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Approximative Compactness and Full Rotundity in Musielak–Orlicz Spaces and Lorentz–Orlicz Spaces

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Abstract. We prove that approximative compactness of a Banach space X is equivalent to the conjunction of reflexivity and the Kadec-Klee property of X. This means that approximative compactness coincides with the drop property defined by Rolewicz in *Studia Math.* 85 (1987), 25 – 35. Using this general result we find criteria for approximative compactness in the class of Musielak–Orlicz function and sequence spaces for both (the Luxemburg norm and the Amemiya norm) as well as critria for this property in the class of Lorentz–Orlicz spaces. Criteria for full rotundity of Musielak–Orlicz spaces are also presented in the case of the Luxemburg norm. An example of a reflexive strictly convex Köthe function space which is not approximatively compact and some remark concerning the compact faces property for Musielak–Orlicz spaces are given.

Keywords. Musielak–Orlicz spaces, Lorentz–Orlicz spaces, Luxemburg norm, Amemyia norm, approximative compactness, reflexivity, Kadec–Klee property, drop property, full rotundity

Mathematics Subject Classification (2000). Primary 46B20, secondary 46E30, 46B40, 46B42, 46B45

1. Introduction

If it is not assumed something different X denotes a real Banach space and $B(X)$, $S(X)$ stand for its unit ball and unit sphere, respectively.

Let us start with the following definition. A nonempty set $C \subset X$ is said to be *approximatively compact* if for any $(x_n) \subset C$ and any $y \in X$ such that $\|x_n - y\| \to \operatorname{dist}(y, C) := \inf\{\|x - y\| : \ x \in C\}$, it follows that (x_n) has a

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Cauchy subsequence. X is called *approximatively* compact if any nonempty, closed and convex set in X is approximatively compact.

Approximative compactness has been introduced by Jefimov and Stechkin in [27]. This property for a Banach space X is strongly related to the approximation theory (see [2]). Namely, it implies that any element $x \in X$ has the best approximant in any nonempty convex and closed subset A of X. We say that $y \in A$ is the best approximant for x in A if $\|x - y\| = \text{dist}(x, C)$. Moreover, approximative compactness of a strictly convex Banach space X guarantees continuity of the function $x \to P_A(x)$, called the metric projection onto A, where $P_A(x) = \{y \in A : \text{ dist}(x, A) = \|x - y\|\}$ for any nonempty, convex and closed subset A of X and any $x \in X$.

A Banach space X is said to have the *Kadec–Klee property* (or property H for short) if for any sequence $(x_n) \subset X$, and $x \in X$ such that $\|x_n\| = \|x\| = 1$, we have $\|x_n - x\| \to 0$ provided $x_n \to x$ weakly. If the weak convergence in this definition is replaced by the local convergence in measure, that is, convergence in measure on any measurable set of finite measure, we obtain the definition of H_μ–property for X.

This property was originally considered by Radon [37] and next by Riesz ([38], [39]), where it has been proven that L^p-spaces $(1 < p < \infty)$ have property H, although $L^1[0, 1]$ has not. Moreover, it has been proved simultaneously that $L^1[0, 1]$ has the *H_μ–property.*

Rolewicz [40] introduced the notions of the drop in a Banach space and the drop property for Banach spaces. For any $x \in X \setminus B(X)$ the drop determined by x is the set

$$D(x, B(X)) = \text{conv}(\{x\} \cup B(X))$$

and X is said to have the *drop property* if for any closed set C, disjoint with $B(X)$, there exists $x \in C$ such that $D(x, B(X)) \cap C = \{x\}$.

Montesinos [34] has shown that a Banach space X has the drop property if and only if X is reflexive and X has property H. We will show that the conjunction of reflexivity and property H is nothing but approximative compactness, whence it follows that drop property and approximative compactness coincide. Since for the most important classes of Banach spaces reflexivity has been already described, to get criteria for approximative compactness, we need only to find criteria for property H. This is a great benefit of our observation. In such a way we are able to find criteria for approximative compactness in Musielak–Orlicz spaces, both for the Luxemburg and the Amemyia norm and in Lorentz–Orlicz sequence spaces equipped with the Luxemburg norm. Conditions for approximative compactness for Orlicz spaces endowed with the Luxemburg norm have been found in [26] in both the function case and the sequence case. For the Orlicz norm it has been done in the function case only. However, in that paper a different method via full rotundity was used. The notion of full rotundity was introduced by Ky Fan and I. Glicksberg [16]. A Banach space X is said to be *fully k-rotund* ($k \geq 2,\ k \in \mathbb{N}$) if every sequence (x_n) in $S(X)$ such that $\|x_n^{(1)} + x_n^{(2)} + \cdots + x_n^{(k)}\| \to k$ as $n \to \infty$ for all its subsequences $(x_n^{(1)}), (x_n^{(2)}), \ldots, (x_n^{(k)})$, is a Cauchy sequence. Moreover, 2-fully rotund Banach spaces are called simply fully rotund spaces.

Approximative compactness for Musielak–Orlicz spaces over non-atomic measure spaces coincide with reflexivity and strict convexity. Example 1 show that it is not a general rule. We construct a reflexive strictly convex Köthe space which is not approximatively compact. For other results of this type see [3, 36].

We say that a Banach space X is *compactly fully k-rotund* if any sequence $(x_n) \subset S(X)$ such that $\|x_n^{(1)} + x_n^{(2)} + \cdots + x_n^{(k)}\| \to k$ as $n \to \infty$ for all its subsequences $(x_n^{(1)}), x_n^{(2)}, \ldots, (x_n^{(k)})$, forms a relatively compact set. It has been proved in [26] that k-fully rotund Banach spaces are approximatively compact.

It is known (see [11]) that a Banach space X is fully k-rotund if and only if it is compactly fully k-rotund and strictly convex. For Musielak–Orlicz sequence spaces equipped with the Luxemburg norm criteria for k-full rotundity were presented in [11]. However, the problem was not solved there completely because an additional assumption concerning the Musielak–Orlicz function, namely condition $(*)$, was assumed. Criteria for 2-full rotundity of Orlicz function spaces have been obtained in [9]. In this paper we present criteria for full k-rotundity and for approximative compactness for Musielak–Orlicz spaces for both the function and the sequence cases and for both the Luxemburg and the Amemyia norms. It is worth noticing that in the sequence case condition $(*)$ is not used in this paper. In the function case condition $(*)$ never appeared. Although full k-rotundity implies approximative compactness, in our paper proofs of the criteria for these properties are presented intependently because criteria for full k-rotundity are presented only for a non-atomic finite measure space and under the additional assumption that $\frac{\Phi(t,u)}{u} \to 0$ as $u \to 0$ for μ-a.e. $t \in T$. We finish the paper with criteria for approximative compactness of Lorentz–Orlicz function (and sequence) spaces.

2. General auxiliary and new results

Let (T, Σ, μ) be a measure space. Set

$$L^o = L^o(T) = \{f : T \to \mathbb{R} : f \text{ is } \Sigma\text{-measurable}\},$$

where measurable functions f and g equal on T μ-a.e. are identified. A Banach space X is called a Banach function lattice or a Köthe function space over the measure space (T, Σ, μ), if it is a subspace of $L^o(T, \Sigma, \mu)$ such that:

1^o if $x \in L^o$, $y \in X$ and $|x(t)| \leq |y(t)|$ for μ-a.e. $t \in T$, then $x \in X$ and $\|x\| \leq \|y\|$,

2^o there exists a positive function $x \in X$ such that $\mathrm{supp}(x) = T$, where $\mathrm{supp}(x) := \{t \in T : \ x(t) \neq 0\}$.

In the case of purely atomic measure we can also use the name Banach sequence lattice or a Köthe sequence space.

Let X be a Banach function lattice. A sequence (x_n) in X is said to be equi-continuous if for any $\varepsilon > 0$ there exist numbers $\sigma = \sigma(\varepsilon) > 0$, $N = N(\varepsilon) \in \mathbb{N}$ and a set $A \in \Sigma$ with $\mu(A) < \infty$ such that $\|x_n \chi_{T \setminus A}\| < \varepsilon$ for any $n > N$ and if $B \subset A$ and $\mu(B) < \sigma$, then $\|x_n \chi_B\| < \varepsilon$ or any $n \in \mathbb{N}$.

A sequence (x_n) in a Banach sequence lattice X is said to be equi-continuous if for any $\varepsilon > 0$ there is $i(\varepsilon) \in \mathbb{N}$ such that $\|(0, \dots, 0, x_n(i(\varepsilon)+1), x_n(i(\varepsilon)+2), \dots)\| < \varepsilon$ for any $n \in \mathbb{N}$.

Theorem 1 ([10, 15]). *Any Banach function and sequence lattice with the Kadec–Klee property is order continuous. The same holds for the property H_μ in place of the Kadec–Klee property.*

For the definition of order continuous elements and order continuity of the norm in Banach lattices see [30] and [32].

Proposition 1 ([19]). *Let X be a Banach function lattice over a measure space (Ω, Σ, μ). If $(x_n) \subset X$, $x \in X$, $x_n \to x$ weakly in X and $x_n \to y$ locally in measure in L^o, then $x = y$.*

Lemma 1 ([31]). *Let E be a Banach function lattice. If $(x_n) \subset E$, $x \in E$ and $x_n \to x$ in E, then there exist $y \in E_+$, $(x_{n_k}) \subset (x_n)$ and $(\varepsilon_{n_k}) \subset \mathbb{R}_+$ with $\varepsilon_{n_k} \downarrow 0$ such that $|x_{n_k}(t) - x(t)| \leq \varepsilon_{n_k} y(t)$ for μ-a.e. $t \in T$ (here E_+ denotes the set of all nonnegative elements from E).*

Theorem 2 ([20], Theorem 5 (2)). *In every reflexive Banach sequence lattice the properties H_μ and H coincide.*

Proposition 2 ([11]). *If X is compactly fully k-rotund, then X is reflexive.*

We start with the following general new result.

Theorem 3. *A Banach space X is approximatively compact if and only if X is reflexive and X has the Kadec–Klee property.*

Proof. Necessity. It is well known (see [41], Corollary 2.4, p. 99) that if all closed subspaces are proximinal, then all linear functionals attain their norm (so X is reflexive). Since approximative compactness of X implies that all closed subspaces of X are proximinal the necessity of reflexivity follows.

Now we will prove necessity of the Kadec–Klee property. Suppose that X is approximatively compact and X has not the Kadec–Klee property. Then there is a sequence $(x_n) \subset X$ and $x \in X$ such that $\|x_n\| = \|x\| = 1$, $x_n \to x$ weakly and (x_n) does not converge to x. Passing to a subsequence, if necessary, we can assume that there exists $d > 0$ such that $\|x_n - x\| \geq d$ for any natural number n. Let $f \in X^*$ be a norming functional for x, that is, $1 = f(x) = \|f\|$. Set $C = \{z \in X : f(z) \geq 1\}$. Obviously C is a nonempty, closed and convex subset of X. Since $\|f\| = 1$, $\|z\| \geq 1$ for any $z \in C$. Hence $\text{dist}(0, C) = 1 = \|x - 0\|$. Since $x_n \to x$ weakly, $f(x_n) \to f(x) = 1$. Setting $z_n = x_n / f(x_n)$, we have that $z_n \in C$, because $f(z_n) = 1$ for any $n \in \mathbb{N}$. Moreover, since $f(x_n) \to 1$ and $\|x_n\| = 1$, we have $\|z_n\| = \|z_n - 0\| \to 1 = \text{dist}(0, C)$. Since X is approximatively compact, (z_n) has a Cauchy subsequence (we will denote it again as (z_n)). Since X is a Banach space, $\|z_n - z\| \to 0$ for some $z \in X$. Hence $z_n \to z$ weakly. But $z_n \to x$ weakly, since $x_n \to x$ weakly and $f(x_n) \to 1$. Consequently $z = x$. Hence $\|z_n - x\| \to 0$, which gives immediately that $\|x_n - x\| \to 0$, a contradiction. This shows that approximative compactness implies the Kadec–Klee property.

Sufficiency. Suppose that X is reflexive and X has the Kadec–Klee property. Let $C \subset X$ be a nonempty, closed and convex set. Assume $y \in X$ and $(x_n) \subset C$ is chosen in such a way that $\|x_n - y\| \to \text{dist}(y, C)$. If $\text{dist}(y, C) = 0$, then $\|x_n - y\| \to 0$ and (x_n) is a Cauchy sequence. So suppose that $\text{dist}(y, C) = d > 0$. Since X is reflexive, passing to a subsequence if necessary, we can assume that (x_n) converges weakly to some $x \in X$. Since C is closed and convex, C is weakly closed. Hence $x \in C$. Moreover,

$$d = \text{dist}(y, C) \leq \|x - y\| \leq \text{liminf}_n \|x_n - y\| = d,$$

which shows that $\|x - y\| = d$. Set $z_n = (x_n - y)/\|x_n - y\|$ and $z = (x - y)/d$. Then $\|z_n\| = \|z\| = 1$, and $z_n \to z$ weakly, since $\|x_n - y\| \to d$ and $x_n \to x$ weakly. By the Kadec–Klee property of X, $\|z_n - z\| \to 0$ and consequently $\|x_n - x\| \to 0$. Hence (x_n) is a Cauchy sequence as required. □

Remark 1. Let X be an approximatively compact Banach space and $V \subset X$ be a nonempty, closed and convex subset of X. Suppose $x \in X$ and $\text{card}(P_V(x) = \{v \in V : \|x - v\| = \text{dist}(x, V)\}) = 1$. Then for any $v_n \in V$ with $\|x - v_n\| \to \text{dist}(x, V)$, we have $\|v_n - v\| \to 0$, where $\{v\} = P_V(x)$.

