]$. *Then under the conditions* (2.9) *and*

$$\lim_{x\to-\infty} q(x) = 0 \quad or \quad \lim_{x\to\infty} q(x) = 0, \tag{2.16}$$

the problem(1.1)–(1.2) *cannot be correctly solvable in* $L_p(\mathbb{R})$.

To conclude the section, we give a statement of the estimates for d.

Theorem 2.11 (§3). *Suppose that there exist a continuous positive function* q_1 *and a function* $q_2 \in L_1^{\mathrm{loc}}(\mathbb{R})$ *such that* $q(x) = q_1(x) + q_2(x)$ *for* $x \in \mathbb{R}$ *and, in addition,* $\varkappa_1(x) \to 0$, $\varkappa_2(x) \to 0$ *as* $|x| \to \infty$, *where*

$$\varkappa_1(x) = \sup_{|z|\le 2\frac{r(x)}{q_1(x)}} \left| \int_0^z \left[\frac{q_1(x+t)}{r(x+t)} - 2\frac{q_1(x)}{r(x)} + \frac{q_1(x-t)}{r(x-t)} \right] dt \right| \tag{2.17}$$

$$\varkappa_2(x) = \sup_{|z|\le 2\frac{r(x)}{q_1(x)}} \left| \int_{x-z}^{x+z} \frac{q_2(t)}{r(t)}\, dt \right|. \tag{2.18}$$

Then the following relations hold:

$$\frac{q_1(x)}{r(x)} d(x) = 1 + \varepsilon(x), \quad |\varepsilon(x)| \le c_1[\varkappa_1(x) + \varkappa_2(x)], \quad |x| \gg 1 \tag{2.19}$$

$$c_2^{-1} \le \frac{q_1(x)}{r(x)} d(x) \le c_2, \quad x \in \mathbb{R}. \tag{2.20}$$

3. Auxiliary results and technical assertions

In the first part of this section, we present various assertions used in the proofs of Theorems 2.1–2.10. In the second part, we prove some of them.

3.1. Statement of auxiliary results.

Theorem 3.1 ([11, Chapter I, §6, no. 6.5]). *Let $AC(\mathbb{R})^{(+)}$ be the set of absolutely continuous functions F on $\mathbb{R}$ such that $F(x) \to 0$ as $x \to \infty$, and let w and v be measurable and positive functions in $\mathbb{R}$. Then the Hardy inequality*

$$\left\|w^{\frac{1}{p}} F\right\|_p \le C \left\|v^{\frac{1}{p}} F'\right\|_p \tag{3.1}$$

holds for all $F \in AC(\mathbb{R})^{(+)}$ if and only if $H^{(p)} < \infty$. Here $H^{(p)} = \sup_{x\in\mathbb{R}} H^{(p)}(x)$,

$$H^{(p)}(x) = \left(\int_{-\infty}^{x} w(t)\,dt\right)^{\frac{1}{p}} \left(\int_{x}^{\infty} v(t)^{-p'/p}\,dt\right)^{\frac{1}{p'}}, \quad x \in \mathbb{R},$$

and $p \in (1,\infty)$, $p' = \frac{p}{p-1}$. Moreover, the following estimates hold for the smallest constant C in (3.1)*:*

$$H^{(p)} \le C \le (p)^{\frac{1}{p}} (p')^{\frac{1}{p'}} H^{(p)}.$$

Theorem 3.2 ([12, Chapter 2, §7]). *Let $p \in (1,\infty)$ and let μ, θ be continuous and positive functions on R. Denote by K the integral operator*

$$(Kf)(t) = \mu(t) \int_t^{\infty} \theta(\xi) f(\xi) d\xi, \quad t \in \mathbb{R}. \tag{3.2}$$

Then the operator $K : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is bounded if and only if $H_p < \infty$. Here $H_p = \sup_{x\in\mathbb{R}} H_p(x)$,

$$H_p(x) = \left[\int_{-\infty}^{x} \mu(t)^p\,dt\right]^{\frac{1}{p}} \cdot \left[\int_x^{\infty} \theta(t)^{p'}\,dt\right]^{\frac{1}{p'}}, \quad p' = \frac{p}{p-1}. \tag{3.3}$$

The following inequalities hold:

$$H_p \le \|K\|_{p\to p} \le (p)^{\frac{1}{p}} (p')^{\frac{1}{p'}} H_p. \tag{3.4}$$

Theorem 3.3 ([9, Ch.V, §2, no. 4–5]). *Let K be the integral operator* (3.2)*. Then*

$$\|K\|_{1\to 1} = \sup_{x\in\mathbb{R}} \theta(x) \int_{-\infty}^{x} \mu(t)\,dt \tag{3.5}$$

$$\|K\|_{C(\mathbb{R})\to C(\mathbb{R})} = \sup_{x\in\mathbb{R}} \mu(x) \int_x^{\infty} \theta(t)\,dt. \tag{3.6}$$

Lemma 3.4. *Let $S_1 = \infty$ (see (2.1). Then the function $d(x)$ is defined for $x \in \mathbb{R}$. Moreover, d is continuous and positive and*

$$|d(x+h) - d(x)| \le |h| \quad \textit{if} \quad |h| \le d(x),\ x \in \mathbb{R}. \tag{3.7}$$

Definition 3.5. Suppose that there are given $x \in \mathbb{R}$, a positive continuous function $\varkappa$, a sequence $\{x_n\}_{n\in N'}$, $N' = \{\pm1, \pm2, \dots\}$. Consider the segments $\Delta_n = [\Delta_n^-, \Delta_n^+]$, $\Delta_n^{\pm} = x_n \pm \varkappa(x_n)$. We say that a sequence of segments $\{\Delta_n\}_{n=1}^{\infty}$ ($\{\Delta_n\}_{n=-\infty}^{-1}$) forms an $R(x,\varkappa)$-covering of $[x,\infty)$ (resp., $(-\infty, x]$) if the following conditions hold:

1) $\Delta_n^+ = \Delta_{n+1}^-$ for $n \ge 1$ (resp., $\Delta_{n-1}^+ = \Delta_n^-$ for $n \le -1$)
2) $\Delta_1^- = x$ (resp., $\Delta_{-1}^+ = x$), $\bigcup_{n\ge1} \Delta_n = [x,\infty)$ (resp., $\bigcup_{n\le -1} \Delta_n = (-\infty, x]$).

Lemma 3.6. *Suppose that a positive continuous function $\varkappa$ satisfies the following relations*

$$\lim_{t\to\infty} (t - \varkappa(t)) = \infty \quad \textit{(resp.,} \lim_{t\to-\infty} (t + \varkappa(t)) = -\infty). \tag{3.8}$$

Then for every $x \in \mathbb{R}$ there exists an $R(x,\varkappa)$-covering of $[x,\infty)$ (resp., an $R(x,\varkappa)$-covering of $(-\infty,x]$).

Lemma 3.7. *Let $S_2 = \infty$ (resp., $S_1 = \infty$, see (2.1)), where*

$$S_2 = \int_0^{\infty} \frac{q(t)}{r(t)}\, dt. \tag{3.9}$$

Then for every $x \in \mathbb{R}$ there exists an $R(x,d)$-covering of $[x,\infty)$ (resp., an $R(x,d)$-covering of $(-\infty,x]$).

3.2. Proofs of auxiliary assertions.

Proof of Theorem 3.2. *Necessity*: Let $p \in (1,\infty)$, and suppose that the operator $K : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is bounded. Denote by $[t_1,t_2]$ the arbitrary finite interval and

$$f_0(\xi) = \begin{cases} \theta(\xi)^{p'-1}, & \text{if } \xi \in [t_1,t_2] \\ 0, & \text{if } \xi \notin [t_1,t_2]. \end{cases}$$

From the continuity of θ, it follows that $f_0 \in L_p(\mathbb{R})$ and

$$\|f_0\|_p^p = \int_{t_1}^{t_2} \theta(\xi)^{p(p'-1)} d\xi = \int_{t_1}^{t_2} \theta(\xi)^{p'} d\xi. \tag{3.10}$$

Moreover,

$$\begin{aligned} \|(Kf_0)\|_p^p &\ge \int_{-\infty}^{t_1} \left[\mu(t) \int_t^{\infty} \theta(\xi) f_0(\xi) d\xi\right]^p dt \\ &\ge \left(\int_{-\infty}^{t_1} \mu(t)^p\, dt\right) \left(\int_{t_1}^{t_2} \theta(\xi)^{p'} d\xi\right)^p. \end{aligned} \tag{3.11}$$

Now using (3.10) and (3.11), we obtain

$$\|K\|_{p\to p} \geq \left(\int_{-\infty}^{t_1} \mu(t)^p\, dt\right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} \theta(\xi)^{p'} d\xi\right)^{\frac{1}{p'}}. \tag{3.12}$$

Since t_1, t_2 are arbitrary, from (3.12) we obtain the lower estimate of (3.4).

Sufficiency: In (3.1), set $w(x) = \mu(x)^p$, $v(x) = \theta(x)^{-p}$. Then $H^{(p)} = H_p < \infty$ and (3.1) holds for every $F \in AC(\mathbb{R})^{(+)}$ by Theorem 3.1. Denote

$$F(x) = \int_x^\infty \theta(s) f(s)\, ds, \quad \forall f \in L_p(\mathbb{R}),\ x \in \mathbb{R}. \tag{3.13}$$

Since $H_p < \infty$, then $\theta \in L_{p'}(x,\infty)$ for every $x \in \mathbb{R}$; by Hölder's inequality the integral (3.13) converges for every $x \in \mathbb{R}$, F is absolutely continuous, $F \in AC(\mathbb{R})^{(+)}$, $F(x) \to 0$ as $x \to \infty$. In addition, almost everywhere

$$-\frac{1}{\theta(x)} F'(x) = f(x), \quad f \in L_p(\mathbb{R}). \tag{3.14}$$

Thus Theorem 3.1 together with (3.14) reduce to (3.4):

$$\|(Kf)\|_p = \|\mu F\|_p \leq (p)^{\frac{1}{p}} (p')^{\frac{1}{p'}} H_p \left\|\frac{1}{\theta} F'\right\|_p = (p)^{\frac{1}{p}} (p')^{\frac{1}{p'}} H_p \|f\|_p. \qquad \square$$

Proof of Lemma 3.4. Clearly, $\Phi(x,0) = 0$, $\Phi(x,d) \to \infty$ as $d \to \infty$. For a fixed $x \in \mathbb{R}$, the function $\Phi(x,d)$ is continuous, non-negative and does not decrease on $(0,\infty)$. Then the equation $\Phi(x,d) = 2$ has at least one solution and therefore the function d is defined.

