Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 25 (2006), 249–255

Fewer Convergence Conditions for the Halley Method

J. A. Ezquerro and M. A. Hernández

Abstract. We present a new semilocal convergence result of Newton-Kantorovich type for Halley's method, where fewer convergence conditions are required than all the existing ones until now.

Keywords. Nonlinear equations in Banach spaces, Halley's method, semilocal convergence theorem, nonlinear integral equation.

Mathematics Subject Classification (2000). Primary 47H17, secondary 45G10, 65J15

1. Introduction

A large number of problems in applied mathematics and engineering are solved by finding the solutions of certain equations. The most commonly used solution methods are iterative: from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Considerable effort has been devoted to the study of iterative methods for the determination of solutions of $F(x) = 0$, where F is a nonlinear operator defined on a non-empty open convex subset Ω of a Banach space X with values in another Banach space Y.

Newton's method is the most best-known iterative method to solve $F(x)=0$, but Halley's process is possibly the second one and perhaps the most rediscovered method in the world (see [3] and [7]). Here we consider the Halley method for solving $F(x) = 0$, which is of R-order three (see [2, 8]) and is based on the algorithm

$$x_{n+1} = x_n - \left[I + \frac{1}{2} L_F(x_n) \left(I - \frac{1}{2} L_F(x_n) \right)^{-1} \right] F(x_n)^{-1} F(x_n), \quad n \geq 0,$$

where $x_0 \in \Omega$, I is the identity operator on X and $L_F(x)$ is the degree of logarithmic convexity [4], defined by $L_F(x) = F'(x)^{-1} F''(x)[F'(x)^{-1} F(x)] \in$

J. A. Ezquerro and M. A. Hernández: Department of Mathematics and Computation, University of La Rioja, Spain; jezquer@dmc.unirioja.es , mahernan@dmc.unirioja.es

$\mathcal{L}(X,X)$, where $\mathcal{L}(X,X)$ is the set of bounded linear operators from X into X, and such that

$$L_F(x)(-) = F'(x)^{-1}F''(x)(F'(x)^{-1}F(x), -),$$

provided that $F'(x_n)^{-1}$ and $\left(I - \frac{1}{2}L_F(x_n)\right)^{-1}$ exist at each step. The operators $F'(x)$ and $F''(x)$ denote the first and the second Fréchet-derivatives of the operator F.

The convergence of the Halley method has been examined extensively by several authors. Basic results concerning the convergence of the method have been published under assumptions of Newton-Kantorovich type. Safiev [6] presented a convergence theorem for the Halley iteration under the following conditions:

$$\|\Gamma_0\| \le \beta, \quad \|\Gamma_0 F(x_0)\| \le \eta, \quad \|F''(x)\| \le M, \quad \text{and} \quad \|F'''(x)\| \le N, \quad x \in \Omega,$$

where it is supposed that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y,X)$ exists at some $x_0 \in \Omega$. Since then, a large number of convergence results concerning this method and related techniques have been published, see [2] and [8], where an abundant list of references can be found.

The convergence conditions given by Safiev can be modified by replacing the strongest one $\|F'''(x)\| \le N$, $x \in \Omega$, with

$$\|F''(x) - F''(y)\| \le K\|x - y\|, \quad K \ge 0,\ x, y \in \Omega, \tag{1}$$

or the milder one

$$\|F''(x) - F''(y)\| \le L\|x - y\|^p, \quad L \ge 0,\ p \in [0,1],\ x, y \in \Omega \tag{2}$$

(see [1, 2, 4, 8]). These two last conditions mean that F'' is Lipschitz continuous in Ω and F'' is (L,p)-Hölder continuous in Ω, respectively. According to this, the number of equations that can be solved by the Halley method is limited. For instance, we cannot analyze the convergence of the Halley process to a solution of equations where sums of operators which satisfy (1) or (2) are involved, as it is shown in the following nonlinear integral equation of mixed Hammerstein type:

$$x(s) = 1 + \int_0^1 G(s,t)\left(x(t)^{\frac{5}{2}} + \tfrac{1}{5}x(t)^3\right) dt, \quad s \in [0,1], \tag{3}$$

where $x \in C[0,1]$, $s, t \in [0,1]$, and the kernel G is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t. \end{cases}$$

Here we reconsider the convergence of Halley's iteration in Banach spaces and a convergence theorem is provided by assuming only that the operator F'' is bounded in the domain Ω, where the solution x^* must exist. Obviously this new convergence theorem requires fewer convergence conditions than all the previous ones appearing in the literature.

To finish, the theoretical significance of the Halley method is used to draw conclusions about the existence and uniqueness of solution and about the region in which it is located, without finding the solution itself. This is sometimes more important than the actual knowledge of the solution (see [5]). And we apply this analysis to equation (3).