Proof. Suppose for the contrary that $\|v_{n_k} - v\| \geq d > 0$ for some subsequence (v_{n_k}). Since X is approximatively compact, there exist $z \in P_V(x)$ and a subsequence of (v_{n_k}) (we will also denote it by (v_{n_k})) such that $\|v_{n_k} - z\| \to 0$. Since $\text{card}P_V(x) = 1$, we have $z = v$, which leads to a contradiction. □

Corollary 1. *Let X, V, $x \in X$ and $v \in V$, be as in Remark 1. If $\|x_n - x\| \to 0$, then $\|v_n - v\| \to 0$ for any $v_n \in P_V(x_n)$. In particular, if X is strictly convex or V is strictly convex, which means that for any $x, z \in \partial(V) = V \setminus Int(V)$ we have $\frac{1}{2}(x + z) \notin \partial(V)$, then $\|P_V(x_n) - P_V(x)\| \to 0$ whenever $x_n \to x$ (here we treat $P_V(x_n)$ and $P_V(x)$ as elements from V).*

Proof. The result follows immediately from Remark 1 and continuity of the function $x \to \mathrm{dist}(x, V)$. □

Remark 2. Let X, V and $x \in X$ be as in Remark 1. If $\|x_n - x\| \to 0$, then

$$\mathrm{diam}(P_V(x_n)) = \sup\{\|y - z\|, y, z \in P_V(x_n)\} \to 0.$$

Proof. The result follows from Corollary 1. □

Example 1. We give an example of a strictly convex, reflexive Köthe function space X which is not approximatively compact. In particular, it means that X does not have property H. Therefore X is not locally uniformly convex (compare with [3, 36]).

Let X be a reflexive Banach space which is not approximatively compact such that there exists a countable set $F = \{f_1, f_2, ...\} \subset S(X^*)$that is total over X. Recall that a set $G \subset X^*$ is called total over X if for any $x \in X$, the condition $g(x) = 0$ for any $g \in G$ implies $x = 0$. By Theorems 11 and 12 from Section 3 we can choose X as a reflexive, non-strictly convex Orlicz space $L_\Phi = (L_\Phi(T, \Sigma, \mu)$, equipped with the Amemiya or the Luxemburg norm, where μ is atomless and σ-finite. L_Φ is then generated by an Orlicz function Φ that is not strictly convex on the whole $\mathbb{R}$ but both Φ and its Young's complement Φ^* satisfy condition Δ_2 (see [26] and Theorem 8 on page 174) . Since μ is σ-finite and atomless, there exists a sequence $\{T_k\}$ of measurable subsets of T of positive and finite measure such that, for any $f \in L_\Phi$ and $k = 1, 2, ...$, if $g_k(f) = 0$, then $f = 0$. Here

$$g_k(f) = \int_{T_k} f(t) d\mu(t).$$

Set $f_k = g_k / \|g_k\|$ for $k = 1, 2, ...$. By definition of g_k, $F = \{f_1, ..., f_k, ...\}$ is total over X. Let us define on X the norm $\|\cdot\|_1$ by

$$\|z\|_1 = \left(\|z\|^2 + \sum_{k=1}^{\infty} \left(\frac{f_k(|z|)}{2^k} \right)^2 \right)^{\frac{1}{2}}.$$

It is clear that $(X, \|\cdot\|_1)$ is a Köthe function space and $\|\cdot\|_1$ is equivalent to $\|\cdot\|$, which shows that $(X, \|\cdot\|_1)$ is reflexive. Moreover, since l_2 is strictly convex, and F is total over X, it is not difficult to check that $(X, \|\cdot\|_1)$ is also strictly convex. We will show that $(X, \|\cdot\|_1)$ is not approximatively compact. Since $(X, \|\cdot\|)$ is not approximatively compact, there exist a sequence $(x_n) \subset S(X)$ and $x \in S(X)$ such that x_n converges to x weakly and $\|x_n - x\| \geq d > 0$. Moreover, by the proofs of Theorem 11 and Theorem 12 in Section 3, we can assume that $x_n \geq 0$ and $x \geq 0$. We will show that $\|x_n\|_1 \to \|x\|_1$. To do this, fix $\varepsilon > 0$ and $k_o \in \mathbb{N}$ such that $\sum_{k=k_o}^{\infty}(2^k)^{-2} < \left(\frac{\varepsilon}{2}\right)^2$. Note that for any $z \in X$,

$$\|z\|_1 \leq \left(\|z\|^2 + \sum_{k=1}^{k_o-1}\left(\frac{f_k(|z|)}{2^k}\right)^2\right)^{\frac{1}{2}} + \left(\sum_{k=k_o}^{\infty}\left(\frac{f_k(|z|)}{2^k}\right)^2\right)^{\frac{1}{2}}$$

and

$$\|z\|_1 \geq \left(\|z\|^2 + \sum_{k=1}^{k_o-1}\left(\frac{f_k(|z|)}{2^k}\right)^2\right)^{\frac{1}{2}} - \left(\sum_{k=k_o}^{\infty}\left(\frac{f_k(|z|)}{2^k}\right)^2\right)^{\frac{1}{2}}.$$

Combining the above inequalities, we get

$$\begin{aligned}|\|x_n\|_1 - \|x\|_1| \leq &\left|\left(\|x_n\|^2 + \sum_{k=1}^{k_o-1}\left(\frac{f_k(|x_n|)}{2^k}\right)^2\right)^{\frac{1}{2}} - \left(\|x\|^2 + \sum_{k=1}^{k_o-1}\left(\frac{f_k(|x|)}{2^k}\right)^2\right)^{\frac{1}{2}}\right| \\ &+ \left(\sum_{k=k_o}^{\infty}\left(\frac{f_k(|x_n|)}{2^k}\right)^2\right)^{\frac{1}{2}} + \left(\sum_{k=k_o}^{\infty}\left(\frac{f_k(|x|)}{2^k}\right)^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\|x_n\| = \|x\| = 1$, $x_n \geq 0$ and $x \geq 0$, $f_k(|x_n|) = f_k(x_n) \to f_k(x) = f_k(|x|)$ for any k, so by the choice of k_o we get that $|\|x_n\|_1 - \|x\|_1| \leq 2\varepsilon$ for $n \geq n_o$. Consequently, $\|x_n\|_1 \to \|x\|_1$. Since $\|x_n - x\| \geq d > 0$, $\|x_n - x\|_1$ does not converge to 0, which shows that $(X, \|\cdot\|_1)$ is not an approximatively compact space.

Remark 3. We say that a Banach space X has compact faces if for any $f \in X^* \setminus \{0\}$ the set $N_f = \{x \in S(X) : f(x) = \|f\|\}$ is compact. We will prove that any approximatively compact Banach space has compact faces. Indeed, by the continuity of the norm, N_f is closed. We will show that it is also compact. For any $x \in N_f$, we have $d(0, N_f) = \|x\| = 1$ and for any $(x_n) \subset N_f$, we have $\|x_n\| = d(0, N_f)$ for any $n \in \mathbb{N}$. Therefore, each sequence $(x_n) \subset N_f$ is a minimizing sequence for 0 with respect to the set N_f. Since X is approximatively compact and N_f is closed and convex, the sequence (x_n) contains a subsequence which converges to some $x \in N_f$, which means that N_f is compact.

In general, the converse is not true. Indeed, the space from Example 1 is reflexive and has compact faces (since it is strictly convex, so the faces are singletons) but it is not approximatively compact. However, for spaces X considered in Theorems 10, 11, 12, 15 and Corollary 2 criteria for approximative compactness can be unified in the following manner: X is approximatively compact if and only if X is reflexive and has compact faces.

3. Approximative compactness and full rotundity of Musielak–Orlicz spaces

We start with notations and definitions that will be used in this section. Let (T, Σ, μ) be a measure space with a nonatomic measure μ. A function $\Phi : T \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ is called a *Musielak–Orlicz* function if

(a) $\Phi(\cdot, u)$ is a Σ-measurable function for any $u \in \mathbb{R}$;
(b) the function $\Phi(t, \cdot)$ is convex, even, continuous at zero and left-continuous on $(0, \infty)$ for μ-almost all $t \in T$;
(c) $\Phi(t, 0) = 0$, $\Phi(t, u_t) < +\infty$ for some $u_t \in (0, +\infty)$ and $\Phi(t, u) \to \infty$ as $u \to \infty$ for almost all $t \in T$.

In the case when $T = \mathbb{N}$ and μ is the counting measure on $2^{\mathbb{N}}$, we can state that a function $\Phi = (\Phi_i)_{i=1}^{\infty}$ is called a Musielak–Orlicz function if

(a) $\Phi_i : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ is convex, even, continuous at zero and left-continuous on $(0, \infty)$ for all $i \in \mathbb{N}$;
(b) $\Phi_i(0) = 0$, $\Phi_i(u_i) < +\infty$ for some $u_i \in (0, +\infty)$ and $\Phi_i(u) \to \infty$ as $u \to \infty$ for all $i \in \mathbb{N}$.

Given a Musielak–Orlicz function Φ, we define $\rho_\Phi : L^o \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\rho_\Phi(f) = \int_T \Phi(t, |f(t)|)\, d\mu(t).$$

Then ρ_Φ is called the modular and the space

$$L_\Phi = \left\{ f \in L^o : \rho_\Phi(\lambda f) < +\infty \text{ for some } \lambda > 0 \right\}$$

is called the *Musielak–Orlicz* space generated by Φ. Analogously, for any real sequence $x = (x_i)_{i=1}^{\infty}$ (the space of such sequences is denoted by l^0) the modular ρ_Φ at x has the form

$$\rho_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(|x_i|),$$

and then the space

$$l_\Phi = \left\{ x \in l^o : \rho_\Phi(\lambda x) < +\infty \text{ for some } \lambda > 0 \right\}$$

is called the *Musielak–Orlicz sequence space.*

We consider two classical norms in Musielak–Orlicz spaces L_Φ (resp. l_Φ): the Luxemburg norm

$$\|x\|_\Phi = \inf \left\{ \lambda > 0 : \rho_\Phi\left(\frac{x}{\lambda}\right) \le 1 \right\}$$

and the Amemiya norm

$$\|x\|_\Phi^A = \inf\left\{k > 0 : \frac{1+\rho_\Phi(kx)}{k}\right\}$$

(see [6] and [35]). In the first case we denote the Musielak–Orlicz space by L_Φ; in the second case by L_Φ^A. Analogously, the respective sequence spaces we denote by l_Φ and l_Φ^A.

Let

$$E_\Phi = \left\{f \in L^o : \rho_\Phi(\lambda f) < +\infty \text{ for all } \lambda > 0\right\}$$

and

$$h_\Phi = \left\{x \in l^o : \text{ for any } \lambda > 0 \text{ there exists } i_\lambda \in \mathbb{N} \text{ s.t.} \sum_{i=i_\lambda}^{\infty} \Phi_i(\lambda|x_i|) < +\infty \right\},$$

where $\mathbb{N}$ denotes the set of natural numbers. Then E_Φ (h_Φ resp.) are called *the subspaces of finite elements.* They are the subspaces of order continuous elements from L_Φ (resp. l_Φ) in fact.

It is said that a Musielak–Orlicz function $\Phi : T\times\mathbb{R}_+ \to \mathbb{R}_+\cup\{+\infty\}$ satisfies the Δ_2-*condition* ($\Phi \in \Delta_2$) if for any $d > 1$ there exist $k > 1$ and $c \in L_1(T)$, $c \geq 0$, such that for any $u \in \mathbb{R}^+$ and μ-a.e. $t \in T$, we have

$$\Phi(t, du) \leq k\Phi(t, u) + c(t).$$

It is said that a Musielak–Orlicz function $\Phi = (\Phi_i)_{i=1}^\infty$ satisfies the δ_2-*condition* ($\Phi \in \delta_2$) if for any $d > 1$ there exists $a > 0$, $k > 1$, $i_o \in \mathbb{N}$ and a nonnegative sequence $\{c_i\} \in l_1$ such that the inequality

$$\Phi_i(du) \leq k\Phi_i(u) + c_i$$

holds for all $i > i_o$ and $u \in \mathbb{R}^+$ satisfying $\Phi_i(u) \leq a$. For some equivalent forms of the Δ_2-condition and the δ_2-condition see [18] and [35].

We denote by Φ^* the function complementary to Φ in sense of Young, i.e., $\Phi^*(t, u) = \sup_{v>0}\{vu - \Phi(t, v)\}$ for any $u \geq 0$ and μ-a.e. $t \in T$ (analogously $\Phi^* = (\Phi_n^*)$, where $\Phi_n^*(u) = \sup_{v>0}\{vu - \Phi_n(v)\}$ for any $u \geq 0$ and all $n \in \mathbb{N}$, in the sequence case).