Let us verify (3.7) for $h \in [0, d(x)]$. The following two relations are obvious:

$$2 = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)}\, dt \geq \int_{(x+h)-(d(x)-h)}^{(x+h)+(d(x)-h)} \frac{q(t)}{r(t)}\, dt, \tag{3.15}$$

$$2 = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)}\, dt \leq \int_{(x+h)-(d(x)+h)}^{(x+h)+(d(x)+h)} \frac{q(t)}{r(t)}\, dt. \tag{3.16}$$

From (3.15) and (3.16), we obtain, respectively,

$$d(x+h) \geq d(x) - h, \quad d(x+h) \leq d(x) + h. \tag{3.17}$$

Clearly, (3.17) is equivalent to (3.7). The case $h \in [-d(x), 0]$ can be treated similarly. From (3.7) it follows that the function $d(x)$ is continuous for $x \in \mathbb{R}$. $\square$

Proof of Lemma 3.6. Let us verify that an $R(x,\varkappa)$-covering exists for $[x,\infty)$ (the case $(-\infty,x]$ can be treated in a similar way). Set $\varphi(t)=t-\varkappa(t)-x$. Then $\varphi(x)=-\varkappa(x)<0$, and by (3.8) there is an $a>x$ such that $\varphi(a)>0$. Since $\varkappa$ is continuous, so is φ, and $\varphi(x)<0$, $\varphi(a)>0$. Hence there is an $x_1\in(x,a)$ such that $\varphi(x_1)=0$, i.e., $x=x_1-\varkappa(x_1)$. Set $\Delta_1^{\pm}=x_1\pm\varkappa(x_1)$, and the segment $\Delta_1=[\Delta_1^-,\Delta_1^+]$ is constructed. The segments Δ_n, $n\ge 2$, with the property $\Delta_n^+=\Delta_{n+1}^-$ are constructed in a similar way. Let us verify that $\bigcup_{n\ge1}\Delta_n=[x,\infty)$. If this is not the case, then there is a $z\in(x,\infty)$ such that $\Delta_n^+<z$ for all $n\ge1$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ is increasing (by construction) and bounded ($x_n<\Delta_n^+<z$, $n\ge1$), it has a limit $x_0\le z$. Moreover, $\infty>z-x\ge 2\sum_{n=1}^{\infty}\varkappa(x_n)$, and hence $\varkappa(x_n)\to0$ for $n\to\infty$. Then $\varkappa(x_0)=0$, a contradiction. The lemma is proved. □

Remark 3.8. Lemma 3.6 is proved by M. O. Otelbaev's method (see [12, Chapter 1, §4], [10, Chapter III, §1]).

Proof of Lemma 3.7. According to Lemma 3.6, it is enough to verify that

$$\lim_{x\to\infty}(x-d(x))=\infty,\quad \left(\lim_{x\to-\infty}(x+d(x))=-\infty\right). \tag{3.18}$$

Equalities (3.18) are checked in the same way. Let us obtain, say, the first one. Assume that $\varliminf_{x\to\infty}(x-d(x))=c<\infty$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n\to\infty$ as $n\to\infty$ and $x_n-d(x_n)\le c+1$ for $n\ge1$. Together with (2.2), this implies

$$2=\int_{x_n-d(x_n)}^{x_n+d(x_n)}\frac{q(t)}{r(t)}\,dt\ge\int_{c+1}^{x_n+d(x_n)}\frac{q(t)}{r(t)}\,dt\to\infty\quad\text{as } n\to\infty,$$

a contradiction. Therefore (3.18) is verified. □

Below we need the following lemma.

Lemma 3.9. *Let* $S_1=\infty$ *(see* (2.1)*). The inequality* $\eta\ge d(x)$ *(resp.,* $0\le\eta\le d(x)$*) holds if and only if*

$$\int_{x-\eta}^{x+\eta}\frac{q(t)}{r(t)}\,dt\ge2\quad\left(\textit{resp.},\ \int_{x-\eta}^{x+\eta}\frac{q(t)}{r(t)}\,dt\le2\right). \tag{3.19}$$

Proof. Necessity: Let $\eta\ge d(x)$. Then by (2.2), we have

$$\int_{x-\eta}^{x+\eta}\frac{q(t)}{r(t)}\,dt\ge\int_{x-d(x)}^{x+d(x)}\frac{q(t)}{r(t)}\,dt=2.$$

Sufficiency. Suppose that (3.19) holds; however, $\eta<d(x)$. Then by (2.2), we get

$$2\le\int_{x-\eta}^{x+\eta}\frac{q(t)}{r(t)}\,dt<\int_{x-d(x)}^{x+d(x)}\frac{q(t)}{r(t)}\,dt=2,$$

a contradiction. Hence $\eta\ge d(x)$. □

Proof of Lemma 2.3. *Necessity*: Let $S_1 = \infty$, $d_0 < \infty$ (see (2.1), (2.8)). Set $a = d_0$. Then

$$B(d_0) = \inf_{x\in\mathbb{R}} \int_{x-d_0}^{x+d_0} \frac{q(t)}{r(t)}\,dt \ge \inf_{x\in\mathbb{R}} \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)}\,dt = 2.$$

Sufficiency: Let $B(a) > 0$ for some $a \in (0,\infty)$, and let k be the smallest natural number such that $(2k+1)B(a) \ge 2$. Let x be an arbitrary point from $\mathbb{R}$, $x_n = x + 2na$, $n = \pm 1, \pm 2, \dots \ \pm k$. Then

$$\int_{x-(2k+1)a}^{x+(2k+1)a} \frac{q(t)}{r(t)}\,dt = \int_{x-a}^{x+a} \frac{q(t)}{r(t)}\,dt + \sum_{n=1}^{k} \int_{x_n-a}^{x_n+a} \frac{q(t)}{r(t)}\,dt + \sum_{n=-k}^{-1} \int_{x_n-a}^{x_n+a} \frac{q(t)}{r(t)}\,dt$$
$$\ge (2k+1)B(a) \ge 2.$$

Lemma 3.9 and the above inequality imply $d(x) \le (2k+1)a$, i.e., $d_0 = \sup_{x\in\mathbb{R}} d(x) \le (2k+1)a < \infty$. Since $d_0 < \infty$, we conclude that $S_1 = \infty$. $\square$

Proof of Theorem 2.11. For $x \in \mathbb{R}$ and $\eta \ge 0$, we get (see (2.2))

$$2\Phi(x,\eta) = 2\frac{q_1(x)}{r(x)}\eta + \int_0^\eta \left[\frac{q_1(x+t)}{r(x+t)} - 2\frac{q_1(x)}{r(x)} + \frac{q_1(x-t)}{r(x-t)}\right] dt + \int_{x-\eta}^{x+\eta} \frac{q_2(t)}{r(t)}\,dt. \tag{3.20}$$

Set $\eta_1(x) = (1 + \varkappa_1(x) + \varkappa_2(x))\frac{r(x)}{q_1(x)}$. Then $\varkappa_1(x) + \varkappa_2(x) \le 1$ for $|x| \gg 1$, and the equation (3.20) implies

$$2\Phi(x,\eta_1(x)) \ge 2(1 + \varkappa_1(x) + \varkappa_2(x)) - \varkappa_1(x) - \varkappa_2(x) \ge 2. \tag{3.21}$$

From here $d(x) \le \eta_1(x)$ for $|x| \gg 1$ (see Lemma 3.10 and (3.21)). Similarly, we set $\eta_2 = (1 - \varkappa_1(x) - \varkappa_2(x))\frac{r(x)}{q_1(x)}$. Then $\eta_2(x) > 0$ for $|x| \gg 1$, and (3.20) implies

$$2\Phi(x,\eta_2(x)) \le 2(1 - \varkappa_1(x) - \varkappa_2(x)) + \varkappa_1(x) + \varkappa_2(x) \le 2.$$

Together with Lemma 3.9, this implies $d(x) \ge \eta_2(x)$ for $|x| \gg 1$. We thus get (2.19). Estimates (2.20) are easily derived from (2.19) taking into account that the functions r, q_1 and d are continuous and positive on $\mathbb{R}$. $\square$

4. Proof of the main results in the case $p \in (1,\infty)$

In this section we prove Theorems 2.1 and 2.4. Below we need the following assertion.

Lemma 4.1. *Let $f \in L_p(\mathbb{R})$, $p \in [1,\infty]$. If the problem* (1.1)–(1.2) *is solvable (not necessarily correctly solvable), then its solution y is unique and can be represented by formula* (1.5).

Proof. Let y be a solution of (1.1)–(1.2). Then

$$\frac{d}{d\xi}\left[y(\xi)\exp\left(-\int_x^\xi \frac{q(s)}{r(s)}\,ds\right)\right] = -\frac{1}{r(\xi)}\exp\left(-\int_x^\xi \frac{q(s)}{r(s)}\,ds\right) f(\xi). \tag{4.1}$$

Let $a \in (0,\infty)$. We integrate (4.1) along the segment $[x, x+a]$ and get

$$\begin{aligned} y(x+a)\exp\left(-\int_x^{x+a} \frac{q(s)}{r(s)}\,ds\right) - y(x) \\ = -\int_x^{x+a} \frac{1}{r(\xi)}\exp\left(-\int_x^\xi \frac{q(s)}{r(s)}\,ds\right) f(\xi)d\xi. \end{aligned} \tag{4.2}$$

In (4.2) we take the limit as $a \to \infty$. From (1.2) it follows that the limit in the left-hand side of (4.2) exists. This proves (1.5). □

Proof of Theorem 2.1. *Necessity.* Suppose that(1.1)–(1.2) is correctly solvable in $L_p(\mathbb{R})$. Then (see (1.5) and (3.2)), its solution y can be written in the form

$$y(x) = (Gf)(x) = (Kf)(x) = \mu(x)\int_x^\infty \theta(t)f(t)\,dt, \quad \forall\, f \in L_p(\mathbb{R}), \tag{4.3}$$

where

$$\mu(x) = \exp\left(\int_0^x \frac{q(s)}{r(s)}\,ds\right), \quad \theta(x) = \frac{1}{r(x)}\exp\left(-\int_0^x \frac{q(s)}{r(s)}\,ds\right), \quad x \in \mathbb{R} \tag{4.4}$$

