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Asymptotic and Pseudo Almost Periodicity of the Convolution Operator and Applications to Differential and Integral Equations

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Abstract. We examine conditions which do ensure the asymptotic almost periodicity (respectively, pseudo almost periodicity) of the convolution function $f * h$ of f with h whenever f is asymptotically almost periodic (respectively, pseudo almost periodic) and h is a (Lebesgue) measurable function satisfying some additional assumptions. Next we make extensive use of those results to investigate on the asymptotically almost periodic (respectively, pseudo almost periodic) solutions to some differential, functional, and integral equations.

Keywords. Almost periodic function, asymptotically almost periodic function, Banach fixed-point principle, convolution operator, differential equation, integral equation, functional equation, pseudo almost periodic function, Zima's fixed-point theorem

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1. Introduction

Given two functions $f, h : \mathbb{R} \mapsto \mathbb{R}$, the convolution function, if it exists, of f with h denoted $f * h$ is defined by

$$(f * h)(t) := \int_{-\infty}^{+\infty} f(\sigma)h(t-\sigma)d\sigma, \quad \forall t \in \mathbb{R}. \tag{1}$$

Several properties of the convolution operation $*$ can be found in most good books in functional analysis. Among others, setting $u = t - \sigma$ in Eq. (1) it is routine to show that the convolution operation is commutative, i.e., $f*h = h*f$.

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Recall that the convolution operation combined with the Fourier or Laplace transforms remains a powerful tool which plays an important role in several fields such as distribution theory, the solvability of some differential equations, functional-differential equations, integral equations, and partial differential equations, data processing, and others.

In [5], it was shown that if $f : \mathbb{R} \mapsto \mathbb{R}$ is almost automorphic and if $g : \mathbb{R} \mapsto \mathbb{R}$ is a (Lebesgue) integrable function, then the convolution function $f * g : \mathbb{R} \mapsto \mathbb{R}$ of f with g is also almost automorphic. (For the definition of the concept of the almost automorphy and related applications, we refer the reader to [11] – [15], [18, 19, 20] and the references therein.)

In this paper we prove similar results for both asymptotically almost periodic and pseudo almost periodic functions. (properties of asymptotically almost periodic and pseudo almost periodic functions can be found in [1], [2], [3], [10], [22], [23], and [24] and the references therein.) More generally, both the asymptotic almost periodicity and pseudo almost periodicity of the convolution operator $\kappa_f : h \mapsto f * h$ is studied whenever f is asymptotically almost periodic (respectively, pseudo almost periodic) and h is a (Lebesgue) measurable function satisfying some additional assumptions. Namely, if f is asymptotically almost periodic (respectively, pseudo almost periodic), we shall examine the following point: under which conditions is the convolution operator $\kappa_f : h \mapsto f * h$ asymptotically almost periodic (respectively, pseudo almost periodic)?

Let $LM(\mathbb{R})$ denote the collection of Lebesgue measurable functions defined from $\mathbb{R}$ the set of real numbers into itself. If f is asymptotically almost periodic (respectively, pseudo almost periodic), one sets

$$\mathbb{A}_f := \{h \in LM(\mathbb{R}) : f * h \text{ is asymptotically almost periodic}\}$$
$$\mathbb{P}_f := \{h \in LM(\mathbb{R}) : f * h \text{ is pseudo almost periodic}\}.$$

As mentioned above, our goal is to describe some of the functions of both $\mathbb{A}_f$ and $\mathbb{P}_f$ for a fixed function f in the collection of asymptotically almost periodic (respectively, pseudo almost periodic) functions. In particular, it will be shown that each function

$$h \in L^1(\mathbb{R}) \tag{2}$$

is also an element of both $\mathbb{A}_f$ and $\mathbb{P}_g$ whenever f is asymptotically almost periodic and g is pseudo almost periodic (Theorem 3.1 and Theorem 3.3). Consequently, $L^1(\mathbb{R}) \subset \mathbb{A}_f \cap \mathbb{P}_g$.

As an application, we shall find conditions which do ensure the asymptotic almost periodicity (respectively, pseudo almost periodicity) of the function defined by

$$F(t) = \int_{-\infty}^{t} K(t, \sigma) f(\sigma) d\sigma, \quad t \in \mathbb{R}, \tag{3}$$

where $K(t, \sigma) = a(t - \sigma)$, for some $a \in LM(\mathbb{R})$, f being asymptotically almost periodic (respectively, pseudo almost periodic). In Section 4, we make extensive use of our results to discuss the existence of asymptotically almost periodic (respectively, pseudo almost periodic) solutions to some differential, functional, and integral equations through both the Banach and Zima's fixed-point theorems.