For any $x \in L_\Phi^A$ or $x \in l_\Phi^A$, we define

$$\begin{aligned} k^*(x) &= \inf\left\{k \geq 0 : I_{\Phi^*}(p \circ k|x|) \geq 1\right\} \\ k^{**}(x) &= \sup\left\{k \geq 0 : I_{\Phi^*}(p \circ k|x|) \leq 1\right\}, \end{aligned}$$

where $p(t, \cdot)$ is the right hand side derivative of $\Phi(t, \cdot)$ on $\mathbb{R}_+$ and $p \circ k|x|(t) := p(t, k|x(t)|)$ for μ-a.e. $t\in T$. Next we define $\overline{K}(x) = [k^*(x), k^{**}(x)]$ if $k^*(x) < \infty$.

This interval has the property that $\|x\|_\Phi^A = \frac{1}{k}(1+\rho_\Phi(kx))$ if and only if $k \in \overline{K}(x)$. If $k^*(x) < \infty$ and $k^{**}(x) = \infty$, we have $\|x\|_\Phi^A = \frac{1}{k}(1+\rho_\Phi(kx))$ for any $k \in [k^*(x), k^{**}(x))$ and $\|x\|_\Phi^A = \lim_{k\to\infty} \frac{1}{k}(1+\rho_\Phi(kx)) = \int_T A(t)|x(t)|d\mu$, where $A(t) = \lim_{k\to\infty}(\frac{1}{u}\Phi(t,u))$.

The dual space of L_Φ is well known. Namely, we have

$$(L_\Phi)^* = L_{\Phi^*} \oplus S,$$

that is, any $x^* \in (L_\Phi)^*$ is uniquely represented in the form $x^* = \xi_v + \phi$, where $v \in L_{\Phi^*}$ and ξ_v is the order continuous functional (order continuous functionals are also called regular functionals) on L_Φ generated by $v \in L_{\Phi^*}$, that is,

$$\xi_v(x) = \int_T v(t)x(t)\,dt \quad (x \in L_\Phi)$$

and $\phi \in S$ is a linear singular functional on L_Φ, that is, $\phi(x) = 0$ for any $x \in E_\Phi$. We denote by RGrad(x), (SGrad(y), respectively) the set of all regular (singular, respectively) support functionals at x (that is, norming functionals for x).

Now, we need to recall some results that will be used in the proofs of our new results.

Lemma 2 ([7]). *Let Φ be a finite-valued Musielak–Orlicz function and μ be a σ-finite measure. Then there exists a sequence $(S_n)_{n=1}^\infty$ of measurable sets of finite measure such that $S_1 \subset S_2 \subset \dots$, $\mu\left(\bigcup_{n=1}^\infty S_n\right) = \mu(T)$ and*

$$L^\infty(\mu/S_n) \hookrightarrow L^\Phi(\mu/S_n) \hookrightarrow L^1(\mu/S_n) \quad (\forall \in \mathbb{N}).$$

Lemma 3 ([28]). *Let Φ be a Musielak–Orlicz function. There exists a sequence $(T_n)_{n=1}^\infty$ of pairwise disjoint, measurable sets of positive measure such that $\bigcup_n T_n = T$ and $\sup\{\Phi(t,u):\ t \in T_n\} < +\infty$ for any $n \in \mathbb{N}$ and $u \in \mathbb{R}_+$.*

Lemma 4 ([8]). *Let $x \in S(L_\Phi)$ and $\rho_\Phi(x) = 1$. Then $x^* \in$ RGrad(x) if and only if*

$$x^*(y) = \frac{\int_T z(t)y(t)d\mu}{\int_T z(t)x(t)d\mu} \quad (\forall\ y \in L_\Phi),$$

where $z \in L_{\Phi^*}$ and $z(t) \in [p_-(t,x(t)), p(t,x(t))]$ for μ-a.e. $t \in T$, where $p_-(t,u)$ and $p(t,u)$ denotes the left and right derivatives of $\Phi(t,\cdot)$ at $u \in \mathbb{R}$.

Lemma 5 ([22]). *Let X denote L_Φ or L_Φ^A. Assume that $\Phi > 0$, $\frac{\Phi(t,u)}{u} \to 0$ as $u \to 0$ for μ-a.e. $t \in T$, $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$. If $(x_n), (y_n) \subset S(X)$ and $\|x_n + y_n\| \to 2$, then for any $\varepsilon > 0$ there exist numbers $\delta = \delta(\varepsilon) > 0$ and $n' = n'(\varepsilon) \in \mathbb{N}$ such that for any $n > n'$ and any set $E \in \Sigma$ the condition $\|y_n\chi_E\| < \delta$ implies $\|x_n\chi_E\| < \varepsilon$.*

Lemma 6 ([6]). *Let T be an interval in $\mathbb{R}$ (the whole $\mathbb{R}$ and other unbounded intervals are not excluded) and (T, Σ, μ) be the Lebesgue measure space. Let $E \in \Sigma$ be a bounded, closed set. Then E can be decomposed into $E_n, F_n \in \Sigma$ ($n \in \mathbb{N}$) such that $F_n \cap E_n = \emptyset$, $E = F_n \cup E_n$, $\mu(E_n) = \mu(F_n) = \frac{1}{2}\mu(E)$ for any $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \int_E v(t)\,(\chi_{E_n}(t) - \chi_{F_n}(t))\,dt = 0$$

for any function v integrable over E.

Theorem 4.

(i) *A Musielak–Orlicz space L_Φ is reflexive (with respect to the Luxemburg and Amemiya norms) if and only if $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$.*

(ii) *A Musielak–Orlicz space l_Φ is reflexive (with respect to the Luxemburg and Amemiya norms) if and only if $\Phi \in \delta_2$ and $\Phi^* = \{\Phi_i^* : i \in \mathbb{N}\} \in \delta_2$.*

Proof. It is evident, but since we do not know any reference to this fact, let us sketch a proof. It is known (see [35]) that if $\Phi \in \Delta_2$, then $(L_\Phi)^* = L^A_{\Phi^*}$. Moreover, by $\Phi^* \in \Delta_2$, we have $(L_\Phi)^{**} = (L^A_{\Phi^*})^* = L_{\Phi^{**}} = L_\Phi$, because $\Phi^{**} = \Phi$. Similar arguments show that Φ, $\Phi^* \in \Delta_2$ imply that L^A_Φ is reflexive.

Moreover, it is known that for any Banach function lattice E, reflexivity of E implies that both E and E^* are order continuous. But order continuity of L_Φ (and so of L^A_Φ as well) is equivalent to $\Phi \in \Delta_2$ (see [17]). In such a way we have shown that reflexivity of L_Φ gives $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$. The proof in the sequence case is analogous. □

Theorem 5 ([23]). *Assume that $\Phi \in \Delta_2$ and $\Phi(t, \cdot)$ is a strictly convex function for almost all $t \in T$. Let $x_n, y_n \in L_\Phi$, $\|x_n\|_\Phi = \|y_n\|_\Phi = 1$. If $\|x_n + y_n\|_\Phi \to 2$, then $x_n - y_n \to 0$ locally in measure.*

Theorem 6 ([23]). *Assume $\Phi^* \in \Delta_2$ and $\Phi(t, \cdot)$ is a strictly convex function for almost all $t \in T$. Set for any $x \in L_\Phi$*

$$K(x) = \big\{k \in [0, +\infty) : \|x\|^A_\Phi = (1 + \tfrac{1}{k}\rho_\Phi(kx))\big\}.$$

Then $K(x) \neq \emptyset$ for any $x \in L_\Phi$ and

$$\sup\big\{k > 0 : k \in K(x),\ \|x\|^A_\Phi = 1\big\} < +\infty.$$

Theorem 7 ([14]). *Let $\Phi = (\Phi_i)$ be a Musielak–Orlicz function. Set, for any $i \in \mathbb{N}$, $b_i = \sup\{u > 0 : \Phi_i(u) < +\infty\}$. Then the Musielak–Orlicz space $(l_\Phi, \|\cdot\|^A_\Phi)$ has the Kadec–Klee property if and only if $\Phi \in \delta_2$ or $\sum_{i=1}^\infty \Phi_i^*(c_i) \le 1$. Here for any $i \in \mathbb{N}$,*

$$c_i = \begin{cases} b_i, & \Phi_i^*(b_i) < 1 \\ (\Phi_i^*)^{-1}(1), & \Phi_i^*(b_i) \ge 1. \end{cases}$$

Theorem 8 ([13]). *The Orlicz space L_Φ^A is strictly convex if and only if*

(i) *Φ is strictly convex,*

(ii) $\lim_{u\to\infty} R(u) = \infty$, *where* $R(u) = A|u| - \Phi(u)$ *and* $A = \lim_{u\to\infty} \frac{\Phi(u)}{u}$.

We start now with our new results in the topic of Section 3.

Theorem 9. *Let $\Phi = (\Phi_i)_{i=1}^{\infty}$ be a Musielak–Orlicz function. Then the Musielak–Orlicz space l_Φ has the Kadec–Klee property with respect to coordinatewise convergence if and only if Φ satisfies the δ_2-condition and for every $i \in \mathbb{N}$ there is $u_i > 0$ such that $\Phi_i(u_i) = 1$.*

Proof. Necessity. For the necessity of the δ_2-condition see [12]. We will show the necessity of the second condition on Φ. Assume that there is $j \in \mathbb{N}$ such that $\Phi_j(b_j) < 1$ (for the definition of b_j see Theorem 7). We will show that l_Φ does not have the Kadec–Klee property with respect to the coordinatewise convergence. Define for $i \neq j$ the numbers

$$c_i = \begin{cases} b_i, & \text{if } \Phi_j(b_j) + \Phi_i(b_i) \le 1 \\ u_i, & \text{where } \Phi_j(b_j) + \Phi_i(u_i) = 1 \text{ otherwise,} \end{cases}$$

the element $x = b_j e_j$ and the sequence $(x_i)_{i>j}$ in l_Φ, where $x_i = b_j e_j + c_i e_i$. Then $x_i \to x$ coordinatewise. Moreover, assuming that $i > j$, we have $\rho_\Phi(x_i) \le 1$, whence $\|x_i\|_\Phi \le 1$. Since $\rho_\Phi(\lambda x_i) > 1$ for all $\lambda > 1$, so $\|x_i\|_\Phi \ge 1$ for any $i > j$. Consequently, $\|x_i\|_\Phi = 1$ for any $i > j$. Similarly we get $\|x\|_\Phi = 1$. If $i \in \mathbb{N}$, $i > j$, is such that $c_i = b_i$, then $\|x_i - x\|_\Phi = 1$. If $i \in \mathbb{N}$, $i > j$, is such that $b_i = u_i$, then $1 \ge \rho_\Phi(x_i - x) = \Phi_i(u_i) = 1 - \Phi_j(b_j)$, whence $\|x_i - x\|_\Phi \ge 1 - \Phi_j(b_j)$. Consequently, $\|x_i - x\|_\Phi \ge \min(1, 1 - \Phi_j(b_j))$ for any $i > j$. Since $\min(1, 1 - \Phi_j(b_j)) > 0$, this shows that l_Φ does not have the Kadec–Klee property with respect to the coordinatewise convergence.

Sufficiency. Fix $\lambda > 1$ and let $k \ge 1$ corresponds to $d = 2\lambda$ in the property δ_2. Let P_J be the projection of a sequence onto its first J coordinates, and let $Q_J = I - P_J$. Let $\varepsilon > 0$. Choose J such that $\rho_\Phi(Q_J x) \le \frac{\varepsilon}{4k}$, so that $\rho_\Phi(P_J x) \ge 1 - \frac{\varepsilon}{4k}$. By coordinatewise convergence, for n large enough we have $\rho_\Phi(\lambda P_J x_n - \lambda P_J x) \le \frac{\varepsilon}{4}$. Also $\rho_\Phi(P_J x_n) \ge 1 - \frac{\varepsilon}{2k}$, so that $\rho_\Phi(Q_J x_n) \le \frac{\varepsilon}{2k}$. The property δ_2 gives now $\rho_\Phi(\lambda Q_J x_n - \lambda Q_J x) \le \varepsilon$, and it follows that $\rho_\Phi(\lambda(x_n - x)) \le 2\varepsilon$ for n large enough. The arbitrariness of $\varepsilon > 0$ gives that $\rho_\Phi(\lambda(x_n - x)) \to 0$ as $n \to \infty$, whence the arbitrariness of $\lambda > 0$ gives that $\|x_n - x\|_\Phi \to$ as $n \to \infty$. □

Corollary 2. *Suppose $\Phi = (\Phi_i)$ is a Musielak–Orlicz function such that for any $i \in \mathbb{N}$ there exists $u_i > 0$ with $\Phi_i(u_i) = 1$. Then l_Φ is approximatively compact if and only if l_Φ is reflexive.*

Proof. The proof follows immediately from Theorems 3 and 9. □

The following example shows that for nonreflexive Musielak–Orlicz sequence spaces equipped with the Luxemburg norm, the property H_μ is essentially stronger than the property H in general. Namely, there exists a Musielak–Orlicz sequence space with property H but without property H_μ.