The operator $G \equiv K : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ in the case (4.3)–(4.4) is bounded, and therefore $H_p < \infty$ (see (3.3)). Clearly, $H_p(x) = M_p(x)$, $x \in \mathbb{R}$, and this implies $M_p = H_p < \infty$. Let us check (2.5). If $S_1 < \infty$, then using (2.3) we obtain for $x \in \mathbb{R}$

$$\begin{aligned} M_p &\ge \left[\int_{-\infty}^x \exp\left(-p\int_t^x \frac{q(s)}{r(s)}\,ds\right)dt\right]^{\frac1p}\left[\int_x^\infty \frac{1}{r(t)^{p'}}\exp\left(-p'\int_x^t \frac{q(s)}{r(s)}\,ds\right)dt\right]^{\frac{1}{p'}} \\ &\ge \exp(-S_1)\sup_{x\in\mathbb{R}}\left[\int_{-\infty}^x 1\,dx\right]^{\frac1p}\left[\int_x^\infty \frac{1}{r(t)^{p'}}\exp\left(-p'\int_x^t \frac{q(s)}{r(s)}\,ds\right)dt\right]^{\frac{1}{p'}} = \infty, \end{aligned}$$

a contradiction. Hence $S_1 = \infty$. Let us check that (2.6) holds for

$$T_{p'} = \infty, \qquad T_{p'} \stackrel{\text{def}}{=} \int_0^\infty \frac{dt}{r(t)^{p'}}. \tag{4.5}$$

Then we get analogously $S_2 = \infty$ (see (3.9)). Let us now turn to (2.6) and prove it ad absurdum. Let $A_{p'} = \infty$ (see (2.6)). Let us show that there exists $F \in L_p(\mathbb{R})$ such that $(GF) \not\to 0$ as $|t| \to \infty$, i.e., (1.2) does not hold for all $f \in L_p(\mathbb{R})$. This contradicts the correct solvability of(1.1)–(1.2) in $L_p(\mathbb{R})$. Let β be a positive number which will be chosen later, and let $a_{p'} = \inf_{x\in\mathbb{R}} A_{p'}(x)$. Clearly, $0 \le a_{p'} < \infty$, and for every integer $k \ge a_{p'} + 1$ there is a point x_k such that

$$k^\beta \le A_{p'}(x_k) \le (k+1)^\beta, \quad k \ge a_{p'} + 1. \tag{4.6}$$

Since $A_{p'}(x)$ is continuous (see Lemma 3.4 and (2.7)), then $|x_k| \to \infty$ as $k \to \infty$. Since $S_1 = S_2 = \infty$ (see (2.1), (3.9)), from (3.18) it follows that one can choose a subsequence $\{x_{k_n}\}_{n=1}^\infty$ such that $|x_{k_n}| \to \infty$ as $n \to \infty$, and the segments

$$\Delta_{k_n} = \left[\Delta_{k_n}^-, \Delta_{k_n}^+\right] = \left[x_{k_n} - d(x_{k_n}), x_{k_n} + d(x_{k_n})\right], \quad n = 1, 2, \dots$$

are disjoint. Let α be another positive number (also to be chosen later), and

$$f_{k_n}(t) = \begin{cases} \frac{1}{(1+k_n)^\alpha} \frac{1}{r(t)^{p'-1}}, & t \in \Delta_{k_n} \\ 0, & t \notin \Delta_{k_n} \end{cases}, \qquad n = 1, 2, \dots \tag{4.7}$$

$$F(t) = \sum_{n=1}^\infty f_{k_n}(t), \qquad t \in \mathbb{R}. \tag{4.8}$$

Let us verify that $F \in L_p(\mathbb{R})$ for $\alpha > \frac{1+\beta}{p}$. Indeed, we have

$$\|F\|_p^p = \sum_{n=1}^\infty \frac{1}{(1+k_n)^{\alpha p}} \int_{\Delta_{k_n}} \frac{dt}{r(t)^{p'}} \le \sum_{n=1}^\infty \frac{1}{n^{p\alpha-\beta}} < \infty. \tag{4.9}$$

Let us estimate $(GF)(t)$ from below for $t = \Delta_{k_n}^- = x_{k_n} - d(x_{k_n})$. By (1.5), (4.8), (2.2) and (4.7), we get

$$\begin{aligned} (GF)(\Delta_{k_n}^-) &\ge \int_{\Delta_{k_n}} \frac{1}{r(t)} f_{k_n}(t) \exp\left(-\int_{\Delta_{k_n}^-}^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &\ge \exp\left(-\int_{\Delta_{k_n}} \frac{q(s)}{r(s)}\,ds\right) \int_{\Delta_{k_n}} \frac{f_{k_n}(t)}{r(t)}\,dt \\ &\ge \exp(-2) \frac{k_n^\beta}{(1+k_n)^\alpha} \\ &\ge c^{-1}(k_n+1)^{\beta-\alpha}. \end{aligned} \tag{4.10}$$

For $\beta \geq \alpha$ we obtain from (4.10) that $(GF)(t) \not\to 0$ as $|t| \to \infty$. Let $\alpha \in \left(\frac{1+\beta}{p}, \beta\right]$, $\beta > (p-1)^{-1}$. Then (4.9) and (4.10) imply that problem (1.1)–(1.2) is correctly non-solvable in $L_p(\mathbb{R})$, a contradiction. Hence $A_{p'} < \infty$. Let now $T_{p'} < \infty$ (see (4.5)). First, let us show that (regardless of the value of $T_{p'}$) there exists $c_0 \in \mathbb{R}$ such that the inequality

$$x - d(x) \geq c_0 \tag{4.11}$$

holds for all $x \geq 0$. Assume the contrary. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n - d(x_n) \leq -n$, $x_n \geq 0$, $n = 1, 2, \ldots$. Together with (2.2) and (2.5), this implies

$$2 = \int_{x_n - d(x_n)}^{x_n + d(x_n)} \frac{q(t)}{r(t)}\, dt \geq \int_{-n}^{0} \frac{q(t)}{r(t)}\, dt \to \infty \quad \text{as } n \to \infty,$$

a contradiction. Hence (4.11) is true. Then the function $A_{p'}(x)$ is absolutely bounded for $x \geq 0$. Indeed, from (4.11) we get for $x \geq 0$

$$A_{p'}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}} \leq \left| \int_{c_0}^{0} \frac{dt}{r(t)^{p'}} \right| + \int_0^{\infty} \frac{dt}{r(t)^{p'}} = c + T_{p'} < \infty. \tag{4.12}$$

Taking into account (4.12), let us assume now that $\tilde{A}_{p'} = \infty$, where $\tilde{A}_{p'} = \sup_{x \leq 0} A_{p'}(x)$. We then can repeat all the arguments from the preceding case ($T_{p'} = \infty$). The only difference is that the initial sequence $\{x_k\}_{k=1}^{\infty}$ (see (4.6)) is known to satisfy the property $x_k \to -\infty$ as $k \to \infty$. Taking into account this remark, we reduce this case ($T_{p'} < \infty$) to the preceding one ($T_{p'} = \infty$). We thus proved the necessity of all the conditions of the theorem.

Sufficiency. Since $M_p < \infty$, then $J_\nu(x) \big|_{\nu = p'} < \infty$ for $x \in \mathbb{R}$. Here (see (2.3)–(2.4))

$$J_\nu(x) = \int_x^{\infty} \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_x^t \frac{q(s)}{r(s)}\, ds \right) dt < \infty, \quad x \in \mathbb{R}. \tag{4.13}$$

Therefore, from Hölder's inequality, it follows that the integral $(Gf)(x)$ converges for $f \in L_p(\mathbb{R})$ and $x \in \mathbb{R}$. Hence, the function $(Gf)(x)$, $x \in \mathbb{R}$ is defined and, obviously, satisfies (1.1) almost everywhere on $\mathbb{R}$. From the equality $M_p(x) = H_p(x)$, $x \in \mathbb{R}$, (see (4.4), (3.3) and (2.4)) and Theorem 3.2, we get (1.4).

Now we consider (1.2). Below we need the following lemma.

Lemma 4.2. *Suppose that conditions* (2.5)–(2.6) *hold. Then* $J_\nu < \infty$ *for any* $\nu > 0$ *where* $J_\nu = \sup_{x \in \mathbb{R}} J_\nu(x)$ *(see* (4.13)*).*

Proof. If $T_{p'} = \infty$ (see (4.5)), then $S_2 = \infty$ (see above), and by Lemma 3.7 there is an $R(x,d)$-covering $\{\Delta_n\}_{n=1}^{\infty}$ of $[x, \infty)$. Then

$$\int_x^t \frac{q(s)}{r(s)}\, ds \geq 2(n-1) \quad \text{for } t \in \Delta_n,\ n \geq 1. \tag{4.14}$$

Indeed, for $n = 1$ estimate (4.14) is obvious, and for $n \geq 2$ we have

$$\int_x^t \frac{q(s)}{r(s)}\,ds = \sum_{k=1}^{n-1} \int_{\Delta_k} \frac{q(s)}{r(s)}\,ds + \int_{\Delta_n^-}^t \frac{q(s)}{r(s)}\,ds \geq \sum_{k=1}^{n-1} 2 = 2(n-1). \tag{4.15}$$

By the properties of $\{\Delta_n\}_{n=1}^{\infty}$ and by (4.15) and (2.6), we get

$$\begin{aligned} J_\nu(x) &= \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_x^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &\leq \sum_{n=1}^{\infty} \exp\big(-2(n-1)\nu\big) \int_{\Delta_n} \frac{dt}{r(t)^{p'}} \\ &\leq A_{p'} \sum_{n=1}^{\infty} \exp\big(-2(n-1)\nu\big) \\ &= c_\nu A_{p'} < \infty. \end{aligned}$$

Let now $T_{p'} < \infty$. Since $S_1 = \infty$ (see (2.5)), by Lemma 3.7 there is an $R(0,d)$-covering $\{\Delta_n\}_{n=1}^{\infty}$ of $(-\infty, 0]$. When estimating $J_\nu(x)$, first consider the case $x < 0$. Then $x \in \Delta_{n_0}$, $n_0 \leq -1$ and (2.2) imply

$$\begin{aligned} &\int_x^0 \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_x^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &\qquad = \exp\left(\nu \int_{\Delta_{n_0}^-}^x \frac{q(s)}{r(s)}\,ds\right) \int_x^0 \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_{\Delta_{n_0}^-}^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &\qquad \leq \exp(2\nu) \int_{\Delta_{n_0}^-}^0 \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_{\Delta_{n_0}^-}^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &\qquad = \exp(2\nu) \sum_{k=n_0}^{-1} \int_{\Delta_k} \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_{\Delta_{n_0}^-}^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &\qquad \leq \exp(2\nu) \sum_{k=n_0}^{-1} \int_{\Delta_k} \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_{\Delta_{n_0}^-}^{\Delta_k^-} \frac{q(s)}{r(s)}\,ds\right) dt \\ &\qquad \leq \exp(2\nu) A_{p'} \sum_{k=n_0}^{-1} \exp\big(-2\nu|n_0 - k|\big) \\ &\qquad \leq \exp(2\nu) A_{p'} \sum_{m=0}^{\infty} \exp(-2\nu m) \\ &\qquad = c(\nu) A_{p'}. \end{aligned} \tag{4.16}$$

Using (4.16), we now get for $x \le 0$

$$\begin{aligned} J_\nu(x) &= \int_x^\infty \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_x^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &= \int_x^0 \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_x^t \frac{q(s)}{r(s)}\,ds\right) dt + \int_0^\infty \frac{1}{r(t)^{p'}} \exp\left(-\nu \int_x^t \frac{q(s)}{r(s)}\,ds\right) dt \\ &\le c(\nu) A_{p'} + \int_0^\infty \frac{dt}{r(t)^{p'}} \\ &= c(\nu) A_{p'} + T_{p'}. \end{aligned}$$

For $x \ge 0$, $J_v(x) \le T_{p'}$ is obvious. Thus, $J_\nu \le c(\nu) A_{p'} + T_{p'} < \infty$. □

Now it is easy to see that $y = (Gf)(x) \to 0$ as $x \to \infty$. Indeed from Hölder's inequality and Lemma 4.2, it follows that

$$0 \le |(Gf)(x)| \le J_1^{\frac{1}{p'}} \left[\int_x^\infty |f(t)|^p\,dt\right]^{\frac{1}{p}} \to 0 \quad \text{as } x \to \infty.$$

To check (1.2) for $x \to -\infty$, we use the following two lemmas.

