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The Plancherel and Hausdorff-Young Type Theorems for an Index Transformation

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Abstract. The Plancherel and Hausdorff–Young type theorems are proved for an integral transformation, which is associated with the product of the modified Bessel functions of different arguments. The transform essentially generalizes Lebedev's transformation involving squares of the modified Bessel functions as kernels.

Keywords. Bessel functions, index transform, Mellin transform, Kontorovich–Lebedev transform, Plancherel theorem, Hausdorff–Young inequality, Parseval equality

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1. Introduction

In this paper we will derive and study the following pair of integral transforms:

$$[\mathcal{G}f](x,y) = \frac{2}{\sqrt{\pi}} \int_0^\infty \tau K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) f(\tau) d\tau \quad (1.1)$$

$$\begin{aligned} f(\tau) = \frac{8\sinh 2\pi\tau}{\pi^3\sqrt{\pi}} \int_0^\infty \int_0^\infty & K_{i\tau}\left(\sqrt{x^2+y^2}-y\right) \\ & \times K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) [\mathcal{G}f](x,y) \frac{dx\,dy}{x}, \end{aligned} \quad (1.2)$$

where $K_\nu(z)$ is the modified Bessel function (cf. in [1, Chapter 9]) of the pure imaginary index (a subscript) $\nu = i\tau$. The convergence of the integrals in the corresponding spaces of functions will be discussed in detail below. Here we note, that we consider transformation (1.1) as an integral operator between two Lebesgue spaces

$$\mathcal{G} : L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh 2\pi\tau}\right) \leftrightarrow L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; \frac{dx\,dy}{x}\right), \quad (1.3)$$

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where the weighted Lebesgue space $L_p(\Omega;\omega(\tau)d\tau),\ 1 \le p < \infty$, is normed by

$$\|f\|_{L_p(\Omega;\omega(\tau)d\tau)} = \left(\int_\Omega |f(\tau)|^p \omega(\tau)d\tau\right)^{\frac{1}{p}} \tag{1.4}$$
$$\|f\|_{L_\infty(\Omega;\omega(\tau)d\tau)} = \text{ess sup}_{\tau\in\Omega}|f(\tau)|.$$

We will show that (1.3) is a one-to-one map. Moreover, using the technique of Mellin and Kontorovich–Lebedev transforms we generalize this result by proving the boundedness of the transformation (1.1) as an operator

$$\mathcal{G} : L_p\left(\mathbb{R}_+; d\tau\right) \leftrightarrow L_q\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{q\nu-1}dx\, dy\right) \tag{1.5}$$

with $1 \le p \le 2,\ p^{-1}+q^{-1} = 1,\ \nu > 0$. These main results are summarized in the Plancherel and Hausdorff -Young type theorems for the transformation (1.1).

As we see below, this surprisingly enough reciprocal formulas arise from the application of Plancherel's theorems for the Mellin and Kontorovich–Lebedev transforms (see in [5, 6, 7]). Since the integration in (1.1) is with respect to the order of the modified Bessel functions, such a class of integral transforms is called index transforms [7]. For instance, if we put $y = 0$ in (1.1) we arrive at the one-dimensional Lebedev transformation with the square of the modified Bessel function [3, 8]

$$[\mathcal{G}f](x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \tau K^2_{i\tau}(x) f(\tau)\, d\tau.$$

By fixing $x > 0$, say $x = 1$, we derive an index transform with the product of the modified Bessel functions

$$[\mathcal{G}f](y) = \frac{2}{\sqrt{\pi}}\int_0^\infty \tau K_{i\tau}\left(\sqrt{1+y^2}-y\right) K_{i\tau}\left(\sqrt{1+y^2}+y\right) f(\tau)\, d\tau.$$

However, when $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ are independent variables, it defines a completely different type of integral transformations which form one-to-one isometric isomorphism between one- and two-dimensional naturally determined weighted L_2-spaces with respect to the measures (1.3). Moreover, as we will show for all $f_1, f_2 \in L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh 2\pi\tau}\right)$ the corresponding $\mathcal{G}f_1, \mathcal{G}f_2$ belong to $L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; \frac{dx\, dy}{x}\right)$ and the Parseval identity holds:

$$\int_0^\infty\int_0^\infty [\mathcal{G}f_1](x,y)\overline{[\mathcal{G}f_2](x,y)}\frac{dx\, dy}{x} = \frac{\pi^3}{4}\int_0^\infty \frac{\tau}{\sinh 2\pi\tau} f_1(\tau)\overline{f_2(\tau)}d\tau. \tag{1.6}$$