Example 2. Let $\Phi_1(t) = t$ for $t \in [0, \frac{1}{2}]$, $\Phi_1(t) = +\infty$ for $t > \frac{1}{2}$ and $\Phi_i(t) = t$ for $i \geq 2$. By Theorem 9, $X = l_\Phi$ has not the H_μ-property. However, X has property H. To prove this fact, it is enough to show that $X = l^1$ and the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_1$ are equivalent, because l^1 has even the Schur property and this property is preserved by equivalent norms. It is obvious that $l^1 = l^\Psi$ and $\|\cdot\|_1 = \|\cdot\|_\Psi$ for the Musielak–Orlicz function $\Psi = (\Psi_i)_{i=1}^\infty$, with $\Psi_i(u) = |u|$ for all $u \in \mathbb{R}$ and $i \in \mathbb{N}$. We have that $\Phi_i(\frac{u}{2}) = \Psi_i(\frac{u}{2})$ for all $u \in [0,1]$ and $i \in \mathbb{N}$, whence we easily to get that $l_\Phi = l^\Psi = l^1$. For any $x \in l^\Psi = l^1$, $x \neq 0$, we have $\rho_\Psi\big(\frac{x}{\|x\|_\Phi}\big) \leq \rho_\Phi\big(\frac{x}{\|x\|_\Phi}\big) \leq 1$, whence $\big\|\frac{x}{\|x\|_\Phi}\big\|_\Psi$ that is, $\|x\|_\Psi \leq \|x\|_\Phi$. On the other hand, since $\frac{x(i)}{2\|x\|_\Psi} \in [-\frac{1}{2}, \frac{1}{2}]$, so $\Phi_i\big(\frac{x(i)}{2\|x\|_\Psi}\big) = \Psi_i\big(\frac{x(i)}{2\|x\|_\Psi}\big)$ for all $i \in \mathbb{N}$, and consequently,

$$\rho_\Phi\left(\frac{x}{\|x\|_\psi}\right) = \rho_\Psi\left(\frac{x}{2\|x\|_\Psi}\right) \leq \frac{1}{2}\,\rho_\Psi\frac{x}{\|x\|_\Psi} \leq \frac{1}{2} < 1,$$

whence $\big\|\frac{x}{2\|x\|_\Psi}\big\|_\Phi \leq 1$, so $\|x\|_\Phi \leq 2\|x\|_\Psi$. In consequence, the equivalence of the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_1$ is proved.

Theorem 10. *The Musielak–Orlicz space l_Φ^A equipped with the Amemiya norm is approximatively compact if and only if it is reflexive, that is, if and only if $\Phi, \Phi^* \in \delta_2$.*

Proof. By Theorem 4, if l_Φ^A is reflexive, then $\Phi \in \delta_2$. By Theorem 7, l_Φ^A has property H. By Theorem 3, l_Φ^A is approximatively compact. The converse also follows from Theorem 3. □

Lemma 7. *Assume that (T, Σ, μ) is a nonatomic, complete and σ-finite measure space and Φ is a Musielak–Orlicz function satisfying the Δ_2-condition. Let $x \in L_\Phi$, $(x_n) \subset L_\Phi$, $x_n \to x$ locally in measure and $\rho_\Phi(x_n) \to \rho_\Phi(x) < \infty$. Then $\|x_n - x\|_\Phi \to 0$ as $n \to \infty$.*

Proof. Let us fix $\varepsilon > 0$. By Lemma 3 there exists a set $A \in \Sigma$ such that $0 < \mu(A) < \infty$, $\chi_A \in L^\infty \cap L^1$ and $\rho_\Phi\left(x\chi_{T\setminus A}\right) < \frac{\varepsilon}{3}$. The sequence $(x_n\chi_A)$ converges in measure to $x\chi_A$. Therefore, there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k}\chi_A \to x\chi_A$ (and consequently $\Phi \circ x_{n_k}\chi_A \to \Phi \circ x\chi_A$) almost everywhere. In consequence (by the fact that $\mu(A) < \infty$) we have $\Phi \circ x_{n_k}\chi_A \to \Phi \circ x\chi_A$ in measure. Let $\delta > 0$ be such that for any $B \in \Sigma \cap A$ satisfying $\mu(B) < \delta$ we have that $\rho_\Phi(x\chi_B) < \frac{\varepsilon}{3}$. By the Jegoroff Theorem we get that there is a $B \subset A$ such that $\mu(B) < \delta$, $x_n\chi_{A\setminus B} \rightrightarrows x\chi_{A\setminus B}$ and $\Phi \circ x_n\chi_{A\setminus B} \rightrightarrows$

$\Phi \circ x\chi_{A\setminus B}$ (the sign $\rightrightarrows$ indicates the uniform convergence). So $\rho_\Phi(x\chi_B) < \frac{\varepsilon}{3}$ whence $\rho_\Phi\left(x\chi_{(T\setminus A)\cup B}\right) < \frac{2\varepsilon}{3}$. Moreover, $\rho_\Phi\left(x_{n_k}\chi_{A\setminus B}\right) \to \rho_\Phi\left(x\chi_{A\setminus B}\right)$. Since $\rho_\Phi(x_{n_k}) \to \rho_\Phi(x)$, so $\rho_\Phi\left(x_{n_k}\chi_{(T\setminus A)\cup B}\right) \to \rho_\Phi\left(x\chi_{(T\setminus A)\cup B}\right) < \frac{2\varepsilon}{3}$. Therefore, there is $n_\varepsilon \in \mathbb{N}$ such that $\rho_\Phi\left(x_{n_k}\chi_{(T\setminus A)\cup B}\right) < \frac{2\varepsilon}{3}$ for $n > n_\varepsilon$. By the facts that $x_n\chi_{A\setminus B} \rightrightarrows x\chi_{A\setminus B}$ and $\rho_\Phi\left(x\chi_{A\setminus B}\right) < \infty$ (which follows by condition Δ_2 and the fact that $x\chi_A \in L^1$), using the Lebesgue dominated convergence theorem, we get that $\rho_\Phi\big(\frac{x_{n_k}-x}{2}\chi_{A\setminus B}\big) \to 0$ as $n \to \infty$. So, there is $m_\varepsilon > n_\varepsilon$ such that $\rho_\Phi\big(\frac{x_{n_k}-x}{2}\chi_{A\setminus B}\big) < \frac{\varepsilon}{3}$ all $n > m_\varepsilon$. In consequence, for all $n > m_\varepsilon$, we have

$$\rho_\Phi\left(\tfrac{x_{n_k}-x}{2}\right) \le \rho_\Phi\left(\tfrac{x_{n_k}-x}{2}\chi_{A\setminus B}\right) + \frac{1}{2}\left[\rho_\Phi\left(x_{n_k}\chi_{(T\setminus A)\cup B}\right) + \rho_\Phi\left(x\chi_{(T\setminus A)\cup B}\right)\right] < \frac{5\varepsilon}{3}.$$

So we have proved that $\rho_\Phi\big(\frac{x_{n_k}-x}{2}\big) \to 0$ as $n \to \infty$. Note that the condition $\rho_\Phi\big(\frac{x_{n_k}-x}{2}\big) \to 0$ is equivalent to $\|\Phi \circ \frac{x_{n_k}-x}{2}\|_{L^1} \to 0$. By Lemma 1, there exists $y \in (L^1)_+$ such that $\Phi \circ \frac{x_{n_k}-x}{2} \le y$. By the Δ_2-condition, we get that for any $\lambda > 0$ there exists $y_\lambda \in L^1$ such that $\Phi \circ \frac{x_{n_k}-x}{2} \le y_\lambda$. Hence, using the fact that $\lambda(x_{n_k}(t) - x(t)) \to 0$ for μ-a.e. $t \in T$ and the Lebesgue dominated convergence theorem, we get $\rho_\Phi(\lambda(x_{n_k} - x)) \to 0$ as $k \to \infty$. This means, by the arbitrariness of $\lambda > 0$, that $\|x_{n_k} - x\|_\Phi \to 0$. Using the double extract subsequence theorem, we have that $\|x_n - x\|_\Phi \to 0$ as $n \to \infty$. □

Theorem 11. *Let $X = L_\Phi^A$. The following conditions are equivalent:*

(i) *X is approximatively compact.*

(ii) *X is reflexive and strictly convex.*

(iii) *$\Phi, \Phi^* \in \Delta_2$ and $\Phi(t,\cdot)$ is strictly convex on $\mathbb{R}$ for μ-a.e. $t \in T$.*

Proof. (ii) ⇔ (iii). By Theorem 4, L_Φ^A is reflexive if and only if $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$. If $\Phi^* \in \Delta_2$, then Φ^* is finitely valued, so $\Phi(t,u)/u \to \infty$ as $u \to \infty$ for μ-a.e. $t \in T$. Therefore, the equivalence of conditions *(ii)* and *(iii)* follows easily by Theorem 8.

(i) ⇔ (ii). We will present two completely different proofs. Although the second proof is simpler than the first one, it concerns only the Lebesgue measure space in $\mathbb{R}$. Since the first proof concerns arbitrary measure space, it is more general.

First proof. Suppose X is approximatively compact. By Theorem 3, X is reflexive. Assume for the contrary that X is not strictly convex. Then there exist $f, g \in X$, $\|f\|_\Phi^A = \|g\|_\Phi^A = 1$, $f \neq g$ such that $\left\|\frac{f+g}{2}\right\|_\Phi^A = 1$. Since X is strictly convex if and only if X_+ is strictly convex (see Theorem 2 in [19]) we can assume, without loss of generality, that $f, g \ge 0$ and $f \neq g$.

First suppose that $K(f) \neq \emptyset$ and $K(g) \neq \emptyset$ (see Theorem 6). Fix $k_f \in K(f)$, $k_g \in K(g)$ and set $l = k_f f$, $m = k_g g$. Define

$$A = \{t \in T : l(t) > m(t)\}, \quad B = \{t \in T : m(t) > l(t)\}, \quad C = T\setminus(A \cup B). \quad (1)$$

Since $f \neq g$ and $\|f\|_\Phi^A = \|g\|_\Phi^A = 1$, we have $\mu(A) > 0$ or $\mu(B) > 0$. Let $F \in X^* = L_{\Phi^*}(T, \Sigma, \mu)$ be a function that defines a support functional for $\frac{f+g}{2}$, that is, $\|F\|_\Phi = 1$ and $F\big(\frac{f+g}{2}\big) = 1$. Then obviously $F(f) = F(g) = 1$ and $F(t) \geq 0$ μ-a.e.. Let $a = \int_T F(t)(l(t) - m(t))\chi_A \, d\mu(t)$ and $b = \int_T F(t)(m(t) - l(t))\chi_B \, d\mu(t)$, where χ_Y denotes the characteristic function of Y. Observe that we have $a, b \geq 0$. We will show that

$$a > 0 \quad \text{or} \quad b > 0. \tag{2}$$

Indeed, if $a = b = 0$, then $k_g = k_g F(g) = F(m) = F(l) = k_f F(f) = k_f$, whence

$$\int_T F(t)\big(f(t) - g(t)\big)\chi_A \, d\mu(t) = 0 \quad \text{and} \quad \int_T F(t)\big(g(t) - f(t)\big)\chi_B \, d\mu(t) = 0,$$

and consequently, by definitions of the sets A and B, we get a contradiction.

Now fix $n \in \mathbb{N}$. Since μ is atomless, we can find a partition $(A_1, ..., A_{2^n})$ of A, and a partition $(B_1, ..., B_{2^n})$ of B such that

$$\int_{A_i} F(t)\big(l(t) - m(t)\big) \, d\mu(t) = \frac{a}{2^n} \quad \text{and} \quad \int_{B_i} F(t)\big(m(t) - l(t)\big) \, d\mu(t) = \frac{b}{2^n}$$

for $i = 1, ..., 2^n$. We can assume that the sets which form the above partitions in the n-th step are partitions of the sets from the $(n-1)$-th step. Define

$$f_{2n-1}(t) = \begin{cases} l(t) & \text{for } t \in C \\ l(t) & \text{for } t \in \bigcup_{j=1}^{2^{n-1}} (A_{2j-1} \cup B_{2j-1}) \\ m(t) & \text{for } t \in \bigcup_{j=1}^{2^{n-1}} (A_{2j} \cup B_{2j}) \end{cases}$$

and

$$f_{2n}(t) = \begin{cases} l(t) & \text{for } t \in C \\ m(t) & \text{for } t \in \bigcup_{j=1}^{2^{n-1}} (A_{2j-1} \cup B_{2j-1}) \\ l(t) & \text{for } t \in \bigcup_{j=1}^{2^{n-1}} (A_{2j} \cup B_{2j}). \end{cases}$$

Observe that

$$\begin{aligned} F(f_{2n-1}) - F(f_{2n}) = & \sum_{j=1}^{2^{n-1}} F\big((f_{2n-1} - f_{2n})\chi_{A_{2j-1} \cup A_{2j}}\big) \\ & + \sum_{j=1}^{2^{n-1}} F\big((f_{2n-1} - f_{2n})\chi_{B_{2j-1} \cup B_{2j}}\big) + F\big((f_{2n-1} - f_{2n})\chi_C\big) \\ = & \sum_{j=1}^{2^{n-1}} \big(F((l - m)\chi_{A_{2j-1}}) + F((m - l)\chi_{A_{2j}})\big) \\ & + \sum_{j=1}^{2^{n-1}} \big(F((l - m)\chi_{B_{2j-1}}) + F((m - l)\chi_{B_{2j}})\big) \\ = & \ 0. \end{aligned}$$