Throughout the paper we denote $\overline{B(x,r)} = \{y \in X; \|y-x\| \leq r\}$ and $B(x,r) = \{y \in X; \|y-x\| < r\}$.

2. A convergence theorem

Initially we give the conditions that the operator F and the starting point x_0 must satisfy to establish a semilocal convergence theorem for Halley's process. Conditions for the existence of a solution x^* of $F(x) = 0$ are given, along with the domains of existence and uniqueness of x^*.

Suppose that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y,X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y,X)$ is the set of bounded linear operators from Y into X.

Theorem 1. *Let $F : \Omega \subseteq X \to Y$ be a twice continuously differentiable operator on a non-empty open convex domain Ω and*

$$\|\Gamma_0 F(x_0)\| \leq \eta \quad \text{and} \quad \|\Gamma_0 F''(x)\| \leq M,\ x \in \Omega\,.$$

If both the conditions $\alpha_0 = M\eta < \frac{4-\sqrt{6}}{5}$ and $\overline{B(x_0, R\eta)} \subseteq \Omega$, where $R = \frac{2-3\alpha_0}{2(1-3\alpha_0+\alpha_0^2)}$, are satisfied, then Halley's method starting from x_0 generates a sequence $\{x_n\}$ that converges to an isolated solution $x^* \in \overline{B(x_0, R\eta)}$ of $F(x) = 0$. Moreover x^* is unique in $\Omega_0 = B(x_0, \frac{2}{M} - R\eta) \cap \Omega$.

Proof. Taking into account the hypotheses we have $\|L_F(x_0)\| \leq \alpha_0$. Moreover, by the Banach lemma, $H(x_0)^{-1}$ exists and $\|H(x_0)^{-1}\| \leq \frac{2}{2-\alpha_0}$, where $H(x) = I - \frac{1}{2}L_F(x)$. Furthermore,

$$\|x_1 - x_0\| \leq \frac{2}{2-\alpha_0}\|\Gamma_0 F(x_0)\| < R\eta$$

and consequently $x_1 \in B(x_0, R\eta)$. By the Banach lemma, since

$$\|I - \Gamma_0 F'(x_1)\| = \left\| \int_0^1 \Gamma_0 F''(x_0 + t(x_1 - x_0))\, dt\, (x_1 - x_0) \right\| \leq \frac{2\alpha_0}{2-\alpha_0} < 1,$$

Γ_1 exists and $\|\Gamma_1 F'(x_0)\| \leq \frac{2-\alpha_0}{2-3\alpha_0}$. So x_1 is well-defined. From the approximation

$$\Gamma_0 F(x_1) = -\frac{1}{2} L_F(x_0) H(x_0)^{-1} \Gamma_0 F(x_0) + \int_0^1 \Gamma_0 F''(x_0 + t(x_1 - x_0))(1-t)\,dt\,(x_1 - x_0)^2,$$

it follows $\|\Gamma_0 F(x_1)\| \leq \frac{\alpha_0(4-\alpha_0)}{(2-\alpha_0)^2}\|\Gamma_0 F(x_0)\|$ and

$$\|\Gamma_1 F(x_1)\| \leq \|\Gamma_1 F'(x_0)\|\|\Gamma_0 F(x_1)\| \leq \frac{\alpha_0(4-\alpha_0)}{(2-\alpha_0)(2-3\alpha_0)}\|\Gamma_0 F(x_0)\|.$$

Next, from $\|L_F(x_1)\| \leq \|\Gamma_1 F''(x_1)\|\|\Gamma_1 F(x_1)\| \leq \frac{\alpha_0^2(4-\alpha_0)}{(2-3\alpha_0)^2} = \alpha_1 < 1$ and the Banach lemma, we obtain that $H(x_1)^{-1}$ exists and $\|H(x_1)^{-1}\| \leq \frac{2}{2-\alpha_1}$. After that, since $\alpha_1 < \alpha_0$,

$$\|x_2 - x_1\| \leq \frac{2}{2-\alpha_1}\|\Gamma_1 F(x_1)\| \leq \frac{2\,\theta}{2-\alpha_0}\|\Gamma_0 F(x_0)\|,$$

where $\theta = \frac{\alpha_0(4-\alpha_0)}{(2-\alpha_0)(2-3\alpha_0)}$. Besides, since $\theta < 1$,

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq \frac{2\,(1+\theta)}{2-\alpha_0}\|\Gamma_0 F(x_0)\| < R\eta.$$