Lemma 4.3. *Let $S_1 = \infty$ (see (2.1)). Then for every $\eta \in (0, \infty)$, there is an $x_0(\eta) \le 0$ such that for every $x \le x_0(\eta)$ the equation in $d \ge 0$*

$$\hat{\Phi}(d) \overset{def}{=} \int_x^{x+d} \frac{q(s)}{r(s)}\,ds = \eta \tag{4.17}$$

has at least one solution $\hat{d}(x, \eta)$ and $x + \hat{d}(x, \eta) \le 0$.

Proof. Clearly, for $d \in [0, \infty)$ the function $\hat{\Phi}(d)$ is continuous, non-negative, and $\hat{\Phi}(0) = 0$. Since $S_1 = \infty$, there is an $x_0(\eta) < 0$ such that

$$\int_{x_0(\eta)}^0 \frac{q(s)}{r(s)}\,ds \ge 2\eta.$$

Set $\mu(x) = -x$ for every $x \le x_0(\eta)$. Then

$$\int_x^{x+\mu(x)} \frac{q(s)}{r(s)}\,ds \ge \int_{x_0(\eta)}^0 \frac{q(s)}{r(s)}\,ds \ge 2\eta > \eta.$$

Thus $\hat{\Phi}(0) = 0$, $\hat{\Phi}(\mu(x)) > \eta$ and therefore, since $\hat{\Phi}$ is continuous, in the segment $[0, \mu(x)]$ there is at least one root $d = \hat{d}(x, \eta)$ of equation (4.17), and since $\hat{d}(x, \eta) \le \mu(x) = -x$, we have $x + \hat{d}(x, \eta) \le 0$. □

Let $d(x,\eta)$, $x \le x_0(\eta)$ be the smallest root of equation (4.17):

$$d(x,\eta) = \inf_{d>0}\left\{ d : \int_x^{x+d} \frac{q(s)}{r(s)}\,ds = \eta,\ x \le x_0(\eta) \right\}. \tag{4.18}$$

Lemma 4.4. *Let $S_1 = \infty$ (see (2.1)). Then*

$$\lim_{x\to-\infty} (x + d(x,\eta)) = -\infty, \quad \eta \in (0,\infty). \tag{4.19}$$

Proof. Let $\overline{\lim}_{x\to-\infty}(x + d(x,\eta)) = c,\ c \in \mathbb{R}$. Then $c \le 0$ (by Lemma 4.3 and (4.18)). Furthermore, there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to -\infty$ as $n \to \infty$ and $x_n + d(x_n,\eta) \ge c - 1$ for all $n \ge 1$. Hence

$$\eta = \int_{x_n}^{x_n + d(x_n,\eta)} \frac{q(s)}{r(s)}\,ds \ge \int_{x_n}^{c-1} \frac{q(s)}{r(s)}\,ds \to \infty \quad \text{for } n \to \infty,$$

a contradiction. Thus $\overline{\lim\limits_{x\to-\infty}}(x+d(x,\eta)) = -\infty$. This equality implies (4.19). $\square$

Fix now $\eta \in (0,\infty)$. Let $x_0(\eta)$ be the number defined in Lemma 4.3 and $x \le x_0(\eta)$. From (4.18), Lemma 4.2 and Hölder's inequality, it follows that

$$\begin{aligned}
0 \le |(Gf)(x)|^p &\le J_1^{\frac{p}{p'}} \int_x^{\infty} |f(t)|^p \exp\left(-\int_x^t \frac{q(s)}{r(s)}\,ds\right) dt \\
&\le c\int_x^{x+d(x,\eta)} |f(t)|^p\,dt + c\exp\left(-\int_x^{x+d(x,\eta)} \frac{q(s)}{r(s)}\,ds\right) \\
&\quad \cdot \int_{x+d(x,\eta)}^{\infty} |f(t)|^p \exp\left(-\int_{x+d(x,\eta)}^{t} \frac{q(s)}{r(s)}\,ds\right) dt \\
&\le c\int_x^{x+d(x,\eta)} |f(t)|^p\,dt + c\exp(-\eta)\|f\|_p^p.
\end{aligned}$$

Since $f \in L_p(\mathbb{R})$, from (4.19) we obtain

$$0 \le \int_x^{x+d(x,\eta)} |f(t)|^p\,dt \le \int_{-\infty}^{x+d(x,\eta)} |f(t)|^p\,dt \to 0 \quad \text{as } x \to -\infty,$$

which implies

$$0 \le \overline{\lim_{x\to-\infty}} |(Gf)(x)|^p \le c\exp(-\eta)\|f\|_p^p. \tag{4.20}$$

In (4.20) the number $\eta \in (0,\infty)$ is arbitrary. Hence in (4.20), we can take the limit $\eta \to \infty$. We get $\overline{\lim}_{x\to-\infty} |(Gf)(x)| = 0$, hence $0 \le \underline{\lim}_{x\to-\infty} |(Gf)(x)| \le \overline{\lim}_{x\to-\infty} |(Gf)(x)| = 0$. The last relations finish the proof of Theorem 2.1. $\square$

Proof of Theorem 2.4. *Necessity*: The necessity of the condition $A_{p'} < \infty$ (see (2.6)) for correct solvability of problem(1.1)–(1.2) in $L_p(\mathbb{R})$ follows from Theorem 2.1.

Sufficiency: Suppose that (2.9) holds. Then $S_1 = \infty$, $d_0 < \infty$, by Lemma 2.3. Then by Lemma 3.7, for every $x \in \mathbb{R}$ there exist $R(x, d)$-coverings $\{\Delta_n\}_{n=-\infty}^{-1}$ for $(-\infty, x]$. Below we need the following lemma.

Lemma 4.5. *Let $S_1 = \infty$ and $d_0 < \infty$ (see* (2.1) *and* (2.8)*). Then $I_\nu < \infty$ for every $\nu > 0$, where $I_\nu = \sup_{x\in\mathbb{R}} I_\nu(x)$,*

$$I_\nu(x) = \int_{-\infty}^{x} \exp\left(-\nu \int_t^x \frac{q(s)}{r(s)}\, ds\right) dt, \quad x \in \mathbb{R}.$$

Proof. If $t \in \Delta_n$, $n \le -1$, then

$$\int_t^x \frac{q(s)}{r(s)}\, ds \ge 2(|n| - 1), \quad n \le -1. \tag{4.21}$$

The proofs of (4.21) and (4.14) are similar. Further

$$\begin{aligned} I_\nu(x) &= \sum_{n=-\infty}^{-1} \int_{\Delta_n} \exp\left(-\nu \int_t^x \frac{q(s)}{r(s)}\, ds\right) dt \\ &\le 2 \sum_{n=-\infty}^{-1} d(x_n) \exp\big(-2(|n| - 1)\nu\big) \\ &\le 2d_0 \sum_{n=1}^{\infty} \exp\big(-2(n-1)\nu\big) \\ &= c(\nu) d_0 < \infty. \end{aligned}$$

□

From Lemmas 4.2 and 4.5, it follows that $M_p < \infty$:

$$M_p(x) = (I_p(x))^{\frac{1}{p}} (J_{p'}(x))^{\frac{1}{p'}} \le (I_p)^{\frac{1}{p}} (J_{p'})^{\frac{1}{p'}} < \infty.$$

Thus the conditions of Theorem 2.1 are satisfied and therefore the problem (1.1)–(1.2) is correctly solvable in $L_p(\mathbb{R})$. □

5. Proof of the main results in the case $p = 1$

In this section, we prove Theorems 2.5 and 2.7.

Proof of Theorem 2.3. *Necessity*: Suppose that problem (1.1)–(1.2) is correctly solvable in $L_1(\mathbb{R})$. By Lemma 4.1, its solution y is of the form (4.3)–(4.4), and therefore (3.5) implies (2.12)–(2.13). Furthermore, if $S_1 < \infty$ (see (2.10), then $M_1 < \infty$ and

$$M_1 = \sup_{x\in\mathbb{R}} \frac{1}{r(x)} \int_{-\infty}^{x} \exp\left(-\int_t^x \frac{q(s)}{r(s)}\,ds\right) dt \geq \exp(-S_1) \sup_{x\in\mathbb{R}} \frac{1}{r(x)} \int_{-\infty}^{x} 1\,dt = \infty,$$

a contradiction, and hence $S_1 = \infty$. Further, let $r_0 = 0$ and let α be a positive number which will be chosen later. Since $r_0 = 0$, from (1.3) it follows that there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $r(x_n) = n^{-\alpha}$, $n = 1, 2, \dots$, and clearly, $|x_n| \to \infty$ as $n \to \infty$. From (1.3) it also follows that there are numbers $\delta_n > 0$, $n = 1, 2, \dots$, such that

$$\frac{1}{2n^\alpha} \leq r(t) \leq \frac{2}{n^\alpha}, \quad t \in [x_n - \delta_n, x_n + \delta_n],\ n = 1, 2, \dots\ .$$