Moreover,

$$\begin{aligned} F(f_{2n-1}+f_{2n}) &= F\big(l\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j-1}\cup B_{2j-1}\cup C}\big) + F\big(m\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j}\cup B_{2j}}\big) \\ &\quad + F\big(m\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j-1}\cup B_{2j-1}\cup C}\big) + F\big(l\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j}\cup B_{2j}}\big) \\ &= F(l)+F(m) \\ &= k_f+k_g. \end{aligned}$$

Consequently, $F(f_{2n-1}) = F(f_{2n}) = \frac{k_f+k_g}{2}$, which gives that $\|f_{2n}\|_\Phi^A \geq \frac{k_f+k_g}{2}$ and $\|f_{2n-1}\|_\Phi^A \geq \frac{k_f+k_g}{2}$. Now we will show that $\|f_{2n-1}\|_\Phi^A = \frac{k_f+k_g}{2}$ and $\|f_{2n}\|_\Phi^A = \frac{k_f+k_g}{2}$. To do this, note that we have by the definition of k_f and k_g and by the equalities $\|f\|_\Phi^A = \|g\|_\Phi^A = 1$ that

$$\begin{aligned} k_f &= 1+\rho_\Phi(k_f f) = 1+\rho_\Phi(l) \\ k_g &= 1+\rho_\Phi(k_g g) = 1+\rho_\Phi(m). \end{aligned} \tag{3}$$

Since $\|f_{2n}\|_\Phi^A \geq \frac{k_f+k_g}{2}$, $\|f_{2n-1}\|_\Phi^A \geq \frac{k_f+k_g}{2}$, and $\|f\|_\Phi^A \leq 1+\rho_\Phi(f)$ for any $f \in L_\Phi$, we get

$$\frac{k_f+k_g}{2} \leq 1+\rho_\Phi(f_{2n}) \tag{4}$$

and

$$\frac{k_f+k_g}{2} \leq 1+\rho_\Phi(f_{2n-1}). \tag{5}$$

Hence

$$\begin{aligned} k_f+k_g &\leq 2+\rho_\Phi(f_{2n})+\rho_\Phi(f_{2n-1}) \\ &= 2+\rho_\Phi\big(l\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j-1}\cup B_{2j-1}\cup C}\big) + \rho_\Phi\big(m\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j}\cup B_{2j}}\big) \\ &\quad + \rho_\Phi\big(m\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j-1}\cup B_{2j-1}\cup C}\big) + \rho_\Phi\big(l\chi_{\bigcup_{j=1}^{2^{n-1}} A_{2j}\cup B_{2j}}\big) \\ &= 2+\rho_\Phi(l)+\rho_\Phi(m) \\ &= k_f+k_g. \end{aligned}$$

This shows that we have equalities in (4) and (5) . Consequently, $\|f_{2n-1}\|_\Phi^A = \frac{k_f+k_g}{2}$ and $\|f_{2n}\|_\Phi^A = \frac{k_f+k_g}{2}$, as required. Observe that by definitions of the functions f_k, and by (2), we have

$$\begin{aligned} \|f_{2n}-f_{2n-1}\|_\Phi^A &= \|\,|f_{2n}-f_{2n-1}|\,\|_\Phi^A \\ &\geq F\big(|f_{2n}-f_{2n-1}|\big) \\ &= \sum_{j=1}^{2^n} F\big(|f_{2n}-f_{2n-1}|\chi_{B_j}\big) + \sum_{j=1}^{2^n} F\big(|f_{2n}-f_{2n-1}|\chi_{A_j}\big) \\ &= a+b > 0. \end{aligned}$$

Also by the definition and by (2),

$$\|f_k - f_j\|_\Phi^A = \| \, |f_k - f_j| \, \|_\Phi^A \geq F\big(|f_k - f_j|\big) \geq \frac{a+b}{2} > 0$$

for functions f_k and f_j corresponding to different partitions. This shows that the sequence $\{f_k\}$ does not contain a norm-convergent subsequence.

Now put $Z = \mathrm{cl}(\mathrm{conv}(\{f_n\}))$, where the closure is taken with respect to the norm topology. Since $F(f_n) = \frac{k_f + k_g}{2}$ for any $n \in \mathbb{N}$, $\|h\|_\Phi^A \geq \frac{k_f+k_g}{2}$ for any $h \in Z$. Consequently, since $\|f_n\|_\Phi^A = \frac{k_f+k_g}{2}$, $\|h\|_\Phi^A = \frac{k_f+k_g}{2}$ for any $h \in Z$. Observe that $\mathrm{dist}(0, Z) = \frac{k_f+k_g}{2} = \|f_n\|_\Phi^A$ for any $n \in \mathbb{N}$, that is, (f_n) is a minimizing sequence in Z for 0. But $\{f_n\}$ does not contain a Cauchy subsequence, which leads to a contradiction.

Now assume that $K(f) \neq \emptyset$ and $K(g) = \emptyset$. Take $k_f \in K(f)$. By convexity of the function $\lambda \to \rho_\Phi(\lambda g)$, we get $1 = \|g\|_\Phi^A = \lim_n \frac{1}{n}\rho_\Phi(ng)$. Note that for any $n \in \mathbb{N}$,

$$\begin{aligned} 2 = \|f+g\|_\Phi^A &\leq \frac{1 + \rho_\Phi\big(\frac{nk_f}{k_f+n}(f+g)\big)}{\frac{nk_f}{k_f+n}} \\ &= \frac{1 + \rho_\Phi\big(\frac{n}{k_f+n}(k_f f) + \frac{k_f}{k_f+n}(ng)\big)}{\frac{nk_f}{k_f+n}} \\ &\leq \frac{1}{n} + \frac{1 + \rho_\Phi(k_f f)}{k_f} + \frac{\rho_\Phi(ng)}{n} \; \to \; 2. \end{aligned}$$

Hence

$$2 = \lim_n \frac{1 + \rho_\Phi\big(\frac{nk_f}{k_f+n}(f+g)\big)}{\frac{nk_f}{k_f+n}}.$$

By $\Phi \in \Delta_2$, the function $\lambda \to \rho_\Phi(\lambda(f+g))$ is convex and continuous. Since $\frac{nk_f}{k_f+n} \to k_f$ as $n \to \infty$, we have

$$\lim_n \frac{1 + \rho\big(\frac{nk_f}{k_f+n}(f+g)\big)}{\frac{nk_f}{k_f+n}} = \frac{1 + \rho\big(k_f(f+g)\big)}{k_f}$$

and consequently,

$$2 = \frac{1 + \rho\big(k_f(f+g)\big)}{k_f}.$$

Since $\|\frac{f+g}{2}\|_\Phi^A = 1$, by the above equality $k_f \in K(\frac{f+g}{2})$. Since $k_f \in K(f)$ and $f \neq \frac{f+g}{2}$, the proof of this case reduces to the proof of the previous one, with $\frac{f+g}{2}$ in place of g.

Finally, assume that $K(f), K(g) = \emptyset$. Starting from f and g instead of l and m, we can construct a sequence $\{f_n\}$ as in the first case of the proof. Analogously as in the first case we can show that $\|f_k\|_\Phi^A \geq F(f_k) = 1$ for any $k \in \mathbb{N}$ and $\|f_n - f_m\|_\Phi^A \geq \frac{a+b}{2}$ for $m \neq n$. Since $K(f), K(g) = \emptyset$, we have $\lim_k \frac{1}{k}\rho_\Phi(kf) = 1$ and $1 = \lim_k \frac{1}{k}\rho_\Phi(kg)$. Now we will show that $\|f_k\|_\Phi^A = 1$ for any $k \in \mathbb{N}$. To do this, assume for the contrary that $\|f_k\|_\Phi^A = 1 + d$ with $d > 0$ for some $k \in \mathbb{N}$. We can assume without loss of generality that $k = 2n$. Observe that

$$\begin{aligned} 2 + d \leq \|f_{2n-1}\| + \|f_{2n}\| &\leq \frac{\rho_\Phi(mf_{2n-1}) + \rho_\Phi(mf_{2n})}{m} + \frac{2}{m} \\ &= \frac{\rho_\Phi(mf) + \rho_\Phi(mg)}{m} + \frac{2}{m} \\ &\leq 2 + \frac{d}{2} \end{aligned}$$

for $m \geq m_o$; a contradiction. Hence $\|f_k\|_\Phi^A = 1$ for any $k \in \mathbb{N}$. Reasoning as in the first case of the proof, we get that X is not approximatively compact.

Now suppose that X is reflexive and strictly convex. By Theorem 4, the functions $\Phi, \Phi^* \in \Delta_2$. The function $\Phi(t, \cdot)$ is strictly convex for μ-a.e. $t \in T$. Now we will show that X has property H. To do this, take a sequence $(f_n) \subset X$, $f \in X$ such that $f_n \to f$ weakly and $\|f_n\|_\Phi^A = \|f\|_\Phi^A = 1$. By Theorem 6, we can find $M > 0$ such that for any $n \in \mathbb{N}$ there exist $k_n \in K(f_n)$ satisfying $k_n < M$. Without loss of generality we can assume that $k_n \to k \in \mathbb{R}$. Since $\|f_n\|_\Phi^A = 1$, $k_n \geq 1$ for any $n \in \mathbb{N}$, which gives $k \geq 1$. Since Φ is strictly convex, $\rho_\Phi(kf) > 0$. We will show that $k - 1 = \rho_\Phi(kf)$. Since $\|f\|_\Phi^A = 1$, we have $k - 1 \leq \rho_\Phi(kf)$. Assume for the contrary that $k - 1 < \rho_\Phi(kf) - d$ for some $d > 0$. Define

$$\rho_1(g) = \frac{\rho_\Phi(g)}{\rho_\Phi(kf) - d}\,.$$

Let $\|\cdot\|_1$ denote the Luxemburg norm associated with ρ_1. Observe that $\rho_1(kf) > 1$ and $\rho_1(k_n f_n) \leq 1$ for n large enough. Hence $\|k_n f_n\|_1 \leq 1$ for n large enough and $\|kf\|_1 > 1$. Let us note that the weak convergence with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_\Phi^A$ coincide. Since $k_n \to k$, by the lower semicontinuity of the norm with respect to the weak topology, we get $\|kf\|_1 \leq \liminf \|k_n f_n\|_1 \leq 1$; a contradiction. Hence

$$\rho_\Phi(kf) = k - 1 = \lim_n k_n - 1 = \lim_n \rho_\Phi(k_n f_n). \tag{6}$$

Now define

$$\rho_2(g) = \frac{\rho_\Phi(g)}{\rho_\Phi(kf)}$$

and let $\|\cdot\|_2$ denote the Luxemburg norm associated with ρ_2. It is obvious that $\|kf\|_2 = 1$. We will show that $\|k_n f_n\|_2 \to 1$. Since $f_n \to f$ weakly , $k_n \to k$, the weak convergence with respect to the norm $\|\cdot\|_\Phi^A$ and the weak convergence with respect to the norm $\|\cdot\|_2$ coincide, liminf $\|k_nf_n\|_2 \geq 1$ by the lower semicontinuity of the norm with respect to the weak topology. Suppose that limsup $\|k_nf_n\|_2 > 1+d$ for some $d > 0$. Passing to a subsequence, if necessary, and applying (6), we get for $n \in \mathbb{N}$, $1 \leq \rho_2\left(\frac{k_nf_n}{1+d}\right) \leq \frac{\rho_2(k_nf_n)}{1+d} \to \frac{1}{1+d} < 1$; a contradiction. Hence $\lim_n\|k_nf_n\|_2 = 1 = \|kf\|_2$. Put $z_n = k_nf_n/\|k_nf_n\|_2$ and $z = kf$. It is easy to see that $\rho_2(z_n) = \|z_n\|_2 = 1$, $\rho_2(z) = \|z\|_2 = 1$. Since $z_n \to z$ weakly, $\|\frac{z_n+z}{2}\|_2 \to 1$. By Theorem 5, $z_n - z \to 0$ locally in measure. We will show that $\rho_2(\frac{z_n-z}{2}) \to 0$. Fix $\varepsilon > 0$. Since $\rho_2(z) = 1$ and μ is atomless and σ-finite, we can find $A_1 \in \Sigma$ of finite and positive measure such that $\rho_2(z\chi_{T\setminus A_1}) < \varepsilon$ and $|z(t)| > 0$ for $t \in A_1$. There exists also $n_o \in \mathbb{N}$ such that $\rho_2(z\chi_{A_o}) < \varepsilon$, where

$$A_o = \Big\{t \in A_1 : \frac{1}{n_o} > \max\big(|z(t)|, \Phi(t,|z(t)|)\big) \text{ or } \min\big(|z(t)|, \Phi(t,|z(t)|)\big) > n_o\Big\}.$$

Put $A = A_1 \setminus A_o$. Since $(z_n - z)\chi_A \to 0$ in measure and $\mu(A) < \infty$, there exists a subsequence (z_{n_k}) of (z_n) such that $z_{n_k} \to z$ μ-a.e. on A. Now choose $\delta > 0$ such that for any $B \in \Sigma$, $B \subset A$, if $\mu(B) < \delta$ then $\rho_2(z\chi_B) < \varepsilon$. By the Jegoroff theorem, we can find $B_o \subset A$ such that $\mu(B_o) < \delta$ and $z_{n_k} \to z$ uniformly on $A \setminus B_o$. Observe that there is $k_o \in \mathbb{N}$ such that for any $k \geq k_o$,

$$\sup\big\{|z_{n_k}(t) - z(t)| : t \in A \setminus B_o\big\} < \frac{1}{n_o}.$$

Hence for any $t \in A \setminus B_o$ and $k \geq k_o$,

$$|z_{n_k}(t) - z(t)| < |z_{n_k}(t)| + |z(t)| < 2|z(t)| + \frac{1}{n_o} < 3|z(t)|.$$