In particular, we have

$$\int_0^\infty\int_0^\infty |[\mathcal{G}f](x,y)|^2\frac{dx\, dy}{x} = \frac{\pi^3}{4}\int_0^\infty \frac{\tau}{\sinh 2\pi\tau} |f(\tau)|^2 d\tau. \tag{1.7}$$

Finally we note in this section, that the formulas (1.1), (1.2) give a new source of index integrals and may successfully collect the related table in [3] (see also in [6, Chapter 10]).

2. Auxiliary definitions and results

It is well known [1] that the modified Bessel function can be defined, in particular, through the inverse Mellin transform of the product of the Euler Gamma-functions (cf. in [7, relation (2.124)])

$$K_{i\tau}(2x) = \frac{1}{8\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(\tfrac{s+i\tau}{2}\right)\Gamma\left(\tfrac{s-i\tau}{2}\right) x^{-s}ds, \quad x>0,\, \gamma>0.$$

Meanwhile, the Mellin direct transform

$$f^{\mathcal{M}}(s) = \int_0^\infty f(x)x^{s-1}dx \tag{2.1}$$

is defined for $f \in L_2(\mathbb{R}_+; x^{2\gamma-1}dx)$ and one-to-one isomorphically maps on $L_2(\gamma - i\infty, \gamma + i\infty)$, where integral (2.1) is convergent in mean with respect to the norm of the latter L_2-space. The inverse operator is given by

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^{\mathcal{M}}(s)x^{-s}ds, \quad s=\gamma+it,\, x>0. \tag{2.2}$$

Here integral (2.2) is convergent in mean by the norm in $L_2(\mathbb{R}_+; x^{2\gamma-1}dx)$. Moreover, the operator $f^{\mathcal{M}}$ is an isometric isomorphism between two mentioned Hilbert spaces and the Parseval equality

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1}dx = \frac{1}{2\pi}\int_{-\infty}^\infty |f^{\mathcal{M}}(\gamma+it)|^2 dt \tag{2.3}$$

holds true. We also consider the Kontorovich–Lebedev transform [5, 7] of the form

$$[KLf](x) = 4\int_0^\infty \tau K_{2i\tau}(2x)f(\tau)\,d\tau. \tag{2.4}$$

As it is proved in [7, Chapter 2], the Kontorovich-Lebedev operator (2.4) is the isomorphism

$$[KLf] : L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh 2\pi\tau}\right) \leftrightarrow L_2\left(\mathbb{R}_+; x^{-1}dx\right),$$

where the integral converges in mean and the Parseval identity

$$\int_0^\infty |[KLf](x)|^2\frac{dx}{x} = 2\pi^2\int_0^\infty \frac{\tau}{\sinh 2\pi\tau}|f(\tau)|^2\,d\tau \tag{2.5}$$

holds. These results will be applied to study mapping properties of the index transform (1.1) $[\mathcal{G}f](x,y)$, which is associated with the product of the modified Bessel functions

$$K_{i\tau}\Big(\sqrt{x^2+y^2}-y\Big)K_{i\tau}\Big(\sqrt{x^2+y^2}+y\Big). \tag{2.6}$$

Furthermore, in Section 3 we will use the formula (2.16.33.10) in [4]

$$\begin{aligned} 2\int_0^\infty x^{s-1}K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)dx \\ = \frac{\sqrt{\pi}}{2}y^{\frac{s}{2}}K_{\frac{s}{2}}(2y)\frac{\Gamma\left(\frac{s}{2}+i\tau\right)\Gamma\left(\frac{s}{2}-i\tau\right)}{\Gamma(\frac{1+s}{2})}, \end{aligned} \tag{2.7}$$

which gives the Mellin transform (2.1) with respect to x of this product for each $y,\tau>0$ and $\gamma=\mathrm{Re}s>0$. Finally in this section we appeal to relation (2.16.52.8) in [4] and we take the Fourier sine transform [5]