Let us recall some definitions and notations that we shall use in the sequel.

2. Asymptotically and pseudo almost periodic functions

Let $BC(\mathbb{R})$ denote the collection of bounded continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$. It is well-known that $BC(\mathbb{R})$ is a Banach space when it is equipped with the sup norm defined by $\|f\|_\infty := \sup_{t\in\mathbb{R}} |f(t)|$ for each $f \in BC(\mathbb{R})$. Let $f \in BC(\mathbb{R})$. Define the linear shift operator σ_τ for some $\tau \in \mathbb{R}$ by $(\sigma_\tau f)(t) := f(t + \tau)$ for each $t \in \mathbb{R}$.

Similarly, $BC(\mathbb{R} \times \Omega)$ where $\Omega \subset \mathbb{R}$ is an open subset denotes the collection of bounded continuous functions $F : \mathbb{R} \times \Omega \mapsto \mathbb{R}$. If $F \in B(\mathbb{R} \times \Omega)$, one defines the function $\sigma_\tau F(\cdot, x)$ for each $x \in \Omega$ by $\sigma_\tau F(t, x) := F(t + \tau, x)$ for each $t \in \mathbb{R}$.

Definition 2.1. A function $f \in BC(\mathbb{R})$ is called *almost periodic* if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the property $\|\sigma_\tau f - f\|_\infty < \varepsilon$.

The number τ above is called an *ε-translation number* of f, and the collection of almost periodic functions will be denoted by $AP(\mathbb{R})$. Similarly,

Definition 2.2. A function $F \in BC(\mathbb{R} \times \Omega)$ is called *almost periodic* in $t \in \mathbb{R}$ *uniformly* in any $K \subset \Omega$ a bounded subset if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains a number τ with the following property: $\|\sigma_\tau F(\cdot, x) - F(\cdot, x)\|_\infty < \varepsilon$ for each $x \in K$.

Here again, the number τ above is called an ε-translation number of F and the class of those functions will be denoted $AP(\mathbb{R} \times \Omega)$. More details on the properties of elements of the class $AP(\mathbb{R})$ (respectively, $AP(\mathbb{R} \times \Omega)$) can be found in the literature, especially in [9, 19]. Throughout the paper, we set

$$AP_0(\mathbb{R}) = \left\{ f \in BC(\mathbb{R}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(s)| ds = 0 \right\}$$

$$AP_0(\mathbb{R} \times \Omega) = \left\{ F \in BC(\mathbb{R} \times \Omega) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(t, s)| dt = 0 \right\}$$

uniformly in $s \in \Omega$.

Definition 2.3. A function $f \in BC(\mathbb{R})$ is called *pseudo almost periodic* if it can be expressed as $f = h + \phi$, where $h \in AP(\mathbb{R})$ and $\phi \in AP_0(\mathbb{R})$. The collection of pseudo almost periodic functions will be denoted by $PAP(\mathbb{R})$.

Let us mention that the functions h and ϕ defined in Definition 2.3 are respectively called the *almost periodic* and the *ergodic perturbation components* of f. Moreover, the decomposition in Definition 2.3 is unique, see, e.g., [22], [23], and [24].

We now equip $PAP(\mathbb{R})$ the collection of pseudo almost periodic functions on $\mathbb{R}$ with the sup norm. It is well-known that $(PAP(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space, see [17] for details. Similarly,

Definition 2.4. A function $F \in BC(\mathbb{R} \times \Omega)$ is called *pseudo almost periodic* in $t \in \mathbb{R}$ *uniformly* in $s \in \Omega$ if it can be expressed as $F = H + \Phi$, where $H \in AP(\mathbb{R} \times \Omega)$ and $\Phi \in AP_0(\mathbb{R} \times \Omega)$. The collection of such functions will be denoted by $PAP(\mathbb{R} \times \Omega)$.

We now define asymptotically almost periodic functions. However details on those functions can be found in [19, 6], or [16]. For our considerations concerning the convolution operator it is more convenient to reformulate [19, Definition 2.5.1] or [6, Definition 5.1] in the following way:

Definition 2.5. A function $f \in BC(\mathbb{R})$ is called *asymptotically almost periodic* if it can be decomposed as $f = h + \phi$, where $h \in AP(\mathbb{R})$ and $\phi \in BC(\mathbb{R})$ with $\lim_{t\to+\infty} |\phi(t)| = \lim_{t\to-\infty} |\phi(t)| = 0$. In this event, h and ϕ are respectively called the *principal and corrective terms* of the function f. The class of such functions will be denoted by $AAP(\mathbb{R})$.