Let $\omega_n = \min\{\delta_n, d(x_n), 1\}$, $n = 1, 2, \dots$. Making δ_n smaller (if necessary), we can choose ω_n, $n = 1, 2, \dots$ so that the segments $\Delta_n =]x_n - \omega_n, x_n + \omega_n]$, $n = 1, 2, \dots$, are disjoint. We introduce the functions

$$f_n(t) = \begin{cases} \frac{1}{2\omega_n}\,\frac{1}{n^\alpha}, & t \in \Delta_n \\ 0, & t \notin \Delta_n \end{cases}, \quad n = 1, 2, \dots$$

$$F(t) = \sum_{n=1}^{\infty} f_n(t), \qquad t \in \mathbb{R}.$$

Let us verify that $F \in L_1(\mathbb{R})$ for $\alpha > 1$. Indeed,

$$\|F\|_1 = \sum_{n=1}^{\infty} \int_{\Delta_n} |f_n(t)|\,dt = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}\,\frac{2\omega_n}{2\omega_n} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty.$$

We now estimate $(GF)(t)$ from below for $t = t_n = x_n - \omega_n$, $n = 1, 2, \dots$:

$$\begin{aligned}(GF)(t_n) &\geq \int_{x_n-\omega_n}^{x_n+\omega_n} \frac{1}{r(t)} \exp\left(-\int_{x_n-\omega_n}^{t} \frac{q(s)}{r(s)}\,ds\right) f_n(t)dt \\ &\geq \frac{1}{2\omega_n \cdot n^\alpha} \int_{x_n-\omega_n}^{x_n+\omega_n} \frac{1}{r(t)} \exp\left(-\int_{x_n-\omega_n}^{x_n+\omega_n} \frac{q(s)}{r(s)}\,ds\right) dt \\ &\geq \frac{\exp(-2)}{2}.\end{aligned}$$

Thus, $(GF)(t) \not\to 0$ as $|t| \to \infty$, a contradiction. Hence $r_0 > 0$.

Sufficiency: Consider the function $y = (Gf)(x)$, $x \in \mathbb{R}$, $f \in L_1(\mathbb{R})$ (see (1.5)). Since $r_0 > 0$ (see (2.11)), the integral $(Gf)(x)$ converges for all $x \in \mathbb{R}$, and we have

$$|(Gf)(x)| \leq \frac{1}{r_0} \int_x^{\infty} \exp\left(-\int_x^t \frac{q(s)}{r(s)}\,ds\right) |f(t)|\,dt \leq \frac{1}{r_0} \int_x^{\infty} |f(t)|\,dt. \tag{5.1}$$

Thus, the function $y = (Gf)$, $x \in \mathbb{R}$ is defined for $f \in L_1(\mathbb{R})$ and, obviously, satisfies (1.1) almost everywhere on $\mathbb{R}$. From (4.3)–(4.4), (3.5) and (2.12)–(2.13), it follows that the operator $G : L_1(\mathbb{R}) \to L_1(\mathbb{R})$ is bounded, and we obtain (1.4). The inequality (5.1) implies (1.2) as $x \to \infty$. The proof of (1.2) as $x \to -\infty$ for $p = 1$ and $p \in (1, \infty)$ is similar (see §4). $\square$

Proof of Theorem 2.7. *Necessity*: If problem(1.1)–(1.2) is correctly solvable in $L_1(\mathbb{R})$, then $r_0 > 0$ by (2.11).

Sufficiency: Suppose that condition (2.9) holds. Then $S_1 = \infty$, $d_0 < \infty$ by Lemma 2.3, and there is a $R(x,d)$-covering $\{\Delta_n\}_{n=-\infty}^{-1}$ of $(-\infty, x]$, $x \in \mathbb{R}$ (see Lemma 3.7). Then (4.21) implies

$$\begin{aligned} M_1 &\le \frac{1}{r_0} \sup_{x\in\mathbb{R}} \sum_{n=-\infty}^{-1} \int_{\Delta_n} \exp\left(-\int_t^x \frac{q(s)}{r(s)}\, ds\right) dt \\ &\le \frac{2}{r_0} \sup_{x\in\mathbb{R}} \sum_{n=-\infty}^{-1} d(x_n) \exp\big(-2(|n|-1)\big) \\ &\le \frac{2d_0}{r_0} \sum_{n=1}^{\infty} \exp\big(-2(n-1)\big) \\ &= \frac{cd_0}{r_0} < \infty. \end{aligned}$$

To complete the proof, it remains to use Theorem 2.5. $\square$

6. Proofs of the main results in the case $p = \infty$

In this section we prove Theorems 2.8 – 2.9.

Proof of Theorem 2.8. *Necessity*: Suppose that problem(1.1)–(1.2) is correctly solvable in $C(\mathbb{R})$ and $y = (Gf)(x)$, $x \in \mathbb{R}$ is its solution (see Lemma 4.1). Then if $f(x) \equiv 1$, $x \in \mathbb{R}$, we obtain $(Gf)(x) = (G1)(x) \to 0$ as $|x| \to \infty$, i.e., $A_0 = 0$.

Sufficiency. Since $A_0 = 0$, $A(x) \in C(\mathbb{R})$ (see (1.3)). Then from (4.3)–(4.4) and (3.6), we obtain that the operator $G : C(\mathbb{R}) \to C(\mathbb{R})$ is bounded. Therefore, the function $y = (Gf)(x)$, $x \in \mathbb{R}$, is defined for $f \in C(\mathbb{R})$ and satisfies (1.4) and, obviously, (1.1) almost everywhere on $\mathbb{R}$. Since $A_0 = 0$, the estimate $|(Gf)(x)| \le A(x)\|f\|_{C(\mathbb{R})}$, $x \in \mathbb{R}$, implies (1.2). It remains to check that $S_1 = \infty$. Let $S_1 < \infty$. If $x \le 0$, then (see below) (6.1) follows from (2.6), and (6.1) implies (6.2) by the condition $A_0 = 0$:

$$A(x) \ge \int_x^\infty \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(s)}{r(s)}\, ds\right) dt \ge \exp(-S_1) \int_x^0 \frac{dt}{r(t)}, \tag{6.1}$$

and hence

$$0 = \lim_{x\to-\infty} A(x) \geq \exp(-S_1) \lim_{x\to-\infty} \int_x^0 \frac{dt}{r(t)} > 0, \tag{6.2}$$

a contradiction. Hence $S_1 = \infty$. Theorem 2.8 is proved. □

Proof of Theorem 2.9. *Necessity*: Let $B(a) > 0$ for some $a \in (0,\infty)$. Then $S_1 = \infty$, $d_0 < \infty$ by Lemma 2.3. Since the problem(1.1)–(1.2) is correctly solvable in $C(\mathbb{R})$, then $A_0 = 0$ and

$$\begin{aligned}
0 = \lim_{|x|\to\infty} A(x-d(x)) &\geq \lim_{|x|\to\infty} \int_{x-d(x)}^{\infty} \frac{1}{r(t)} \exp\left(-\int_{x-d(x)}^{t} \frac{q(s)}{r(s)}\,ds\right) dt \\
&\geq \lim_{|x|\to\infty} \exp\left(-\int_{x-d(x)}^{x+d(x)} \frac{q(s)}{r(s)}\,ds\right) \cdot \int_{x-(d(x)}^{x+d(x)} \frac{dt}{r(t)} \\
&= \exp(-2) \lim_{|x|\to\infty} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \geq 0.
\end{aligned}$$

Thus $\tilde{A}(x) \to 0$ as $|x| \to \infty$.

Sufficiency: Let $B(a) > 0$ for some $a \in (0,\infty)$ and $\tilde{A}_0 = 0$. Then $S_1 = \infty$, $d_0 < \infty$ by Lemma 2.3, $S_2 = \infty$ (because $d_0 < \infty$), there is a $R(x,d)$-covering $\{\Delta_n\}_{n=1}^{\infty}$ of $[x,\infty)$, $x \in \mathbb{R}$, (see Lemma 3.7) and, finally, $\tilde{A} = \|\tilde{A}\|_{C(\mathbb{R})} < \infty$. Furthermore, for a given $\varepsilon > 0$, there is an $x_0(\varepsilon) \gg 1$ such that $\tilde{A}(x) \leq \frac{\varepsilon}{4}$ for $|x| \geq x_0(\varepsilon)$. Let n_0 be a natural number such that $4\tilde{A}\exp(-2n_0) \leq \varepsilon$. Then the following relations are fulfilled for $|x| \geq \tilde{x}_0(\varepsilon) = x_0(\varepsilon) + 2n_0 d_0$:

$$\bigcup_{k=1}^{n_0} \Delta_k \subseteq [x, x+2n_0d_0], \quad [x, x+2n_0d_0] \cap [-x_0(\varepsilon), x_0(\varepsilon)] = \emptyset.$$

Consequently, for $|x| \geq \tilde{x}_0(\varepsilon)$, we obtain

$$\begin{aligned}
A(x) &= \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{r(t)} \exp\left(-\int_x^t \frac{q(s)}{r(s)}\,ds\right) dt \\
&\leq \sum_{n=1}^{\infty} \exp\big(-2(n-1)\big) \int_{\Delta_n} \frac{dt}{r(t)} \\
&= \sum_{n=1}^{n_0} \exp\big(-2(n-1)\big) \int_{\Delta_n} \frac{dt}{r(t)} + \sum_{n=n_0+1}^{\infty} \exp\big(-2(n-1)\big) \int_{\Delta_n} \frac{dt}{r(t)} \\
&\leq \frac{\varepsilon}{4} \sum_{n=1}^{\infty} \exp\big(-2(n-1)\big) + \tilde{A}\exp(-2n_0) \sum_{n=1}^{\infty} \exp\big(-2(n-1)\big) \leq \varepsilon.
\end{aligned} \tag{6.3}$$

From (6.3) it follows that $A_0 = 0$. It remains to apply Theorem 2.8. □

7. Correct non-solvability of the boundary value problem

In this section we prove Theorem 2.10.

Proof of Theorem 2.10. By Lemma 2.3 we have $S_1 = \infty$, $d_0 < \infty$. Let $q(x) \to 0$ as $x \to \infty$ (the second case of (2.16) can be considered in a similar way). Then for a given $\varepsilon > 0$, there is $x_0(\varepsilon)$ such that $q(x) \le \varepsilon$ for $x \ge x_0(\varepsilon)$. Below $x \ge x_0(\varepsilon) + d_0$. Obviously, $q(t) \le \varepsilon$ for $t \ge x - d(x)$ since $x - d(x) \ge x_0(\varepsilon) + d_0 - d(x) \ge x_0(\varepsilon)$. Let the problem (1.1)–(1.2) be correctly solvable for some $p \in [1, \infty]$. Further we study separately the following cases: I) $p \in (1, \infty)$; II) $p = 1$; III) $p = \infty$. In each case, we obtain a contradiction which proves the statement of the theorem.