$$(F_sf)(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(\tau)\sin x\tau\,d\tau \tag{2.8}$$

in order to derive the following integral representation for the kernel (2.6)

$$\begin{aligned} \tau K_{i\tau}\Big(\sqrt{x^2+y^2}-y\Big)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right) \\ = \frac{x^2}{2}\int_0^\infty \frac{K_1\left(2\sqrt{y^2+x^2\cosh^2(\frac{t}{2})}\right)}{\sqrt{y^2+x^2\cosh^2(\frac{t}{2})}}\sinh t\sin t\tau\,dt. \end{aligned} \tag{2.9}$$

3. The Plancherel type theorem

In this section we give a sketch of proof of the Plancherel type theorem for the transformation (1.1). The complete proof is recently published in [10, Theorems 3, 4].

We begin to consider transformation (1.1) $[\mathcal{G}f_N](x,y)$ by taking

$$f_N(\tau)=f(\tau)\in L_2\big(\mathbb{R}_+;\tau[\sinh 2\pi\tau]^{-1}d\tau\big),$$

which vanishes outside of the interval $(\frac{1}{N},N)$. Then in view of the uniform estimate for the modified Bessel function [7, formula (1.100)]

$$|K_{i\tau}(u)|\le e^{-\delta|\tau|}K_0(u\cos\delta),\quad u>0,\,\delta\in\left[0,\tfrac{\pi}{2}\right), \tag{3.1}$$

it follows that integral (1.1) exists in the Lebesgue sense. Hence we may calculate with respect to x the Mellin transform (2.1) of the function $[\mathcal{G}f_N](x,y)$. For this we apply formula (2.7) and interchange the order of integration by virtue of Fubini's theorem. Therefore we obtain for $y > 0, s = \gamma + it, \gamma > 0$

$$[\mathcal{G}f_N]^{\mathcal{M}}(s,y) = \frac{y^{\frac{s}{2}} K_{\frac{s}{2}}(2y)}{\Gamma(\frac{1+s}{2})} \int_0^\infty \tau \Gamma\left(\tfrac{s}{2} + i\tau\right) \Gamma\left(\tfrac{s}{2} - i\tau\right) f_N(\tau)\, d\tau.$$

Using (2.1) we see that the latter integral is the composition of the Kontorovich-Lebedev and Mellin transforms (2.4) and (2.1), respectively. In fact, it can be verified by substituting instead of the product of gamma-functions the value of the Mellin transform (2.1) of the modified Bessel function (cf. relation (2.16.2.2) in [4]). Then appealing to (3.1) and the Fubini theorem we invert the order of integration in the obtained iterated integral and arrive at the composition representation

$$[\mathcal{G}f_N]^{\mathcal{M}}(s,y) = \frac{y^{\frac{s}{2}} K_{\frac{s}{2}}(2y)}{\Gamma(\frac{1+s}{2})} [KLf_N]^{\mathcal{M}}(s) \tag{3.2}$$

with $s = \gamma + it, \gamma > 0$,

$$[KLf_N]^{\mathcal{M}}(s) = \int_{\frac{1}{N}}^N \tau \Gamma\left(\tfrac{s}{2} + i\tau\right) \Gamma\left(\tfrac{s}{2} - i\tau\right) f(\tau)\, d\tau.$$

Theorem 1. *Let $f \in L_2(\mathbb{R}_+; \tau[\sinh 2\pi\tau]^{-1} d\tau)$. Then, as $N \to \infty$, the integral*

$$[\mathcal{G}f_N](x,y) = \frac{2}{\sqrt{\pi}} \int_{\frac{1}{N}}^N \tau K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) f(\tau)\, d\tau \tag{3.3}$$

converges in mean to $[\mathcal{G}f](x,y)$ with respect to the norm of $L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; \frac{dx\,dy}{x}\right)$, and

$$\begin{aligned} f_N(\tau) = \frac{8}{\pi^3\sqrt{\pi}} \sinh 2\pi\tau \int_{\frac{1}{N}}^N \int_{\frac{1}{N}}^N K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) \\ \times K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) [\mathcal{G}f](x,y) \frac{dx\,dy}{x} \end{aligned} \tag{3.4}$$