We can now construct the decreasing scalar sequence

$$\alpha_n = \frac{\alpha_{n-1}^2(4-\alpha_{n-1})}{(2-3\alpha_{n-1})^2}, \quad n = 1, 2, 3\ldots,$$

and following an inductive procedure we can replace x_1 by x_2, x_2 by x_3 and, in general, x_{n-1} by x_n to confirm that Γ_n exists and the next:

$$\|\Gamma_n F'(x_{n-1})\| \leq \frac{2-\alpha_{n-1}}{2-3\alpha_{n-1}}$$
$$\|\Gamma_{n-1} F(x_n)\| \leq \frac{\alpha_{n-1}(4-\alpha_{n-1})}{(2-\alpha_{n-1})^2}\|\Gamma_{n-1} F(x_{n-1})\|$$
$$\|\Gamma_n F(x_n)\| \leq \frac{\alpha_{n-1}(4-\alpha_{n-1})}{(2-\alpha_{n-1})(2-3\alpha_{n-1})}\|\Gamma_{n-1} F(x_{n-1})\| \leq \theta^n \|\Gamma_0 F(x_0)\| < \eta$$
$$\|L_F(x_n)\| \leq \left(\prod_{i=1}^n \|\Gamma_i F'(x_{i-1})\|\right)\|\Gamma_0 F''(x_n)\|\|\Gamma_n F(x_n)\| \leq \alpha_n$$
$$H(x_n)^{-1} \text{ exists and } \|H(x_n)^{-1}\| \leq \frac{2}{2-\alpha_n}$$
$$\|x_{n+1} - x_n\| \leq \frac{2}{2-\alpha_n}\|\Gamma_n F(x_n)\|$$
$$\|x_{n+1} - x_0\| \leq \frac{2}{2-\alpha_0}\,\frac{1-\theta^{n+1}}{1-\theta}\,\|\Gamma_0 F(x_0)\| < R\eta.$$

The proof of the previous items is similar to that mentioned above for the case $n = 1$.

Then we can derive that $\{x_n\}$ is a Cauchy sequence. Observe

$$\begin{aligned}\|\Gamma_n F(x_n)\| &\leq \frac{\alpha_{n-1}(4-\alpha_{n-1})}{(2-\alpha_{n-1})(2-3\alpha_{n-1})}\|\Gamma_{n-1}F(x_{n-1})\| \leq \dots \\ &\leq \left(\prod_{i=0}^{n-1}\frac{\alpha_i(4-\alpha_i)}{(2-\alpha_i)(2-3\alpha_i)}\right)\|\Gamma_0 F(x_0)\| \leq \theta^n\eta,\end{aligned}$$

and consequently

$$\|x_{n+1}-x_n\| \leq \frac{2}{2-\alpha_n}\|\Gamma_n F(x_n)\| \leq \frac{2}{2-\alpha_0}\,\theta^n\eta.$$

Hence

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \|x_{n+m}-x_{n+m-1}\| + \|x_{n+m-1}-x_{n+m-2}\| + \dots + \|x_{n+1}-x_n\| \\ &\leq \frac{2\,\theta^n\,(1-\theta^m)}{(2-\alpha_0)(1-\theta)}\eta\end{aligned}$$

so that $\{x_n\}$ is convergent and $\lim_n x_n = x^*$. If $n=0$, $\|x_m - x_0\| \leq \frac{2}{2-\alpha_0}\frac{1-\theta^m}{1-\theta}\eta < R\eta$ and $x_m \in B(x_0, R\eta)$, $m \geq 1$.

Now, by letting $n \to \infty$ in $\|\Gamma_n F(x_n)\| \leq \theta^n \|\Gamma_0 F(x_0)\|$, it follows that $\|\Gamma_n F(x_n)\| \to 0$. Since $\|\Gamma_0 F(x_n)\| \leq \|\Gamma_0 F'(x_n)\|\|\Gamma_n F(x_n)\|$ and the sequence $\{\|\Gamma_0 F'(x_n)\|\}$ is bounded, we obtain $\|F(x_n)\| \to 0$ and $F(x^*) = 0$ by the continuity of F.

To show the uniqueness of x^*, we suppose that z^* is another solution of $F(x) = 0$ in $\Omega_0 = B(x_0, \frac{2}{M} - R\eta) \cap \Omega$. From the approximation

$$0 = \Gamma_0(F(z^*) - F(x^*)) = \int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\,dt\,(z^* - x^*),$$

and the fact that the operator $P = \int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\,dt$ is invertible, the equality $z^* = x^*$ follows. Observe that P is invertible since

$$\begin{aligned}&\int_0^1 \|\Gamma_0(F'(x^* + t(z^* - x^*)) - F'(x_0))\|\,dt \\ &\qquad \leq M\int_0^1 \|x^* + t(z^* - x^*) - x_0\|\,dt \\ &\qquad \leq M\int_0^1 \big((1-t)\|x^* - x_0\| + t\|z^* - x_0\|\big)\,dt \\ &\qquad < \frac{M}{2}\left(R\eta + \frac{2}{M} - R\eta\right) = 1,\end{aligned}$$

and the Banach lemma for the operator P holds. □

3. Application

We now illustrate the previous study with an application to nonlinear integral equation of mixed Hammerstein type (3), where the domains of existence and uniqueness are provided.