`Case I)`: Since $A_{p'} < \infty$, then (2.2), (2.7) and (2.15) imply

$$2 \le \varepsilon \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \le c\varepsilon d(x)^{\frac{1}{p}} \left[\int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}} \right]^{\frac{1}{p'}} \le c\varepsilon d(x)^{\frac{1}{p}} A_p^{\frac{1}{p'}}.$$

Thus, $d(x) \to \infty$ as $x \to \infty$, a contradiction, since $d_0 < \infty$.

`Case II)`: Since $r_0 > 0$ (see (2.11)), then from (2.2) it follows that

$$2 = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)}\, dt \le \varepsilon \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \le \frac{2\varepsilon}{r_0} d(x).$$

Thus, $d(x) \to \infty$ as $x \to \infty$, a contradiction, since $d_0 < \infty$.

`Case III)`: From (2.2) and (2.14), it follows that

$$2 = \int_{x-d(x)}^{x+d(x)} \frac{q(t)}{r(t)} \le \varepsilon \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)}.$$

Thus, $\tilde{A}(x) \nrightarrow 0$ as $x \to \infty$, a contradiction, since $\tilde{A} = 0$. □

8. Example

Let $\alpha, \beta \in \mathbb{R}$, $\gamma > 0$, and

$$r(x) = e^{\alpha|x|}, \quad q(x) = e^{\beta|x|} + e^{\beta|x|} \cos e^{\gamma|x|}, \quad x \in \mathbb{R}. \tag{8.1}$$

In this section, we study a condition of correct solvability in $L_p(\mathbb{R})$, $p \in [1, \infty]$, for problem(1.1)–(1.2) in the case (8.1) (below for brevity, we write “problem (8.1)”).

8.1. Necessary conditions for correct solvability.

Lemma 8.1. *The problem* (8.1)*, is correctly solvable in* $L_p(\mathbb{R})$*,* $p \in [1,\infty]$*, only if* $\beta \geq \alpha$ *and* $\beta \geq 0$*. In the latter case,* $S_1 = \infty$*,* $d_0 < \infty$ *(see* (2.1)*,* (2.8)*).*

Proof. By Theorems 2.1, 2.5 and 2.8, the problem(1.1)–(1.2) is correctly solvable in $L_p(\mathbb{R})$, $p \in [1,\infty]$ only if $S_1 = \infty$. It is clear that in our case $S_1 = \infty$ only if $\beta \geq \alpha$. We shall show that in this case (2.9) is fulfilled. Let $a = \frac{3}{2\gamma}$, $x \geq a$ and ξ is some point on the segment $[x-a, x+a]$. We denote

$$f(t) = e^{\theta t} + e^{\theta t}\cos e^{\gamma t}, \quad t \geq 0,\ \theta = \beta - \alpha \geq 0. \tag{8.2}$$

Below we use the mean value theorem [13, §12.3] and get

$$\begin{aligned} F(x) = \int_{x-a}^{x+a} \frac{q(t)}{r(t)}\,dt &= \int_{x-a}^{x+a} f(t)\,dt \\ &\geq \int_{x-a}^{x+a} (1+\cos e^{\gamma t})\,dt \\ &= 2a + \frac{1}{\gamma}\int_{x-a}^{x+a} e^{-\gamma t}[\gamma e^{\gamma t}\cos e^{\gamma t}]\,dt \\ &= 2a + \frac{1}{\gamma e^{x-a}}\left(\sin e^{\gamma t}\Big|_{x-a}^{\xi}\right) \\ &\geq 2a - \frac{2}{\gamma} = \frac{1}{\gamma} > 0. \end{aligned}$$

For $z \in [0,a]$, the function $F(z)$ is continuous and positive. Therefore $F_0(a) = \min_{z\in[0,a]} F(z) > 0$. Let $\alpha = \min\{\gamma^{-1}, F_0(a)\}$. Then $F(z) \geq \alpha$ for all $z \geq 0$. The case $z \leq 0$ can be considered in a similar way. Thus, (2.9) holds. By Lemma 2.3, we have $d_0 < \infty$. Moreover, $\beta \geq 0$ by Theorem 2.10, and therefore the lemma is proved. ☐

8.2. Estimates of the auxiliary function of growth. Below we assume that conditions $\beta > \alpha$, $\beta \geq 0$ hold. We shall establish inequalities for d on $\mathbb{R}$ in three separate cases: $\theta < \gamma$, $\theta = \gamma$, $\theta > \gamma$. We need the following notation. Let the functions φ and ψ be positive and continuous on $\mathbb{R}$. We write $\varphi \asymp \psi$ if there is a $c \in [1,\infty)$ such that $c^{-1}\varphi(x) \leq \psi(x) \leq c\varphi(x)$, $x \in \mathbb{R}$. Moreover, in the proofs, we consider only the case $x \geq 0$ where the functions r and q in (8.1) are even.

Lemma 8.2. *Let* $0 \leq \theta \leq \gamma$*,* $\theta = \beta - \alpha$*. Then*

$$d(x) \asymp e^{-\theta|x|}, \quad x \in \mathbb{R}. \tag{8.3}$$

Proof. To apply Theorem 2.11, set

$$q_1(x)=e^{\beta|x|},\quad q_2(x)=e^{\beta|x|}\cos(e^{\gamma|x|}),\quad x\in\mathbb{R}.$$

Let $\theta<\gamma$. Let us estimate $\varkappa_1(x)$ (see (2.17)) for $x\to\infty$:

$$\begin{aligned}\varkappa_1(x)&=\sup_{|z|\le 2e^{-\theta x}}\left|\int_0^z\left[e^{\theta(x+t)}-2e^{\theta x}+e^{\theta(x-t)}\right]dt\right|\\&=e^{\theta x}\sup_{|z|\le 2e^{-\theta x}}\left|\int_0^z\left[\theta^2t^2+\dots\right]dt\right|\le ce^{-2\theta x}.\end{aligned}$$

Below, when estimating $\varkappa_2(x)$, $x\gg 1$, we use the mean value theorem [13, §12.3]:

$$\begin{aligned}\varkappa_2(x)&=\sup_{|z|\le 2e^{-\theta x}}\left|\int_{x-z}^{x+z}e^{\theta t}\cos(e^{\gamma t})\,dt\right|\\&=\sup_{|z|\le 2e^{-\theta x}}\left|\int_{x-z}^{x+z}\frac{e^{(\theta-\gamma)t}}{\gamma}\left[\gamma e^{\gamma t}\cos(e^{\gamma t})\right]dt\right|\\&\le ce^{(\theta-\gamma)x}.\end{aligned}\tag{8.4}$$

Since $\theta<\gamma$, we have $\varkappa_1(x)\to 0$, $\varkappa_2(x)\to 0$ as $x\to\infty$. By Theorem 2.11 this implies (8.3). Consider now the cases $\theta=0$ and $\theta=\gamma$. If $\theta=0$, then $\varkappa_1(x)=0$ and (8.4) holds. Therefore, we obtain (8.3) as above. Let $\theta=\gamma$. We set $\eta(x)=(1+\gamma^{-1})e^{-\gamma x}$, $x\gg 1$. Then

$$\begin{aligned}\int_{x-\eta(x)}^{x+\eta(x)}f(t)\,dt&=e^{\gamma x}\frac{e^{\gamma\eta(x)}-e^{-\gamma\eta(x)}}{\gamma}+\frac{\sin(e^{\gamma x})}{\gamma}\bigg|_{x-\eta(x)}^{x+\eta(x)}\\&\ge\frac{e^{\gamma x}}{\gamma}\left[2\gamma\eta(x)+2\frac{(\gamma\eta(x))^3}{3!}+\dots\right]-\frac{2}{\gamma}\ge 2.\end{aligned}$$

From here $d(x)\le\eta(x)$ for $x\gg 1$ (see (3.19)). Let $\varepsilon=\min\{4^{-1},(2\gamma)^{-1}\}$, $\eta(x)=\varepsilon e^{-\gamma x}$. Then, for $x\gg 1$, we obtain

$$\int_{x-\eta(x)}^{x+\eta(x)}f(t)\,dt\le 2\int_{x-\eta(x)}^{x+\eta(x)}e^{\gamma t}\,dt=\frac{4}{\gamma}e^{\gamma x}\left[\gamma\eta(x)+\frac{(\gamma\eta(x))^3}{3!}+\cdots\right]<2.$$

Hence $d(x)\ge\eta(x)$ for $x\gg 1$ (see (3.19)). This implies (8.3). □

Lemma 8.3. *Let $\theta>\gamma$. Denote by $\{x_k\}_{k=-\infty}^{\infty}$ a sequence of points such that $|x_k|=\gamma^{-1}\ln[(2|k|+1)\pi]$, $k=0,\pm1,\pm2,\dots.$ Then*

$$d_k\overset{def}{=}d(x_k)\asymp(2|k|+1)^{-\frac{\theta+2\gamma}{3\gamma}},\quad k=0,\pm1,\pm2,\dots.\tag{8.5}$$

Proof. Let $k \gg 1$, $d \in [0,1]$, $t \in \omega_k = [x_k - d, x_k + d]$. Since $f(x_k) = f'(x_k) = 0$ (see (8.2)), by Taylor's formula we obtain

$$\int_{\omega_k} f(t)\,dt = \frac{f''(x_k)}{3} d^3 + \frac{1}{3!}\int_{\omega_k} f'''(\xi(t))(t-x_k)^3\,dt. \tag{8.6}$$

The following relations are obvious:

$$f''(x_k) = \gamma^2[(2k+1)\pi]^{\frac{\theta+2\gamma}{\gamma}} \asymp k^{\frac{\theta+2\gamma}{\gamma}}, \qquad k \gg 1, \tag{8.7}$$

$$|f'''(\xi)| \le ce^{(\theta+3\gamma)\xi} \le ce^{(\theta+3\gamma)x_k} \le ck^{\frac{\theta+3\gamma}{\gamma}}, \quad k \gg 1,\ \xi \in \omega_k, \tag{8.8}$$

and hence

$$\left|\int_{\omega_k} f'''(\xi(t))(t-x_k)^3\,dt\right| \le ck^{\frac{\theta+3\gamma}{\gamma}} d^4. \tag{8.9}$$

Let $\mu \in (0,\infty)$ and $\eta_k = \mu[f''(x_k)]^{-\frac{1}{3}}$. Then due to (8.6), (8.7) and (8.9), we get for $\mu = (\frac{9}{2})^{\frac{1}{3}}$ and $k \gg 1$

$$\int_{x-\eta_k}^{x+\eta_k} f(t)\,dt \le \frac{3}{2} + ck^{-\frac{\theta-\gamma}{3\gamma}} < 2.$$

Hence, $d_k \ge \eta_k$ by Lemma 3.9. Similarly, we have for $\mu = 9^{\frac{1}{3}}$

$$\int_{x_k-d}^{x_k+d} f(t)\,dt \ge 3 - ck^{-\frac{\theta-\gamma}{3\gamma}} \ge 2, \quad k \gg 1,$$

and hence $d_k \le \eta_k$ by Lemma 3.9. The relation (8.5) then follows. □

Lemma 8.4. *The following inequalities hold:*

$$d(x) \le cd_k \quad \text{for } x \in [x_k, x_{k+1}],\ k = 0, \pm 1, \pm 2, \dots . \tag{8.10}$$

To prove (8.10) (for $x \ge 0$), we need Lemmas 8.5 – 8.10 below. The case $x \le 0$ can be studied in a similar way.