converges in mean to $f(\tau)$ with respect to the norm of $L_2(\mathbb{R}_+; \tau[\sinh 2\pi\tau]^{-1} d\tau)$. Moreover, almost for all $\tau \in \mathbb{R}_+$ and $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, respectively, the following reciprocal formulas take place:

$$\begin{aligned} f(\tau) = \frac{8}{\pi^3\sqrt{\pi}} \frac{\sinh 2\pi\tau}{\tau} \frac{d}{d\tau} \int_0^\infty \int_0^\infty \int_0^\tau \xi K_{i\xi}\left(\sqrt{x^2+y^2} - y\right) \\ \times K_{i\xi}\left(\sqrt{x^2+y^2} + y\right) [\mathcal{G}f](x,y)\, d\xi \frac{dx\,dy}{x}, \end{aligned} \tag{3.5}$$

$$[\mathcal{G}f](x,y) = \frac{2}{\sqrt{\pi}}\frac{\partial}{\partial x \partial y}\int_0^\infty \int_0^x \int_0^y \tau K_{i\tau}\left(\sqrt{u^2+v^2}-v\right) \times K_{i\tau}\left(\sqrt{u^2+v^2}+v\right) f(\tau)\, du\, dv\, d\tau. \tag{3.6}$$

Finally, for all $f_1, f_2 \in L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh 2\pi\tau}\right)$ *and the corresponding* $\mathcal{G}f_1, \mathcal{G}f_2 \in L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; \frac{dx\, dy}{x}\right)$ *the Parseval identity* (1.6) *holds. In particular for* $f_1 \equiv f_2$ *it takes the form* (1.7).

Sketch of the proof. First we estimate the following double integral

$$I_N(\gamma) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_0^\infty \left|[\mathcal{G}f_N]^{\mathcal{M}}(\gamma+it,y)\right|^2 dy\, dt. \tag{3.7}$$

Making use of representation (3.2) we substitute it in (3.7). This leads to the iterated integral in the form

$$I_N(\gamma) = \frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{[KLf_N]^{\mathcal{M}}(\gamma+it)}{\Gamma(\frac{1+\gamma+it}{2})}\right|^2 dt \int_0^\infty y^\gamma K_{\frac{\gamma+it}{2}}(2y) K_{\frac{\gamma-it}{2}}(2y)\, dy. \tag{3.8}$$

But the integral with respect to y in (3.8) can be calculated in view of the formula (2.16.33.2) in [4]. Thus, we insert in (3.8) the corresponding result and employ the Mellin–Parseval identity (2.3). Consequently, the integral I_N can be written as

$$I_N(\gamma) = \frac{\sqrt{\pi}\Gamma(\gamma+\frac{1}{2})}{8\Gamma(\gamma+1)}\int_0^\infty \left|[KLf_N](x)\right|^2 x^{2\gamma-1} dx. \tag{3.9}$$

However on the other hand, we represent I_N in terms of the square of norm of $[\mathcal{G}f](x,y)$ in the space $L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{2\gamma-1}dx\, dy\right)$. Indeed, by the use of (2.3) we have

$$I_N(\gamma) = \int_0^\infty \int_0^\infty \left|[\mathcal{G}f_N](x,y)\right|^2 x^{2\gamma-1} dx\, dy. \tag{3.10}$$

Combining (3.9) and (3.10) we arrive at the equality

$$\int_0^\infty \int_0^\infty \left|[\mathcal{G}f_N](x,y)\right|^2 x^{2\gamma-1} dx\, dy = \frac{\sqrt{\pi}\Gamma(\gamma+\frac{1}{2})}{8\Gamma(\gamma+1)}\int_0^\infty \left|[KLf_N](x)\right|^2 x^{2\gamma-1} dx. \tag{3.11}$$

So the Parseval identity (1.7) for the L_2-sequence $\{f_N\}$ will follow from (3.11) by formal substitution $\gamma = 0$ and using the equality (2.5). We note that (1.6) is now an immediate consequence of the equality (1.7) and the parallelogram identity.