Solving (3) is equivalent to solve $F(x) = 0$, where $F : \Omega \subseteq C[0,1] \to C[0,1]$,

$$[F(x)](s) = x(s) - 1 - \int_0^1 G(s,t)\left(x(t)^{\frac{5}{2}} + \tfrac{1}{5}x(t)^3\right) dt, \quad s \in [0,1], \tag{4}$$

and Ω is a suitable non-empty open convex domain. Observe that the first and the second Fréchet derivatives of the previous operator are

$$[F'(x)y](s) = y(s) - \int_0^1 G(s,t)\left(\tfrac{5}{2}x(t)^{\frac{3}{2}} + \tfrac{3}{5}x(t)^2\right) y(t)\, dt$$

and

$$[F''(x)yz](s) = -\int_0^1 G(s,t)\left(\tfrac{15}{4}x(t)^{\frac{1}{2}} + \tfrac{6}{5}x(t)\right) z(t)y(t)\, dt. \tag{5}$$

Notice that F'' does not satisfy (1) neither (2), but the conditions of Theorem 1 are, so that a solution of equation (3) can be approximated by the Halley iteration.

Using the max-norm and taking into account that a solution x^* of (3) in $C[0,1]$ must satisfy

$$\|x^*\| - \frac{1}{8}\|x^*\|^{\frac{5}{2}} - \frac{1}{40}\|x^*\|^3 - 1 \le 0,$$

i.e., $\|x^*\| \le \rho_1 = 1.28982\ldots$ and $\|x^*\| \ge \rho_2 = 2.28537\ldots$, where ρ_1 and ρ_2 are the positive roots of the real equation $z - \frac{1}{8}z^{\frac{5}{2}} - \frac{1}{40}z^3 - 1 = 0$. Taking now into account (4), it is needed that $x^* \ge 0$, then if we look for a solution such that $\|x^*\| < \rho_1$, we can consider for example $\Omega = B(1,1) \subseteq C[0,1]$ as a non-empty open convex domain.

Choosing $x_0(s) = 1$, we have $\|I - F'(x_0)\| \le \frac{31}{80} < 1$, Γ_0 is defined, $\|\Gamma_0\| \le \frac{80}{49}$ and $\|\Gamma_0 F(x_0)\| \le \frac{12}{49} = \eta$. From (5), it follows

$$\|\Gamma_0 F''(x)\| \le 1.24838\ldots = M \quad \text{and} \quad \alpha_0 = 0.305726\ldots < \frac{4-\sqrt{6}}{5}.$$

Moreover, $B(x_0, R\eta) = B(1, 0.752111\ldots) \subseteq B(1,1) = \Omega$.

Therefore, every condition of Theorem 1 holds and equation (3) has then a solution x^* in the domain $\{u \in C[0,1];\ \|u-1\| \le 0.752111\ldots\}$, which is unique in $\{u \in C[0,1];\ \|u-1\| < 0.849966\ldots\}$.

Acknowledgement. Preparation of this paper was partly supported by the Ministry of Education and Science (MTM 2005-03091), the University of La Rioja (API-04/13) and the Government of La Rioja (ACPI 2003/2004).

References

[1] Argyros, I. K., The Halley method in Banach spaces and the Ptak error estimates. *Rev. Acad. Cienc. Zaragoza* 52 (1997)(2), 31 – 41.

[2] Chen, D. and Qian Q. S., A note on the Halley method in Banach spaces. *Appl. Math. Comput.* 58 (1993), 215 – 224.

[3] Ezquerro, J. A., Gutiérrez, J. M., Hernández, M. A. and Salanova, M. A., El método de Halley: posiblemente, el método más redescubierto del mundo. In: *Margarita mathematica* (eds.: L. Español and J. L. Varona). Logroño: University of La Rioja 2001, pp. 205 – 220.

[4] Hernández, M. A. and Salanova, M. A., Indices of convexity and concavity: Application to Halley method. *Appl. Math. Comput.* 103 (1999), 27 – 49.

[5] Kantorovich, L. V. and Akilov, G. P., *Functional analysis.* Oxford: Pergamon Press 1982.

[6] Safiev, R. A., On some iterative processes (in Russian). *Zh. Vychisl. Mat. i Mat. Fiz.* 4 (1964), 139 – 143. (Translated into English by L.B. Rall as MRC Technical Summary Report, No. 649, Univ. Wisconsin-Madison, 1966.)

[7] Traub, J. F., *Iterative Methods for the Solution of Equations.* Englewood Cliffs (NJ): Prentice Hall 1964.

[8] Yamamoto, T., On the Method of Tangent Hyperbolas in Banach Spaces. *J. Comput. Appl. Math.* 21 (1988), 75 – 86.

Received July 20, 2004; revised April 7, 2005