Lemma 8.5. *For all $k = 0, 1, 2, \dots,$ the function $f(t)$ (see (8.2)) has a unique extremum (maximum) on the interval (x_k, x_{k+1}). If $\tilde z_k$ is a corresponding extreme point, then*

$$z_k < \tilde z_k < z_k + \mu k^{-2} \quad \text{for } k \gg 1. \tag{8.11}$$

Here μ is some absolutely positive constant, $z_k = \gamma^{-1}\ln[(2k+2)\pi]$.

Proof. The equality $f'(t) = 0$ can be easily brought to the form

$$\gamma\Big(\cos\frac{1}{2}e^{\gamma t}\Big)^2 e^{-\gamma t}\varphi(t) = 0, \quad \varphi(t) \overset{\text{def}}{=} \theta\gamma^{-1}e^{-\gamma t} - \operatorname{tg}\Big(\frac{1}{2}e^{\gamma t}\Big). \tag{8.12}$$

Since $f(t) > 0$, $t \in (x_k, x_{k+1})$ and $f(x_k) = f(x_{k+1}) = 0$, then $f(t)$ has maximum on the interval (x_k, x_{k+1}). Furthermore, the first two factors in (8.12) are positive on (x_k, x_{k+1}), and $\varphi'(t) < 0$ for $t \in (x_k, x_{k+1})$, $\varphi(x_k)\varphi(x_{k+1}) < 0$. This means that $\varphi(t)$ has a unique root on (x_k, x_{k+1}), as desired. The lower estimate (8.11) follows from an inequality $f'(z_k) > 0$. Let $k \gg 1$, $\hat{z}_k = z_k + \mu k^{-2}$. By Taylor's formula, we obtain (8.13) with $\xi \in (z_k, \hat{z}_k)$

$$f'(\hat{z}_k) = f'(z_k)\left[1 + \frac{f''(z_k)}{f'(z_k)}\,\frac{\mu}{k^2} + \frac{1}{2!}\,\frac{f'''(\xi)}{f'(z_k)}\,\frac{\mu^2}{k^4}\right]. \tag{8.13}$$

Together with the obvious relations

$$f'(z_k) \asymp k^{\frac{\theta}{\gamma}}, \quad |f''(z_k)| \asymp k^{\frac{\theta+2\gamma}{\gamma}}, \quad f''(z_k) < 0, \quad k \gg 1, \tag{8.14}$$

we use (8.13) and (8.8) and obtain

$$f'(\hat{z}_k) \le f'(z_k)\big(1 - c^{-1}\mu + c\mu^2 k^{-1}\big).$$

Clearly, $f'(\hat{z}_k) < 0$ for $\mu = 2c$ and $k \gg 1$. Therefore, the upper estimate (8.11) is true. □

Lemma 8.6. *For all $k \gg 1$, the following inequality holds:*

$$f(x_k + d_k) \ge f(x_k - d_k). \tag{8.15}$$

Proof. The inequality (8.15) is equivalent to (see (8.2))

$$e^{\theta d_k}\big[1 + \cos e^{\gamma(x_k+d_k)}\big] \ge e^{-\theta d_k}\big[1 + \cos e^{\gamma(x_k-d_k)}\big], \quad k \gg 1. \tag{8.16}$$

From (8.16) and the obvious equalities

$$\begin{aligned}\cos\left[e^{\gamma(x_k+d_k)}\right] &= -\cos\left[(2k+1)\pi(e^{\gamma d_k} - 1)\right], \quad k \gg 1\\ \cos\left[e^{\gamma(x_k-d_k)}\right] &= -\cos\left[(2k+1)\pi(e^{-\gamma d_k} - 1)\right], \quad k \gg 1,\end{aligned}$$

it follows that (8.16) is equivalent to

$$e^{\theta d_k} \ge \left|\frac{\sin\left[\left(k+\frac{1}{2}\right)\pi(e^{\gamma d_k} - 1)e^{-\gamma d_k}\right]}{\sin\left[\left(k+\frac{1}{2}\right)\pi(e^{\gamma d_k} - 1)\right]}\right|, \quad k \gg 1. \tag{8.17}$$

In this connection, we note that

$$0 < \big(k + \tfrac{1}{2}\big)\pi(e^{\gamma d_k} - 1) \le ckd_k \le ck^{-\frac{\theta-\gamma}{3\gamma}} \to 0 \quad \text{as } k \to \infty.$$

This means that in (8.17), the arguments of both sines tend to zero (as $k \to \infty$) and are positive. Therefore, (8.17) follows from the monotonicity of the function $\sin x$ in the neighborhood of the point $x = 0$. □

Lemma 8.7. *The following equalities hold:*

$$\lim_{k\to\infty} \frac{\tilde z_k - x_k}{d_k} = \infty, \quad \lim_{k\to\infty} \frac{x_{k+1} - \tilde z_k}{d_k} = \infty.$$

Proof. Below we use (8.5) and (8.11) to get

$$\frac{\tilde z_k - x_k}{d_k} > \frac{z_k - x_k}{d_k} = \frac{1}{\gamma}\,\frac{\ln\left(1+\frac{1}{2k+1}\right)}{d_k} \geq ck^{\frac{\theta-\gamma}{3\gamma}} \to \infty$$

$$\frac{x_{k+1} - \tilde z_k}{d_{k+1}} \geq \frac{x_{k+1} - z_k - \mu k^{-2}}{d_{k+1}} \geq \frac{1}{\gamma}\,\frac{\ln\left(1+\frac{1}{2k+2}\right)}{d_k} - \frac{\mu}{k^2 d_k} \geq c^{-1}k^{\frac{\theta-\gamma}{3\gamma}} \to \infty$$

as $k \to \infty$. □

Corollary 8.7.1. *For all $k \gg 1$, the function $f(t)$ decreases monotonically on the segment $[x_k - d_k, x_k]$ and increases monotonically on the segment $[x_k, x_k + d_k]$.*

Lemma 8.8. *The following equality holds:*

$$\lim_{k\to\infty} \frac{f(\tilde z_k)}{f(z_k)} = 1. \tag{8.18}$$

Proof. It is obvious that $f(\tilde z_k) = f(z_k) + f'(\xi)(\tilde z_k - z_k)$, $\xi \in (z_k, \tilde z_k)$ and $|f'(\xi)| \leq ce^{(\theta+\gamma)\xi} \leq ce^{(\theta+\gamma)z_k} \leq ck^{\frac{\theta+\gamma}{\gamma}}$ for $k \gg 1$. From here and taking into account (8.11), we obtain

$$\left|\frac{f(\tilde z_k)}{f(z_k)} - 1\right| \leq c\,\frac{|f'(\xi)|}{f(z_k)}\,\frac{1}{k^2} \leq ck^{\frac{\theta+\gamma}{\gamma} - \frac{\theta}{\gamma} - 2} = \frac{c}{k} \to 0 \quad \text{as } k \to \infty.$$ □

Lemma 8.9. *For $k \gg 1$, the following inequalities hold:*

$$f(\tilde z_k + d_k) \geq f(x_k + d_k), \quad f(\tilde z_k - d_{k+1}) \geq f(x_{k+1} + d_{k+1}).$$

Proof. Both inequalities can be verified in the same way. We prove, for example, the first one. Below we use Taylor's formula, the estimates of type (8.14) and equality (8.18):

$$f(\tilde z_k + d_k) \geq f(\tilde z_k)\left(1 - \frac{|f''(\xi)|}{2f(\tilde z_k)}\,d_k^2\right) \geq c^{-1}f(z_k)\left(1 - ck^{\frac{\theta+2\gamma}{\gamma} - \frac{\theta}{\gamma} - \frac{2}{3}\,\frac{\theta+2\gamma}{\gamma}}\right) \geq c^{-1}k^{\frac{\theta}{\gamma}},$$

for $k \gg 1$. Below we again use Taylor's formula $f(x_k + d_k) = \frac{f''(\xi)}{2!}\,d_k^2$, $\xi \in (x_k, x_k + d_k)$, and estimates of type (8.7) to get

$$f(x_k + d_k) = \frac{f''(\xi)}{2!}\,d_k^2 \leq ck^{\frac{\theta+2\gamma}{\gamma} - \frac{2}{3}\,\frac{\theta+2\gamma}{\gamma}} = ck^{\frac{\theta+2\gamma}{3\gamma}}, \quad k \gg 1.$$

Based on what we have found, we obtain for $k \gg 1$

$$f(\tilde z_k + d_k) \geq c^{-1}k^{\frac{\theta}{\gamma}} \geq ck^{\frac{\theta+2\gamma}{3\gamma}} \geq f(x_k + d_k).$$ □

Lemma 8.10. *The following inequality is true:*

$$f(x_k - 2d_k) \geq f(x_k + d_k), \qquad k \gg 1. \tag{8.19}$$

Proof. The following relations are obvious:

$$\begin{aligned} \frac{2f(x_k-2d_k)}{f''(x_k)\,d_k^2} &= 4 - \frac{8}{3}\,\frac{f'''(\xi_1)}{f''(x_k)}\,d_k \geq 4 - \frac{8}{3}\,\frac{|f'''(\xi_1)|}{f''(x_k)}\,d_k \\ \frac{2f(x_k+d_k)}{f''(x_k)d_k^2} &= 1 + \frac{f'''(\xi_2)}{3f''(x_k)}\,d_k \ \leq 1 + \frac{|f'''(\xi_2)|}{3f''(x_k)}\,d_k. \end{aligned}$$

Here $\xi_1 \in (x_k - 2d_k, x_k)$, $\xi_2 \in (x_k, x_k + d_k)$. Clearly, (8.19) holds if

$$4 - \frac{8}{3}\,\frac{|f'''(\xi_1)|}{f''(x_k)}d_k \geq 1 + \frac{|f'''(\xi_2)|}{f''(x_k)}d_k, \quad k \gg 1. \tag{8.20}$$

From (8.5), (8.7) and (8.8), it follows that (8.20) is indeed true for all $k \gg 1$. $\square$

Proof of Lemma 8.4. Below we consider (8.10) separately in the cases I) $x \in [x_k, \tilde z_k]$ and II) $x \in [\tilde z_k, x_{k+1}]$. Furthermore, we assume that $k \geq k_0 \gg 1$. Here k_0 is chosen so that for $k \geq k_0$ it is possible to use Lemmas 8.5 – 8.10. It should be noted that inequality (8.10) is obvious for $k \leq k_0$.