Further, we have

$$\int_0^\infty \int_0^\infty \left|[\mathcal{G}f_N](x,y) - [\mathcal{G}f_M](x,y)\right|^2 \frac{dx\,dy}{x} = \frac{\pi^3}{4}\left(\int_{\frac{1}{M}}^{\frac{1}{N}} + \int_N^M\right) \frac{\tau}{\sinh(2\pi\tau)} |f(\tau)|^2 d\tau. \tag{3.12}$$

Since the right-hand side of (3.14) tends to zero as $M \to \infty, N \to \infty$, so it does the left-hand side. That is, $[\mathcal{G}f_N](x,y)$ converges in mean to a function, $[\mathcal{G}f](x,y)$ say, of the class $L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{-1}dx\,dy\right)$, which satisfies the Parseval identity (1.7).

On the other hand, for two functions f, θ we have (see (1.6)) that

$$\int_0^\infty \int_0^\infty [\mathcal{G}; \tau f(\tau)](x,y)\overline{[\mathcal{G}; \tau\theta(\tau)](x,y)}\frac{dxdy}{x} = \frac{\pi^3}{4}\int_0^\infty \frac{\xi}{\sinh 2\pi\xi} f(\xi)\overline{\theta(\xi)}d\xi. \tag{3.13}$$

Putting

$$\theta(\xi) \equiv \theta_\tau(\xi) = \begin{cases} 1, & \text{if } \xi \in [0,\tau] \\ 0, & \text{if } \xi > \tau, \end{cases}$$

and differentiating through with respect to τ in the equality (3.13) we obtain for almost all $\tau \in \mathbb{R}_+$ that

$$f(\tau) = \frac{8\sinh 2\pi\tau}{\pi^3\sqrt{\pi}\tau}\frac{d}{d\tau}\int_0^\infty \int_0^\infty \int_0^\tau \xi K_{i\xi}\left(\sqrt{x^2+y^2} - y\right) \times K_{i\xi}\left(\sqrt{x^2+y^2} + y\right)[\mathcal{G}; \tau f(\tau)](x,y)\, d\xi \frac{dx\,dy}{x}. \tag{3.14}$$

Now, analogously we set $\mathcal{G}_N(x,y) = [\mathcal{G}; \tau f(\tau)](x,y)$ and it is equal to zero outside of the square $[\frac{1}{N}, N] \times [\frac{1}{N}, N]$. Hence evidently it converges to $[\mathcal{G}; \tau f(\tau)](x,y)$ by the norm of the space $L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; \frac{dx\,dy}{x}\right)$ (cf. in (1.4) with $p = 2, \Omega = \mathbb{R}_+ \times \mathbb{R}_+$, with respect to the measure $x^{-1}dx\,dy$). Moreover, substituting $\mathcal{G}_N(x,y)$ into (3.14) we may differentiate through the integral sign by virtue of the uniform convergence. Thus we arrive at (3.4), and via Parseval identity (1.7) its left-hand side converges to the limit function $\psi(\tau)$. It is proved [10] that $\psi(\tau) = f(\tau)$ almost for all $\tau \in \mathbb{R}_+$.

Now we show that apart from sets of measure zero, there is a one-to-one correspondence between $[\mathcal{G}f](x,y)$ and $f(\tau)$. Indeed, for the sequence $f_N(\tau)$ integral (3.3) has a finite range of integration and converges absolutely and uniformly by $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+, 0 < r = \sqrt{x^2+y^2} \le R$. Therefore, we integrate by x and y in (3.3), and making the interchange of the order of integration we arrive at the equality

$$\int_0^x\int_0^y [\mathcal{G}f_N](u,v)\,du\,dv = \frac{2}{\sqrt{\pi}}\int_0^\infty \tau \int_0^x\int_0^y K_{i\tau}\left(\sqrt{u^2+v^2}-v\right) \times K_{i\tau}\left(\sqrt{u^2+v^2}+v\right) f_N(\tau)\,du\,dv\,d\tau. \tag{3.15}$$