Consequently, by the Δ_2-condition and the Lebesgue dominated convergence theorem,

$$\rho_2((z_{n_k} - z)\chi_{A\setminus B_o}) \to 0. \tag{7}$$

Moreover, $\Phi \circ z_{n_k} \to \Phi \circ z$ coordinatewise μ-a.e. on $A \setminus A_o$. By the Jegoroff theorem, we can find $B_1 \subset A \setminus B_o$ such that $\mu(B_1) < \delta$ and $\Phi \circ z_{n_k} \to \Phi \circ z$ uniformly on $A \setminus (B_o \cup B_1)$ and $\rho_2(z\chi_{B_1}) < \varepsilon$. Reasoning as above gives

$$\rho_2\big(z_{n_k}\chi_{A\setminus(B_o\cup B_1)}\big) \to \rho_2\big(z\chi_{A\setminus(B_o\cup B_1)}\big). \tag{8}$$

Since ρ_2 is orthogonally additive and $\rho_2(z_{n_k}) = \rho_2(z) = 1$, we get by (8)

$$\rho_2\big(z_{n_k}\chi_{(T\setminus A)\cup(B_o\cup B_1)}\big) \to \rho_2\big(z\chi_{(T\setminus A)\cup(B_o\cup B_1)}\big). \tag{9}$$

By (7) and (9), we have

$$\begin{aligned}\rho_2\Big(\frac{z_{n_k} - z}{2}\Big) &\leq \frac{1}{2}\Big(\rho_2\big((z_{n_k} - z)\chi_{A\setminus(B_o\cup B_1)}\big) + \rho_2\big(z_{n_k}\chi_{(T\setminus A)\cup(B_o\cup B_1)}\big) \\ &\quad + \rho_2\big(z\chi_{(T\setminus A)\cup(B_o\cup B_1)}\big)\Big) \\ &\leq 4\varepsilon.\end{aligned}$$

Hence $\rho_2\left(\frac{z_{n_k}-z}{2}\right) \to 0$ and consequently $\rho_\Phi\left(\frac{z_{n_k}-z}{2}\right) \to 0$. By the Δ_2-condition we get, $\|z_{n_k} - z\|_\Phi^A \to 0$. The above reasoning implies that $\|z_n - z\|_\Phi^A \to 0$, which completes the proof.

Second proof. By Theorem 3, we need only to prove that the Kadec–Klee property for L_Φ^A is equivalent to the facts that the Musielak–Orlicz function Φ is strictly convex and $\Phi \in \Delta_2$.

Let $\Phi \in \Delta_2$ and $\Phi(t,\cdot)$ be strictly convex for μ-a.e $t \in T$. Assume that $x \in S(L_\Phi^A)$, $(x_n) \subset S(L_\Phi^A)$ and $x_n \to x$ weakly. By reflexivity of L_Φ^A we have that $\Phi^* \in \Delta_2$, which implies that $\lim_{u\to\infty} \frac{\Phi(t,u)}{u} = \infty$ for μ-a.e. $t \in T$. Then we have that for any $x \in L_\Phi^A \backslash \{0\}$ there exists $k > 0$ such that $\|x\|_\Phi^A = \frac{1}{k}(1+\rho_\Phi(kx))$. Moreover, by the assumption that Φ is strictly convex, we know that such a number $k > 0$ is only one (otherwise if $k, l \in K(x)$ and $k \neq l$, then one can easily prove that Φ must be affine between $kx(t)$ and $lx(t)$). So $K(x) = \{k\}$ and similarly there are $k_n > 0$ $(n = 1, 2, \dots)$ such that $K(x_n) = \{k_n\}$, which yields that $k_n - 1 = \rho_\Phi(k_n x_n)$ for any $n \in \mathbb{N}$ and $k - 1 = \rho_\Phi(kx)$. We may assume that $(k_n)_{n=1}^\infty$

is a bounded sequence, because otherwise considering $\left(\frac{x_n+x}{2}\right)_{n=1}^\infty$ in place of $(x_n)_{n=1}^\infty$, we have $\left\|\frac{x_n+x}{2}\right\|_\Phi \to 1$ (which follows be the weak convergence of $\frac{x_n+x}{2}$ to x) and $\left\|\frac{x_n+x}{2} - x\right\|_\Phi \to 0$ if and only if $\|x_n - x\|_\Phi \to 0$. It is easy to show that $\frac{2k_n k}{k_n+k} \in K\left(\frac{x_n+x}{2}\right)$ for any $n \in \mathbb{N}$. The sequence $\left(\frac{2k_n k}{k_n+k}\right)_{n=1}^\infty$ is bounded in any case (also when (k_n) is unbounded). It follows by strict convexity of Φ that $k_n x_n \to kx$ locally in measure (see [23]). Since $(k_n)_{n=1}^\infty$ is a bounded sequence, there are $k' > 0$ and a subsequence (k_{n_l}) of (k_n) such that $k_{n_l} \to k'$ as $l \to \infty$. By $x_n \xrightarrow{w} x$, we get that $k_{n_l} x_{n_l} \xrightarrow{w} k'x$. Since $k_{n_l} x_{n_l} \to kx$ locally in measure, we have $k'x = kx$ (see Proposition 1). In consequence $k' = k$, whence we get that $k_{n_l} \to k$. Therefore $\rho_\Phi(k_{n_l} x_{n_l}) \to \rho_\Phi(kx)$. We may assume without loss of generality that this holds for $k_n x_n$ in place of $k_{n_l} x_{n_l}$. By Lemma 7, we have $\|k_n x_n - kx\|_\Phi \to 0$, and finally $\|x_n - x\|_\Phi \to 0$, because $k_n \to k$ as $n \to \infty$.

Now we will show that condition $\Phi \in \Delta_2$ and strict convexity of Φ are necessary for the property H of L_Φ^A. Condition $\Phi \in \Delta_2$ is necessary by Theorem 1. It remains to prove the necessity of strict convexity of Φ. Assume that Φ is not strictly convex on $\mathbb{R}$. Then there are a set $C \in \Sigma$ of positive measure and an interval $[a, b]$ such that $0 < a < b$ and $\Phi(t,\cdot)$ is affine on $[a, b]$ for μ-a.e. $t \in C$. We will apply Lemma 6. Take $\delta > 0$ such that $\delta < \frac{b-a}{4}$. We can construct a function $y \in S(L_\Phi^A)$ such that $ky(t) \in [a+\delta, b-\delta]$ for $t \in G \in \Sigma$, where G is bounded and closed, $\mu(G) > 0$ and $\mu(T \setminus G) > 0$. Let $c = \frac{a+b}{2}$ and $\int_G \Phi^*(t, p(c))dt < 1$. We can find $d > 0$ and $G_1 \subset T \setminus G$ such that

$$\int_G \Phi^*(t, p(c))dt + \int_{G_1} \Phi^*(t, p(d))dt = 1.$$

Then defining

$$x = c\chi_G + d\chi_{G_1},$$

we have $\|x\|_{\Phi}^{A} = 1 + \rho_{\Phi}(x)$. In consequence, $y := x/(1 + \rho_{\Phi}(x)) \in S(L_{\Phi}^{A})$ and $1+\rho_{\Phi}(x) \in K(y)$. Denoting $k = 1+\rho_{\Phi}(x)$, we see that $ky(t) = c$ for any $t \in G$, that is, $ky(t) = c \in [a + \delta, b - \delta]$ for any $t \in G$. Let $(T_n^{'})_{n=1}^{\infty}$ be a sequence of bounded and closed sets such that $0 < \mu(T_n^{'}) < \infty$, $T_n^{'} \subset T_{n+1}^{'}$, $\chi_{T_n^{'}} \in E_{\Phi}$ for any $n \in \mathbb{N}$ and $\mu(T \setminus \bigcup_n T_n^{'}) = 0$ (the sets $T_n^{'}$ can be constructed using the sums of the sets T_n from Lemma 3). Define $y_n = y + \frac{\delta}{k}\chi_{E_n} - \frac{\delta}{k}\chi_{F_n}$, where $(E_n), (F_n)$ are the sequences decomposing G as in Lemma 6 (with G in place of E). We have for any $n \in \mathbb{N}$,

$$\rho_{\Phi}(ky_n) = \rho_{\Phi}(ky + \delta\chi_{E_n} - \delta\chi_{F_n}) = \rho_{\Phi}(ky) = \rho_{\Phi}(x)$$

by the fact that Φ is affine on the interval $[a, b]$ and the values of $ky+\delta\chi_{E_n}-\delta\chi_{F_n}$ belongs to $[a, b]$ for $t \in G$. Consequently,

$$\|y_n\|_{\Phi}^{A} \leq \frac{1}{k}(1 + \rho_{\Phi}(kx_n)) = \frac{1}{k}(1 + \rho_{\Phi}(ky)) = \frac{1}{k}(1 + \rho_{\Phi}(x)) = 1.$$

Moreover, by Lemma 6 and the facts that $y_n-y = \delta\chi_{E_n}-\delta\chi_{F_n}$ and the functions from L_{Φ}^{A} are integrable on the sets $V_n = T_n^{'} \cap S_n$, where S_n are as in Lemma 2, we deduce that $y_n - y \to 0$ weakly, that is $y_n \to y$ weakly. In particular, taking $x^* \in \mathrm{Grad}(y)$, we have $x^*(y_n) \to x^*(y) = 1$, whence $\|y_n\|_{\Phi}^{A} \to 1 = \|y\|_{\Phi}^{A}$. However,

$$\rho_{\Phi}(y_n - y) = \int_G \Phi(t, \delta)dt > 0,$$

whence

$$\|x_n - x\|_{\Phi}^{A} \geq \|x_n - x\|_{\Phi} \geq \min\left(1, \int_G \Phi(t, \delta)dt\right) > 0,$$

which shows that L_{Φ}^{A} has not the Kadec–Klee property. □

Theorem 12. *Let $X = L_{\Phi}$, where μ is a σ-finite, atomless measure. Then X is approximatively compact if and only if X is reflexive and strictly convex.*

Proof. Similarly as in the previous theorem, we will present two different proofs.

First proof. In general, the proof goes on the same line as the proof of Theorem 11. First suppose that X is approximatively compact but not strictly convex. Let $f, g \in X$, $f \neq g$, be such that $1 = \|f\|_{\Phi} = \|g\|_{\Phi} = \|\frac{f+g}{2}\|_{\Phi}$. Since X is strictly convex if and only if X_+ is strictly convex (see [20]), we can assume that $f, g \geq 0$. Let $F \in X^*$ be a support functional for $\frac{f+g}{2}$. Since X is approximatively compact, X is reflexive. Hence $F \in L_{\Phi^*}(T, \Sigma, \mu)$ and $\|F\|_{\Phi}^{A} = 1$. Now we proceed analogously as in the proof of Theorem 11. Starting from f and g instead of l and m (see page 177), we construct the sequence of functions $\{f_n\}$. We can show as in Theorem 11 that $F(f_k) = 1$, which implies that $\|f_n\|_{\Phi} \geq 1$ for any $k \in \mathbb{N}$, whence $\rho(f_k) \geq 1$ for any $k \in \mathbb{N}$. Also by the orthogonal additivity of ρ_{Φ},

$$\rho(f_{2n-1}) + \rho(f_{2n}) = \rho(f) + \rho(g) = 2,$$

Therefore $\rho(f_{2n-1}) = \rho(f_{2n}) = 1$, that is, $\|f_k\|_\Phi = 1$ for any $k \in \mathbb{N}$. Moreover

$$\|f_k - f_m\|_\Phi = \||f_k - f_m|\|_\Phi \geq F(|f_k - f_m|) \geq \frac{a+b}{2} > 0,$$

where a, b are as in Theorem 11. Put $Z = \mathrm{cl}\,(\mathrm{conv}(\{f_n\}))$. Reasoning as in the proof of Theorem 11, we get a contradiction with approximative compactness of X.

Now suppose that X is reflexive and strictly convex. We will show that X has property H. To do this, take $f_n \in X$, $f \in X$ of norm one of any $n \in \mathbb{N}$ such that $f_n \to f$ weakly. Let $F \in S(X^*)$ be such that $F(f) = \|f\| = 1$. Then $F\left(\frac{f_n+f}{2}\right) \to 1$, whence $\|\frac{f_n+f}{2}\| \to 1$. By Theorem 5, $f_n - f \to 0$ locally in measure. Reasoning as in Theorem 11, replacing ρ_2 by ρ_Φ, we get that $\rho_\Phi\left(\frac{f_{n_k}-f}{2}\right) \to 0$ for some subsequence $\{f_{n_k}\}$. By the Δ_2-condition, we get that $\|f_{n_k} - f\|_\Phi \to 0$. Hence, by the double extract subsequence theorem, we get that $\|f_n - f\|_\Phi \to 0$, which shows that X has the property H. By Theorem 3, X is approximatively compact, as required.