`Case I):` If $x \in [x_k, x_k + 2d_k]$, then $[x_k - d_k, x_k + d_k] \subseteq [x - 3d_k, x + 3d_k]$; and therefore

$$\int_{x-3d_k}^{x+3d_k} f(t)\,dt \geq \int_{x_k-d_k}^{x_k+d_k} f(t)\,dt = 2.$$

From here $d(x) \leq 3d_k$ by Lemma 3.10. If $x \in [x_k+2d_k, \tilde z_k]$, then $[x-d_k, x+d_k] \subseteq [x_k + d_k, \tilde z_k + d_k]$. Hence if $t \in [x - d_k, x + d_k]$ and $\xi \in [x_k - d_k, x_k + d_k]$, then $f(t) \geq f(\xi)$ by Lemmas 8.6, 8.9 and Corollary 8.7.1. Thus, we get

$$\int_{x-d_k}^{x+d_k} f(t)\,dt \geq \int_{x_k-d_k}^{x_k+d_k} f(\xi)d\xi = 2,$$

and $d(x) \leq d_k$ by Lemma 3.10.

`Case II):` If $x \in [x_{k+1} - 3d_{k+1}, x_{k+1}]$, then $[x_{k+1} - d_{k+1}, x_{k+1} + d_{k+1}] \subseteq [x - 4d_{k+1}, x + 4d_{k+1}]$; and we have

$$\int_{x-4d_{k+1}}^{x+4d_{k+1}} f(t)\,dt \geq \int_{x_{k+1}-d_{k+1}}^{x_{k+1}+d_{k+1}} f(t)\,dt = 2.$$

From here it follows that $d(x) \leq 4d_{k+1}$ by Lemma 3.10. If $x \in [\tilde z_k, x_{k+1} - 3d_{k+1}]$, then $[x-d_{k+1}, x+d_{k+1}] \subseteq [\tilde z_k - d_{k+1}, x_{k+1} - 2d_{k+1}]$. Hence if $t \in [x-d_{k+1}, x+d_{k+1}]$

and $\xi \in [x_{k+1} - d_{k+1}, x_{k+1} + d_{k+1}]$, then $f(t) \geq f(\xi)$ by Lemmas 8.6, 8.9, and 8.10 and by Corollary 8.7.1. From here we obtain

$$\int_{x-d_{k+1}}^{x+d_{k+1}} f(t)\,dt \geq \int_{x_{k+1}-d_{k+1}}^{x_{k+1}+d_{k+1}} f(\xi)d\xi = 2$$

and $d(x) \leq d_{k+1}$ by Lemma 3.10. It remains to note that $d_k \asymp d_{k+1}$ (see (8.5)). □

8.3. Precise conditions for correct solvability of problem (8.1)**.** Below we study problem (8.1) with the help of Theorems 2.4, 2.7 and 2.9 and Lemmas 2.3 and 8.1. Since the requirement $d_0 < \infty$ in the case (8.1) is fulfilled "by necessity" (see Lemma 8.1), then $r(t) \asymp r(x)$ for $|t-x| \leq d(x)$, $x \in \mathbb{R}$ (see (2.8) and (8.1)). We use these relations together with conditions $\beta \geq \alpha$, $\beta \geq 0$ (see Lemma 8.1), without additional stipulation.

Theorem 8.11. *Let $p \in (1, \infty)$. Then problem* (8.1) *is correctly solvable in $L_p(\mathbb{R})$ if and only if one of the following conditions hold:*

1) $\beta = \alpha = 0$ (8.21)

2) $\beta > 0,\ \gamma \geq \beta - \alpha \geq 0,\ p \geq 1 - \frac{\alpha}{\beta}$ (8.22)

3) $\beta > 0,\ \beta - \alpha > \gamma,\ \beta + 2\alpha + 2\gamma > 0,\ p > 1 - \frac{3\alpha}{\beta+2\alpha+2\gamma}$. (8.23)

Proof. Let $\gamma \geq \beta - \alpha$. Then (2.7) and (8.3) imply:

$$A_{p'}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)} \asymp \frac{d(x)}{e^{\alpha p'|x|}} \asymp \frac{1}{e^{(\beta-\alpha+\alpha p')|x|}}.$$

Therefore, $A_{p'} < \infty$ if and only if $\beta - \alpha + \alpha p' \geq 0$. Then, by Theorem 2.4, we have the following relations:

$$\left.\begin{array}{l} \beta \geq \alpha,\ \beta \geq 0 \\ \beta - \alpha - \gamma \leq 0 \\ \beta - \alpha + \alpha p' \geq 0 \end{array}\right\} \Rightarrow \begin{cases} \text{a) if } \beta = 0 \ \Rightarrow\ \alpha \leq 0,\ \alpha(p'-1) \geq 0 \ \Rightarrow \beta = \alpha \\ \text{b) if } \beta > 0 \ \Rightarrow\ 0 \leq \beta - a \leq \gamma,\ p \geq 1 - \frac{\alpha}{\beta}. \end{cases}$$

Thus conditions (8.21) and (8.22) are obtained. Let now $\beta - \alpha > \gamma$. We use (2.6), (2.7) and (8.5) to obtain

$$A_{p'} \geq \sup_{|k|\geq 0} A_{p'}(x_k) \geq c^{-1} \sup_{|k|\geq 0} \frac{d_k}{e^{\alpha p'|x_k|}} \geq c^{-1} \sup_{|k|\geq 0} (2|k|+1)^{-\frac{\theta+2\gamma}{3\gamma} - \frac{\alpha p'}{\gamma}}.$$

where $A_{p'}(x_k) = \sup_{|k|\geq 0} \int_{x_k-d_k}^{x_k+d_k} \frac{dt}{e^{\alpha p'|t|}}$. On the other hand, if $x \in [x_k, x_{k+1}]$, then from (8.10) and (8.5) it follows that

$$A_{p'}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{e^{\alpha p'|t|}} \leq c\frac{d(x)}{e^{\alpha p'|x|}} \leq c\frac{d_k}{e^{\alpha p'|x_k|}} \leq c(2|k|+1)^{-\frac{\theta+2\gamma}{3\gamma} - \frac{\alpha p'}{\gamma}}$$

and hence

$$A_{p'} \le c \sup_{|k|\ge 0} (2|k|+1)^{-\frac{\theta+2\gamma}{3\gamma}-\frac{\alpha p'}{\gamma}}.$$

Thus, $A_{p'} < \infty$ if and only if the following relations are fulfilled:

$$\left.\begin{matrix}\beta \ge 0,\ \beta-\alpha>\gamma \\ \frac{\beta-\alpha+2\gamma}{3\gamma}+\frac{\alpha p'}{\gamma}\ge 0\end{matrix}\right\} \quad\Rightarrow\quad \begin{cases}\beta>0,\ \beta-\alpha>\gamma\\ \beta+2\alpha+2\gamma>0\\ p\ge 1-\frac{3\alpha}{\beta+2\alpha+2\gamma}\end{cases}$$

which implies (8.23). It remains to quote Theorem 2.4. □

Theorem 8.12. *The problem* (8.1) *is correctly solvable in* $L_1(\mathbb{R})$ *if and only if* $\beta \ge \alpha \ge 0$.

Proof. This statement follows from Lemma 8.1 and Theorem 2.7. □

Theorem 8.13. *The problem* (8.1) *is correctly solvable in* $C(\mathbb{R})$ *if and only if either one of the following conditions hold:*

1) $\beta > 0,\ \gamma \ge \beta-\alpha \ge 0$ (8.24)
2) $\beta > 0,\ \beta-\alpha>\gamma,\ \beta+2\alpha+2\gamma>0.$ (8.25)

Proof. Let $\gamma \ge \beta-\alpha$. Then as above, we obtain

$$\int_{x-d(x)}^{x+d(x)} \frac{dt}{e^{\alpha|t|}} \asymp \frac{d(x)}{e^{\alpha|x|}} \asymp \frac{1}{e^{\beta|x|}}, \quad x\in\mathbb{R}.$$

Therefore, $\tilde{A}_0 = 0$ (see Theorem 2.9) if and only if $\beta>0$ and (8.24) is fulfilled. Let $\beta-\alpha>\gamma$. If $\tilde{A}_0 = \lim_{|x|\to\infty}\int_{x-d(x)}^{x+d(x)}\frac{dt}{e^{\alpha|t|}} = 0$, then (see (8.5))

$$0 = \lim_{|k|\to\infty}\int_{x_k-d_k}^{x_k+d_k}\frac{dt}{e^{\alpha|t|}} \ge c^{-1}\lim_{|k|\to\infty}\frac{d_k}{e^{\alpha|x_k|}} \ge c^{-1}\lim_{|k|\to\infty}(2|k|+1)^{-\frac{\beta-\alpha+2\gamma}{3\gamma}-\frac{\alpha}{\gamma}} \ge 0.$$

This implies $\beta+2\alpha+2\gamma>0$. On the other hand, for $x\in[x_k,x_{k+1}]$ by (8.5) and (8.10), we obtain

$$\int_{x-d(x)}^{x+d(x)}\frac{dt}{e^{\alpha|t|}} \le c\frac{d(x)}{c^{\alpha|x|}} \le c\frac{d_k}{e^{\alpha|x_k|}} \le c(2|k|+1)^{-\frac{\beta-\alpha+2\gamma}{3\gamma}-\frac{\alpha}{\gamma}},$$

and therefore the condition $\beta+2\alpha+2\gamma>0$ must fulfill the equality $\tilde{A}_0=0$. Thus, $\tilde{A}_0=0$ if and only if

$$\left.\begin{matrix}\beta \ge 0,\ \beta-\alpha>\gamma \\ \beta+2\alpha+2\gamma>0\end{matrix}\right\} \quad\Rightarrow\quad \begin{cases}\beta>0,\ \beta-\alpha>\gamma\\ \beta+2\alpha+2\gamma>0\end{cases}$$

which implies (8.25). It remains to use Theorem 2.9. □

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