If $N \to \infty$, then for each fixed $x > 0, y > 0$ the left-hand side of (3.15) tends to the value

$$\int_0^x\int_0^y [\mathcal{G}f](u,v)\,du\,dv$$

as a bounded linear functional. Moreover, since (see [10])

$$\int_0^x\int_0^y K_{i\tau}\left(\sqrt{u^2+v^2}-v\right) K_{i\tau}\left(\sqrt{u^2+v^2}+v\right) du\,dv \in L_2(\mathbb{R}_+;\tau\sinh 2\pi\tau\,d\tau),$$

then as it is easily seen via the Schwarz inequality the integral in the right-hand side of (3.15) converges absolutely. Moreover, making $N \to \infty$ for almost all positive x and y after differentiation of both sides in (3.15) we obtain formula (3.6). So this correspondence is unique in L_2-sense. Similarly, since for each $\tau > 0$

$$\mathcal{K}_\tau(x,y) = \frac{2}{\sqrt{\pi}}\int_0^\tau \xi K_{i\xi}\left(\sqrt{x^2+y^2}-y\right) K_{i\xi}\left(\sqrt{x^2+y^2}+y\right) d\xi \in L_2\left(\mathbb{R}_+\times\mathbb{R}_+;\tfrac{dx\,dy}{x}\right),$$

then in the same manner we observe that integral (3.5) converges absolutely and represents an inversion formula for the transformation (3.6). $\square$

4. The Hausdorff-Young type theorem

In this section we study transformation (1.1) as an integral operator (1.5). We will prove an analog of the boundedness Hausdorff-Young theorem [2] for this transformation and we will estimate its norm for general p by using the Riesz-Thorin interpolation theorem.

First we begin to prove the boundedness of the operator (1.1) for the case $p = q = 2,\ \nu > 0$. We employ the following inequality from [9], which estimates the square of norms for the Kontorovich-Lebedev transformation (2.4)

$$\int_0^\infty x^{2\nu-1}\left|[KLf](x)\right|^2 dx \le \frac{\pi^{\frac{3}{2}}4^{1-2\nu}}{\Gamma(\frac{1}{2}+2\nu)}\int_0^\infty \tau^2\left|\Gamma(2(\nu+i\tau))\right|^2|f(\tau)|^2 d\tau. \tag{4.1}$$

Assuming that $f \in L_2(\mathbb{R}_+;d\tau)$ we apply the reduction formula for gamma-functions [7, formula (1.23)] $\Gamma(z+1) = z\Gamma(z)$ and the elementary inequality

[7, formula (1.26)] $|\Gamma(z)| \le \Gamma(\mathrm{Re} z)$ to estimate the integral in the right-hand side of (4.1) as follows:

$$\int_0^\infty \tau^2 \left|\Gamma(2(\nu+i\tau))\right|^2 |f(\tau)|^2 d\tau \le \int_0^\infty \tau^2 \frac{\left|\Gamma(1+2(\nu+i\tau))\right|^2}{4(\nu^2+\tau^2)} |f(\tau)|^2 d\tau$$
$$\le \frac{1}{4}\left[\Gamma(1+2\nu)\right]^2 \int_0^\infty |f(\tau)|^2 d\tau.$$

Thus combining with (4.1) we obtain

$$\int_0^\infty x^{2\nu-1} \left|[KLf](x)\right|^2 dx \le \frac{\pi^{\frac{3}{2}} 4^{-2\nu} \left[\Gamma(1+2\nu)\right]^2}{\Gamma(\frac{1}{2}+2\nu)} \int_0^\infty |f(\tau)|^2 d\tau.$$

Meantime taking into account equality (3.11) we immediately arrive at the estimate

$$\int_0^\infty\int_0^\infty \left|[\mathcal{G}f](x,y)\right|^2 x^{2\nu-1} dx\, dy = \frac{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}{8\Gamma(\nu+1)} \int_0^\infty x^{2\nu-1} \left|[KLf](x)\right|^2 dx$$
$$\le \frac{\pi^2 2^{-4\nu-3} \Gamma(\frac{1}{2}+\nu) \left[\Gamma(1+2\nu)\right]^2}{\Gamma(\frac{1}{2}+2\nu)\Gamma(1+\nu)} \int_0^\infty |f(\tau)|^2 d\tau.$$