Second proof (for $T \subset \mathbb{R}$ and the Lebesgue measure in T). Basing on Theorem 3, it is enough to show that L_Φ has the Kadec–Klee property if and only if $\Phi \in \Delta_2$ and Φ is strictly convex. The necessity of $\Phi \in \Delta_2$ follows by Theorem 1 and the fact that L_Φ is order continuous if and only if $\Phi \in \Delta_2$ (see [17]). Let us prove now the necessity of strict convexity of Φ. Suppose that Φ is not strictly convex. Then there are a set $C \in \Sigma$ of positive measure and an interval $[a,b]$ such that $0 < a < b < \infty$ and $\Phi(t,\cdot)$ is affine on $[a,b]$ for μ-a.e. $t \in C$. Let $D \subset C$ be a measurable set such that $\mu(D) > 0$, $\mu(C \setminus D) > 0$ and

$$\int_D \Phi(t,c)\,d\mu \leq 1,$$

where $c = \frac{a+b}{2}$. Let $0 < \delta < \frac{a+b}{4}$ and $(E_n), (F_n)$ be the sequences decomposing D as in Lemma 6, with D in place of E. There is $d > 0$ such that

$$x = c\chi_D + d\chi_E,$$

with $E \subset (C \setminus D) \cap \Sigma$ satisfies $\|x\|_\Phi = 1$ (in the case when $\Phi(t,\cdot)$ is finitely valued on some measurable set in $T \setminus C$, then we can even find d and E such that $\rho_\Phi(x) = 1$). Define $x_n = x + \delta\chi_{E_n} - \delta\chi_{F_n}$. Then $\rho_\Phi(x_n\chi_D) = \rho_\Phi(x\chi_D)$ and $x_n\chi_E = x\chi_E$, whence it follows that $\|x_n\|_\Phi = \|x\|_\Phi$. By Lemma 6 we know that $x_n - x \to 0$ weakly. Moreover, by the fact that $\Phi(t,\cdot)$ is affine on $[c-\delta, c+\delta]$ for any $t \in C$, we have $\rho_\Phi(x_n - x) = \int_C \Phi(t,c)\,d\mu > 0$. Since $0 < \int_C \Phi(t,c)d\mu \leq 1$, we have $\|x_n - x\|_\Phi \geq \int_C \Phi(t,c)d\mu$, which means that $x_n - x \not\to 0$ in norm.

If $\|x_n\|_\Phi = \|x\|_\Phi = 1$ and $x_n \to x$ weakly, then taking $x^* \in \mathrm{Grad}(x)$, we have $x^*\left(\frac{x_n+x}{2}\right) \to x^*(x) = 1$, whence $\left\|\frac{x_n+x}{2}\right\|_\Phi \to 1$. In consequence (see [23]), we have that $x_n \to x$ locally in measure. Therefore, applying Lemma 7 finishes the proof. □

Theorem 13. *Let μ be non-atomic, $\mu(T) < \infty$ and Φ be a Musielak–Orlicz function such that $\frac{\Phi(t,u)}{u} \to 0$ as $u \to 0$ for μ-a.e. $t \in T$. Then the Musielak–Orlicz space L_Φ is fully k-rotund if and only if $\Phi(t,\cdot)$ are strictly convex functions for μ-a.e. $t \in T$ and $\Phi \in \Delta_2$, $\Phi^* \in \Delta_2$.*

Proof. Necessity. The necessity of the conditions $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$ follows by Proposition 2. Moreover, full k-rotundity of L_Φ implies its approximative compactness (see [26]). We will show that strict convexity of the functions $\Phi(t,\cdot)$ (for μ-a.e $t \in T$) is the necessary condition for approximative compactness of L_Φ. Suppose that there exists a set $A \in \Sigma$ such that $\mu(A) > 0$ and the functions $\Phi(t,\cdot)$ are affine on some intervals, for any $t \in A$. Let $Q = (q_n)_1^\infty$ denote the set of all rational numbers. For any $k \in \mathbb{N}$, we define the sets

$$A_k = \big\{t \in A : \Phi(t,\cdot) \text{ is affine on } [a_k, b_k];\ a_k, b_k \in Q\big\}.$$

Since $A = \bigcup_{k=1}^\infty A_k$, so there exists an $l \in N$ such that $\mu(A_l) > 0$, that is, there exist an interval $[a,b] \subset (0,\infty)$ and a set $B \subset A$ such that $\mu(B) > 0$ and $\Phi(t,\cdot)$ is affine on $[a,b]$ for any $t \in B$. Moreover, we can assume that

$$p_-(t,a) = p_+(t,b) \quad \text{ for } \mu - a.e.\ t \in T, \tag{10}$$

(considering in the opposite case a subinterval of $[a,b]$). Similarly we can assume without loss of generality that $\int_B \Phi\left(t, \frac{a+b}{2}\right) d\mu < 1$, considering a subset of B if necessary. Indeed, in the opposite case, using the facts that $\Phi\left(t, \frac{a+b}{2}\right)$ is a nonnegative, measurable and integrable function we define on $\Sigma \cap B$ a measure ν such that $\nu(C) = \int_C \Phi\left(t, \frac{a+b}{2}\right) d\mu$ for any $C \in \Sigma \cap B$ (if such a function is not integrable we use intersections of the sets C with the sums of the sets from Lemma 3). Then the measure ν is nonatomic and $\nu(B) \geq 1$, so there exists a set $D \subset B$ such that $D \in \Sigma$ and $\nu(D) < 1$.

Let $c \in R$ and $E \subset T \setminus B$ be such that

$$\int_B \Phi\left(t, \tfrac{a+b}{2}\right) d\mu + \int_E \Phi(t,c) d\mu = 1.$$

We denote $B_1^0 = B$. Since the function $\Phi(t,b) - \Phi(t,a)$ is nonnegative, measurable and integrable, we can define on $\Sigma \cap B$ a nonatomic measure κ by the formula $\kappa(C) = \int_C [\Phi(t,b) - \Phi(t,a)] d\mu$. Then $\kappa(B) > 0$, so there exist sets $B_1^1, B_2^1 \in \Sigma \cap B$ such that $\kappa(B_1^1) = \kappa(B_2^1)$ and $\kappa(B_1^1 \cap B_2^1) = 0$. Hence

$$\int_{B_1^1} \Phi(t,b) d\mu + \int_{B_2^1} \Phi(t,a) d\mu = \int_{B_2^1} \Phi(t,b) d\mu + \int_{B_1^1} \Phi(t,a) d\mu.$$

Let $x_1 = a\chi_{B_1^1} + b\chi_{B_2^1} + c\chi_E$. Then

$$\rho_\Phi(x_1) = \int_{B_1^1} \Phi(t,a) d\mu + \int_{B_2^1} \Phi(t,b) d\mu + \int_E \Phi(t,c)\, d\mu$$

$$= \frac{1}{2}\bigg(\int_{B_1^1}\Phi(t,a)d\mu + \int_{B_2^1}\Phi(t,b)d\mu + \int_{B_2^1}\Phi(t,a)d\mu + \int_{B_1^1}\Phi(t,b)d\mu\bigg)$$
$$+ \int_E \Phi(t,c)d\mu$$
$$= \frac{1}{2}\left(\int_B \Phi(t,a) + \int_B \Phi(t,b)\right) + \int_E \Phi(t,c)d\mu$$
$$= \int_B \Phi\left(t, \tfrac{a+b}{2}\right)d\mu + \int_E \Phi(t,c)d\mu$$
$$= 1.$$

Similarly we decompose the sets B_i^n, $n \geq 2$, $i = 1, \ldots, 2^n$, into subsets B_{2i-1}^{n+1}, B_{2i}^{n+1} such that $B = \bigcup_{i=1}^{2^n} B_i^n$, $\kappa(B_{2i-1}^{n+1} \cap B_{2i}^{n+1}) = 0$, $B_i^n = B_{2i-1}^{n+1} \cup B_{2i}^{n+1}$ and $\kappa(B_{2i-1}^{n+1}) = \kappa(B_{2i}^{n+1})$.

Define the sets $C_1^n = \bigcup_{k=1}^{2^{n-1}} B_{2k-1}^n$, $C_2^n = \bigcup_{k=1}^{2^{n-1}} B_{2k}^n$ and the elements

$$x_n = a\chi_{C_1^n} + b\chi_{C_2^n} + c\chi_E.$$

Then $\rho_\Phi(x_n) = 1$ (whence $\|x_n\|_\Phi = 1$)and $\kappa(C_1^n) = \kappa(C_2^n)$ for any $n \in \mathbb{N}$. Moreover, by (10) and Lemma 4, there exists $z \in L_\Psi$ generating a common support functional x^* for all x_n. Let us consider $x \in D = \overline{\text{conv}}(x_n)$. Then $x = \sum_{i=1}^l a_i x_i$ for some $l \in N$ and $\sum_{i=1}^l a_i = 1$. It gives, by the triangle inequality, that $\|x\|_\Phi \leq 1$. Moreover $x^*(x) = \sum_{i=1}^l a_i x^*(x_i) = 1$. Then $1 = x^*(x) \leq \|x^*\|\|x\|_\Phi = \|x\|_\Phi$, which implies $\|x\|_\Phi \geq 1$. Therefore, we have $\|x\| = 1$ for any $x \in D$, whence $d(0,D) = 1 = \|x_n\|$ for any $n \in N$. Now we will show that $\rho_\Phi(x_n - x_m) > \frac{1}{4}\kappa(B) > 0$. Let $m, n \in \mathbb{N}$, $n < m$. Then

$$C_1^n = C_1^n \cap B = C_1^n \cap (C_1^m \cup C_2^m) = (C_1^n \cap C_1^m) \cup (C_1^n \cap C_2^m).$$

Therefore, $C_1^n \setminus C_1^m = C_1^n \setminus (C_1^n \cap C_1^m) = C_1^n \cap C_2^m$. Similarly, $C_2^n \setminus C_2^m = C_2^n \cap C_1^m$. Moreover, $(C_1^n \setminus C_1^m) \cap (D_2^n \setminus C_2^m) = \emptyset$. So, by symmetry of the decomposing,

$$\frac{1}{4}\kappa(B) = \frac{1}{2}\kappa(C_1^n) = \kappa(C_1^n \cap C_1^m) = \kappa(C_1^n \cap C_2^m) = \kappa(C_1^n \setminus C_1^m).$$

Therefore, $\kappa(C_1^n \setminus C_1^m) = \frac{1}{4}\kappa(B)$. Moreover, $x_n - x_m = a\chi_{C_1^n \setminus C_1^m} + b\chi_{C_1^n \setminus C_1^m}$, so

$$\rho_\Phi(x_n - x_m) = \rho_\Phi\left(a\chi_{C_1^n \setminus C_1^m}\right) + \rho_\Phi\left(b\chi_{C_1^n \setminus C_1^m}\right) > \frac{1}{4}\kappa(B) > 0,$$

because

$$\begin{aligned}\kappa(C_1^n \setminus C_1^m) &= \rho_\Phi\left(b\chi_{C_1^n \setminus C_1^m}\right) - \rho_\Phi\left(a\chi_{C_1^n \setminus C_1^m}\right)\\ &< \rho_\Phi\left(b\chi_{C_1^n \setminus C_1^m}\right) + \rho_\Phi\left(a\chi_{C_1^n \setminus C_1^m}\right)\\ &= \rho_\Phi(x_n - x_m).\end{aligned}$$

This implies that the sequence (x_n) has no Cauchy subsequence, which means that L_Φ is not approximatively compact.

Sufficiency. Assume that $(x_n) \subset S(L_\Phi)$, $\left\|\frac{x_n^{(1)}+x_n^{(2)}}{2}\right\|_\Phi \to 1$ for every subsequences (x_n^1), (x_n^2) of (x_n), the function $\Phi(t,\cdot)$ is strictly convex for μ-a.e. $t \in T$, $\Phi \in \Delta_2$ and $\Psi \in \Delta_2$. Denote, for simplicity of notations, the sequences $(x_n^{(1)}), (x_n^{(2)})$ by $(x_l), (x_m)$, respectively. We will show that (x_n) is equi-continuous. Suppose the contrary. Then, since $\mu(T) < \infty$, we have

$$\begin{gathered}\exists\, \varepsilon_0 > 0 \;\; \forall\, \sigma > 0 \;\; \forall\, k \in \mathbb{N} \;\; \exists\, n_k = n_k(\sigma) > k \;\; \exists\, B_k = B_k(\sigma) \in \Sigma \\ \text{such that } \|x_{n_k}\chi_{B_k}\|_\Phi \geq \varepsilon_0 \text{ whenever } \mu(B_k) < \sigma.\end{gathered} \tag{11}$$

By Lemma 5, we get

$$\begin{gathered}\forall\, \varepsilon > 0 \;\; \exists\, \delta = \delta(\varepsilon) > 0 \;\; \exists\, n' = n'(\varepsilon) \in \mathbb{N} \;\; \forall\, l, m > n' \;\; \forall\, E \in \Sigma \\ \text{we have } \|x_l\chi_E\|_\Phi < \varepsilon \text{ whenever } \|x_m\chi_E\|_\Phi < \delta.\end{gathered} \tag{12}$$

Let us fix $k \in \mathbb{N}$ and $\delta > 0$. By $\Phi \in \Delta_2$, the element x_{k+1} has order continuous norm, whence there exists $\sigma > 0$ such that $\|x_{k+1}\chi_A\|_\Phi < \delta$ whenever $A \in \Sigma$ and $\mu(A) < \sigma$. Now, using condition (11) to this value $\sigma > 0$, we get

$$\begin{gathered}\exists\, \varepsilon_0 > 0 \;\; \forall\, \delta > 0 \;\; \forall\, k \in \mathbb{N} \;\; \exists\, l_k = n_k, m_k = k+1 > k \;\; \exists\, B_k \in \Sigma \\ \text{such that } \|x_{m_k}\chi_{B_k}\|_\Phi < \delta \;\text{ and }\; \|x_{l_k}\chi_{B_k}\|_\Phi \geq \varepsilon_0.\end{gathered} \tag{13}$$

We get a contradiction, because condition (13) is just the condition opposite to condition (12). So the sequence (x_n) is equi-continuous. Moreover, $x_n - x \xrightarrow{\mu} 0$ for some $x \in L_\Phi$ (see [23]). We can assume, without loss of generality, that $x_n(t) \to x(t)$ for μ-a.e. $t \in T$. Since $\mu(T) < \infty$ and (x_n) is equi-continuous, there exist $n(\varepsilon) \in \mathbb{N}$ and $a(\varepsilon) > 0$ such that $\|x_n\chi_E\|_\Phi < \frac{\varepsilon}{3}$ for any $n > n(\varepsilon)$ whenever $E \in \Sigma$ and $\mu(E) < a(\varepsilon)$. Since L_Φ is order continuous (by $\Phi \in \Delta_2$), there exists $b(\varepsilon) > 0$ such that $\|x\chi_E\|_\Phi < \frac{\varepsilon}{3}$ for any $E \in \Sigma$ with $\mu(E) < b(\varepsilon)$. By the Jegoroff Theorem there exists $A \in \Sigma$ such that $\mu(A) < \frac{1}{2}\min(a(\varepsilon), b(\varepsilon))$ and $x_n - x \to 0$ uniformly on $T \setminus A$, which yields that there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|x_n(t) - x(t)| \leq 1$ for any $t \in T \setminus A$ and for any $n > n_1(\varepsilon)$. Then $\Phi(t, x_n(t) - x(t)) \leq \Phi(t, 1)$ for any $t \in T \setminus A$. Let $(T_i)_{i=1}^\infty$ be the sequence from Lemma 3 and $m \in \mathbb{N}$ be such that $\mu((T \setminus A) \setminus S_m) < \frac{1}{2}\min(a(\varepsilon), b(\varepsilon))$ for $S_m = \bigcup_{i=1}^m T_i$, $S_m \subset T \setminus A$. Then $\mu(T \setminus S_m) < \min(a(\varepsilon), b(\varepsilon))$ and $\rho_\Phi(\chi_{S_m}) < \infty$. By the Lebesgue dominated convergence theorem we have that $\rho_\Phi((x_n - x)\chi_{S_m}) \to 0$ as $n \to \infty$. Condition $\Phi \in \Delta_2$ implies that $\|(x_n - x)\chi_{S_m}\|_\Phi \to 0$, whence there exists $n_2(\varepsilon) \geq \max(n(\varepsilon), n_1(\varepsilon))$ such that $\|x_n - x\|_\Phi\chi_{S_m} < \frac{\varepsilon}{3}$ for any $n > n_2(\varepsilon)$. Finally,

$$\begin{aligned}\|x_n - x\|_\Phi &\leq \|(x_n - x)\chi_{S_m}\|_\Phi + \left\|(x_n - x)\chi_{T\setminus S_m}\right\|_\Phi \\ &< \|x_n\chi_{S_m}\|_\Phi + \left\|x_n\chi_{T\setminus S_m}\right\|_\Phi + \left\|x\chi_{T\setminus S_m}\right\|_\Phi < \varepsilon\end{aligned}$$

for any $n > n_2(\varepsilon)$. By the arbitrariness of $\varepsilon > 0$, this finishes the proof. □

4. Lorentz–Orlicz spaces

Let (I, σ, m) be the Lebesgue measure space with $I = (0,1)$ or $I = (0,\infty)$. Let $\Phi : [0,\infty] \to [0,\infty]$ be an Orlicz function (i.e., Musielak–Orlicz function which does not depend on the parameter t) and $\omega : I \to (0,\infty)$ be a weight function (i.e., nonincreasing and locally integrable function with respect to the measure m and such that $\int_0^\infty \omega\, dm = \infty$ if $I = (0,\infty)$). For $x \in L^o$, x^* denotes the nonincreasing rearrangement of x defined by

$$x^*(t) = \inf\{\lambda > 0 : \mu_x(\lambda) \leq t\}$$

for any $t > 0$ (by convention $\inf(\emptyset) = \infty$), where $\mu_x(\lambda) = \mu(\{s \in T : |x(s)| > \lambda\})$ for any $\lambda > 0$. The Orlicz–Lorentz function space $\Lambda_{\Phi,\omega}$ is defined by

$$\Lambda_{\Phi,\omega} = \left\{ x \in L^0(m) : \int_I \Phi(\lambda x^*)\omega\, dm < \infty \text{ for some } \lambda > 0 \right\}.$$

In the case of counting measure on $2^{\mathbb{N}}$ the Orlicz–Lorentz sequence space $\lambda_{\Phi,\omega}$ is defined by

$$\lambda_{\Phi,\omega} = \left\{ x = \big(x(k)\big) \in c^0 : \sum_{k=1}^{\infty} \Phi\big(\lambda x^*(k)\big)\, \omega(k) < \infty \text{ for some } \lambda > 0 \right\}.$$

Here $\omega = (\omega(k))$ is a weight sequence, that is, a nonincreasing sequence of positive reals such that $\sum_{k=1}^\infty \omega(k) = \infty$. In this case x^* is nothing but the permutation of x such that x^* is a nondecreasing sequence.

It is easy to check that $\Lambda_{\Phi,\omega}$ (resp. $\lambda_{\Phi,\omega}$) is a symmetric function space (resp. symmetric sequence space) with the Fatou property, if it is equipped with the norm

$$\|x\|_\Phi^\omega = \inf\left\{ \lambda > 0 : \ \rho_\Phi^\omega\left(\frac{x}{\lambda}\right) \leq 1 \right\},$$

where $\rho_\Phi^\omega(x) = \int_T \Phi(x^*(t))\omega(t)\, d\mu$ (resp. $\rho_\Phi^\omega(x) = \sum_{n=1}^\infty \Phi(x^*(n))\omega(n)$ in the sequence case). The symmetry of the space means the fact that if x and g are equimeasurable, that is, $\mu_x = \mu_g$, then $\|x\|_{\Phi,\omega} = \|g\|_{\Phi,\omega}$.

Now we consider the Kadec–Klee property and approximative compactness in Lorentz–Orlicz function and sequence spaces. In the sequence case we assume that the weight sequence (ω_n) belongs to c_0. We say that an Orlicz function Φ satisfies condition Δ_2 at zero ($\Phi \in \Delta_2(0)$ for short) if there are positive constants K, a such that $\Phi(a) > 0$ and the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in \ [0, a]$.

Theory of Lorentz spaces and Lorentz–Orlicz spaces is very important and popular mainly because of its applications to the interpolation theory. We refer the readers to [1, 4, 5, 32] and [33].

Theorem 14. *Suppose that $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex function. Then the Lorentz–Orlicz sequence space $\lambda_{\Phi,\omega}$ has the Kadec–Klee property if and only if $a(\Phi) = 0$, Φ satisfies the Δ_2-condition at zero and $\sum_{n=1}^{\infty} \omega_n = +\infty$.*

Proof. Necessity. Suppose that $a(\Phi) > 0$. Take $b > 0$ such that $\Phi(b)\omega_1 = 1$ and define $x = be_1$, $x_n = x + a(\Phi)e_{n+1}$ $(n \in \mathbb{N})$. It is evident that $x \geq 0$ and $x_n \geq 0$ for any $n \in \mathbb{N}$. We will show that $x_n \to x$ weakly. Define $z = a(\Phi)\sum_{n=1}^{\infty} e_{n+1}$. We have $z \geq 0$, $\rho_\Phi(z) = 0$ and $\rho_\Phi(\lambda z) = +\infty$ for any $\lambda > 1$, whence $\|z\|_\Phi^\omega = 1$. Therefore $z \in \lambda_{\Phi,\omega}$. In consequence, for any $x^* \in (\lambda_{\Phi,\omega})^*$, $x^* \geq 0$, we have $0 \leq x^*(z) < \infty$ and for any $k \in \mathbb{N}$:

$$0 \leq a(\Phi)\sum_{n=1}^{k} x^*(e_{n+1}) = x^*\left(a(\Phi)\sum_{n=1}^{k} e_{n+1}\right) \leq x^*(z) < \infty$$

whence $\sum_{n=1}^{\infty} x^*(e_{n+1}) < \infty$, and so $x^*(x_n) \to 0$ as $n \to \infty$. Since any $x^* \in (\lambda_{\Phi,\omega})^*$ can be written as a difference of two nonnegative functionals from $(\lambda_{\Phi,\omega})^*$, this shows that $x_n \to x$ weakly. Since $\rho_\Phi^w(x) = \Phi(b)w_1 = 1$, we get $\|x\|_\Phi^w = 1$. Analogously, since $\rho_\Phi^w(x_n) = \Phi(b)w_1 + \Phi(a_\Phi)w_2 = 1$, we get $\|x_n\|_\Phi^w = 1$. But $\|x_n - x\|_\Phi^w = a(\Phi)\|e_1\|_\Phi^w > 0$ for any $n \in \mathbb{N}$, which shows that $\lambda_{\Phi,w}$ does not have the Kadec–Klee property.

Suppose now that Φ does not satisfy the Δ_2-condition at zero or $\sum_{n=1}^{\infty}\omega_n < \infty$. Then $\lambda_{\Phi,\omega}$ contains an order isometric copy of l^∞ (see [25]), so $\lambda_{\Phi,w}$ does not have the Kadec–Klee property.

Sufficiency. Assume now that $a(\Phi) = 0$, $\Phi \in \Delta_2(0)$ and $\sum_{n=1}^{\infty} \omega_n = \infty$. Then, taking into account that in the sequence case weak convergence implies pointwise convergence of sequences, we get in the same way as in [24] in the function case that $\lambda_{\Phi,\omega}$ has the Kadec–Klee property. □

Theorem 15.

(i) *If Φ is an Orlicz function vanishing only at zero and $\omega : I \to \mathbb{R}_+$ is a weighted function that is strictly decreasing on I, then the Lorentz–Orlicz space $\Lambda_{\Phi,\omega}$ is approximatively compact if and only if $\Lambda_{\Phi,\omega}$ is reflexive, that is, Φ and Φ^* satisfy condition $\Delta_2(\infty)$ if $m(I) < \infty$ and condition $\Delta_2(\mathbb{R}_+)$ if $m(I) = \infty$, and $\int_I \omega(t)dm = \infty$ if $m(I) = \infty$,*

(ii) *If Φ is an Orlicz function vanishing only at zero and (ω_n) is a weighted sequence from c_0, then the Lorentz–Orlicz space $\lambda_{\Phi,\omega}$ is approximatively compact if and only Φ and Φ^* satisfy condition $\Delta_2(0)$ and $\sum_{n=1}^{\infty} \omega_n = \infty$.*

Proof. Sufficiency. (i) By the assumptions, $\Lambda_{\Phi,\omega}$ is reflexive (see [21])) and it has the Kadec–Klee property (see [19], Theorem 18, p. 327). So, by Theorem 3, $\Lambda_{\Phi,\omega}$ is approximatively compact.

(ii) By the assumptions we know (see [19]) that $\lambda_{\Phi,\omega}$ is reflexive and that (see [19], Theorem 2) $\lambda_{\Phi,\omega}$ has property H_μ (= the Kadec–Klee property with respect to the coordinatewise convergence), so by reflexivity of the space, it has also property H. By Theorem 3, $\lambda_{\Phi,\omega}$ is approximatively compact.

Necessity. By [21], the assumptions are necessary for reflexivity of $\Lambda_{\Phi,\omega}$ and $\lambda_{\Phi,\omega}$, respectively. By [19], Theorems 18 and 2, the assumptions are necessary for property H of $\Lambda_{\Phi,\omega}$ and $\lambda_{\Phi,\omega}$, respectively. By Theorem 3, the assumptions are necessary for approximative compactness of the spaces $\Lambda_{\Phi,\omega}$ and $\lambda_{\Phi,\omega}$, respectively. □

Remark 4. Note that by Theorem 15 and the criteria for strict convexity of $\Lambda_{\Phi,\omega}$ from [19], in the case when the weighted function ω is strictly decreasing on I, approximative compactness of the Lorentz–Orlicz $\Lambda_{\Phi,\omega}$ does not imply strict convexity of $\Lambda_{\Phi,\omega}$, in contrast to Musielak–Orlicz function spaces.

Remark 5. It follows from the results of this paper that in reflexive strictly convex Musielak–Orlicz spaces and in Lorentz–Orlicz spaces that are reflexive and strictly convex, the metric projections from the space onto its nonempty, convex and closed subsets are continuous.

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