Finally, invoking the duplication formula for gamma-function (cf. in [7, formula (1.30)] $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(\frac{1}{2}+z)$ we find

$$\int_0^\infty\int_0^\infty \left|[\mathcal{G}f](x,y)\right|^2 x^{2\nu-1} dx\, dy \le \frac{\pi \left[\Gamma(\frac{1}{2}+\nu)\right]^3 \Gamma(1+\nu)}{8\Gamma(\frac{1}{2}+2\nu)} \int_0^\infty |f(\tau)|^2 d\tau. \quad (4.2)$$

Therefore, transformation (1.1) is a bounded operator (1.5) of type $(2,2)$ and from (4.2) (see (1.4)) we obtain

$$\|\mathcal{G}f\|_{L_2(\mathbb{R}_+\times\mathbb{R}_+; x^{2\nu-1}dx\, dy)} \le \sqrt{\frac{\pi}{8}} \left[\Gamma(\tfrac{1}{2}+\nu)\right]^{\frac{3}{2}} \left[\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+2\nu)}\right]^{\frac{1}{2}} \|f\|_{L_2(\mathbb{R}_+;d\tau)}. \quad (4.3)$$

Inequality (4.3) implies that the norm $\|\mathcal{G}\|_{2,2}$ of the operator $\mathcal{G}$ in this case is such that

$$\|\mathcal{G}\|_{2,2} \le \sqrt{\frac{\pi}{8}} \left[\Gamma(\tfrac{1}{2}+\nu)\right]^{\frac{3}{2}} \left[\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+2\nu)}\right]^{\frac{1}{2}}.$$

Now we prove that transformation (1.1) is of type $(1,\infty)$. We have

Theorem 2. *The index transformation* (1.1) *is a bounded operator*

$$\mathcal{G} : L_1(\mathbb{R}_+; d\tau) \leftrightarrow L_{\infty,\nu}(\mathbb{R}_+\times\mathbb{R}_+), \quad \nu > 0,$$

with the norm $\|\mathcal{G}\|_{1,\infty} \leq \frac{2}{\sqrt{\pi}} \sup_{x>0} |x^\nu K_0(2x)|$, *i.e., for all* $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$|x^\nu [\mathcal{G}f](x,y)| \leq \frac{2}{\sqrt{\pi}} \sup_{x>0} |x^\nu K_0(2x)| \, \|f\|_1, \quad \nu > 0,$$

where the space $L_{\infty,\nu}(\mathbb{R}_+ \times \mathbb{R}_+)$ *is normed by*

$$\|f\|_{L_{\infty,\nu}(\mathbb{R}_+\times\mathbb{R}_+)} = \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+} |x^\nu f(x,y)|.$$

Proof. By taking integral representation (2.9) for the product of the modified Bessel function we substitute it in (1.1) and change the order of integration via Fubini's theorem and the absolute convergence of the corresponding iterated integral. Indeed, appealing to the differentiation and asymptotic properties for the modified Bessel functions [1, Chapter 9] we find for $x > 0$

$$\frac{x^2}{2} \int_0^\infty |f(\tau)| \int_0^\infty \frac{K_1\left(2\sqrt{y^2 + x^2\cosh^2(\frac{t}{2})}\right)}{\sqrt{y^2 + x^2\cosh^2(\frac{t}{2})}} \sinh t |\sin t\tau| \, dt \, d\tau$$

$$\leq \frac{x^2}{2} \int_0^\infty |f(\tau)| \int_0^\infty \frac{K_1\left(2\sqrt{y^2 + x^2\cosh^2(\frac{t}{2})}\right)}{\sqrt{y^2 + x^2\cosh^2(\frac{t}{2})}} \sinh t \, dt \, d\tau$$

$$= \int_0^\infty |f(\tau)| \int_0^\infty \frac{d}{dt}\left[-K_0\left(2\sqrt{y^2 + x^2\cosh^2(\tfrac{t}{2})}\right)\right] dt \, d\tau$$

$$= K_0\left(2\sqrt{y^2 + x^2}\right) \|f\|_1$$

$$\leq K_0(2x) \|f\|_1 < \infty.$$

So, invoking (2.8) we can write transformation (1.1) in the form

$$[\mathcal{G}f](x,y) = \frac{x^2}{\sqrt{2}} \int_0^\infty \frac{K_1\left(2\sqrt{y^2 + x^2\cosh^2(\frac{t}{2})}\right)}{\sqrt{y^2 + x^2\cosh^2(\frac{t}{2})}} \sinh t (F_s f)(t) \, dt.$$

Consequently,

$$|x^\nu [\mathcal{G}f](x,y)| \leq \sqrt{2} \sup_{t>0} |(F_s f)(t)| \, |x^\nu K_0(2x)| \, \|f\|_1 = \frac{2}{\sqrt{\pi}} \sup_{x>0} |x^\nu K_0(2x)| \, \|f\|_1,$$

which completes the proof of Theorem 2. □

Now we are ready to prove an analog of the Hausdorff–Young theorem (cf. [2]) for transformation (1.1).

Theorem 3. *The index transformation* (1.1) *is a bounded operator*

$$\mathcal{G} : L_p(\mathbb{R}_+; d\tau) \leftrightarrow L_q\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{\nu q-1} dx\, dy\right), \quad \nu > 0,$$

where $1 \le p \le 2$ *and* $p^{-1} + q^{-1} = 1$. *It satisfies the norm inequality*

$$\begin{aligned} &\|\mathcal{G}f\|_{L_q(\mathbb{R}_+\times\mathbb{R}_+;x^{\nu q-1}dx\,dy)} \\ &\quad \le 2^{1-\frac{5}{q}} \pi^{\frac{2}{q}-\frac{1}{2}} \left[\Gamma(\tfrac{1}{2}+\nu)\right]^{\frac{3}{q}} \left[\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+2\nu)}\right]^{\frac{1}{q}} \left[\sup_{x>0} |x^\nu K_0(2x)|\right]^{1-\frac{2}{q}} \|f\|_{L_p(\mathbb{R}_+;d\tau)}, \end{aligned} \tag{4.4}$$

and therefore

$$\|\mathcal{G}\|_{p,q} \le 2^{1-\frac{5}{q}} \pi^{\frac{2}{q}-\frac{1}{2}} \left[\Gamma(\tfrac{1}{2}+\nu)\right]^{\frac{3}{q}} \left[\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+2\nu)}\right]^{\frac{1}{q}} \left[\sup_{x>0} |x^\nu K_0(2x)|\right]^{1-\frac{2}{q}}. \tag{4.5}$$

Proof. In fact, by inequality (4.3) we observe that transformation (1.1) is of type $(2,2)$. Meantime Theorem 2 states that this operator is of type $(1,\infty)$. Hence by the Riesz–Thorin convexity theorem [2] the index transformation (1.1) is an operator of type (p,q), i.e., it maps the space $L_p(\mathbb{R}_+; d\tau)$ into $L_q(\mathbb{R}_+ \times \mathbb{R}_+; x^{\nu q-1} dx\, dy)$, where $q^{-1} = \frac{\theta}{2}$, $0 \le \theta \le 1$. This means that $2 \le q \le \infty$ and we find

$$\begin{aligned} &\|\mathcal{G}f\|_{L_q(\mathbb{R}_+\times\mathbb{R}_+;x^{\nu q-1}dx\,dy)} \\ &\quad \le \left[\sqrt{\frac{\pi}{8}} \left[\Gamma(\tfrac{1}{2}+\nu)\right]^{\frac{3}{2}} \left[\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+2\nu)}\right]^{\frac{1}{2}}\right]^\theta \left[\frac{2}{\sqrt{\pi}} \sup_{x>0} |x^\nu K_0(2x)|\right]^{1-\theta} \|f\|_{L_p(\mathbb{R}_+;d\tau)} \\ &\quad = 2^{1-\frac{5}{q}} \pi^{\frac{2}{q}-\frac{1}{2}} \left[\Gamma(\tfrac{1}{2}+\nu)\right]^{\frac{3}{q}} \left[\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+2\nu)}\right]^{\frac{1}{q}} \left[\sup_{x>0} |x^\nu K_0(2x)|\right]^{1-\frac{2}{q}} \|f\|_{L_p(\mathbb{R}_+;d\tau)}. \end{aligned}$$

Thus we get inequality (4.4). This immediately implies (4.5). Theorem 3 is proved. □

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