As in the proof of [19, Theorem 2.5.4] or in the proof of [6, Theorem 5.3] one can show the following:

Theorem 2.6. *The decomposition of an asymptotically almost periodic function is unique.*

As for $PAP(\mathbb{R})$, one equips $AAP(\mathbb{R})$ the collection of asymptotically almost periodic functions on $\mathbb{R}$ with the sup norm on $\mathbb{R}$. As in [16], it can be readily shown that $(AAP(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

3. Asymptotic and pseudo almost periodicity of the convolution function

Theorem 3.1. *Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *be a pseudo almost periodic function and let* $g \in L^1(\mathbb{R})$. *Then the convolution function* $f * g \in PAP(\mathbb{R})$, *i.e.,* $g \in \mathbb{P}_f$.

Proof. Since f is continuous and $g \in L^1(\mathbb{R})$, it is not hard to see that the function $t \mapsto (f * g)(t)$ is continuous. Moreover, $|(f * g)(t)| \leq \|f\|_\infty \|g\|_1$ for each $t \in \mathbb{R}$, where $\|g\|_1$ is the L^1-norm of g, and therefore, $f * g \in BC(\mathbb{R})$.

It remains to prove that $f * g$ is pseudo almost periodic. First, notice that when $g \equiv 0$ there is nothing to prove. From now on, we suppose $g \not\equiv 0$.

Since f is pseudo almost periodic, there exist $h \in AP(\mathbb{R})$ and $\phi \in AP_0(\mathbb{R})$ such that $f = h + \phi$, and hence $f * g = h * g + \phi * g$. To complete the proof we first show that $h * g \in AP(\mathbb{R})$, and next that $\phi * g \in AP_0(\mathbb{R})$.

Clearly, $h * g \in BC(\mathbb{R})$. Now since $h \in AP(\mathbb{R})$, for every $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that for all $\delta \in \mathbb{R}$ there exists $\tau \in [\delta, \delta + l_\varepsilon]$ with

$$|h(\sigma + \tau) - h(\sigma)| \leq \frac{\varepsilon}{\|g\|_1} \quad \text{for each} \quad \sigma \in \mathbb{R}.$$

In particular, the following holds:

$$|h(t - s + \tau) - h(t - s)| \leq \frac{\varepsilon}{\|g\|_1} \quad \text{for each } \sigma = t - s \in \mathbb{R}. \tag{4}$$

Since $(h * g)(t + \tau) - (h * g)(t) = \int_{-\infty}^{+\infty} \{h(t - \sigma + \tau) - h(t - \sigma)\} g(\sigma) d\sigma$ for each $t \in \mathbb{R}$, using Eq. (4) and the the assumption $g \in L^1(\mathbb{R})$ it readily follows that $\|\sigma_\tau(h * g) - (h * g)\|_\infty \leq \varepsilon$, and hence $h * g \in AP(\mathbb{R})$.

It remains to show that $\phi * g \in AP_0(\mathbb{R})$. To this end, note that since $\phi \in AP_0(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, we have that $\phi * g \in BC(\mathbb{R})$. By assumption, $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)| \, dt = 0$. Now setting

$$J(T) := \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{+\infty} |\phi(t - s)| \; |g(s)| \, ds \; dt$$

it follows that

$$\begin{aligned}
\frac{1}{2T} \int_{-T}^{T} |(\phi * g)(t)| \; dt &\leq J(T) \\
&= \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{+\infty} |\phi(t - s)| \, |g(s)| \, ds \, dt \\
&= \int_{-\infty}^{+\infty} |g(s)| \left(\frac{1}{2T} \int_{-T}^{T} |\phi(t - s)| \, dt \right) ds \\
&= \int_{-\infty}^{+\infty} |g(s)| \left(\frac{1}{2T} \int_{-T-s}^{T-s} |\phi(r)| \, dr \right) ds \\
&= \int_{-\infty}^{+\infty} |g(s)| \, \phi_T(s) \, ds,
\end{aligned}$$

where $\phi_T(u) = \frac{1}{2T}\int_{-T-u}^{T-u} |\phi(r)|\,dr$. Clearly, $\phi_T(u) \mapsto 0$ as $T \mapsto \infty$. Next, since ϕ_T is bounded and $g \in L^1(\mathbb{R})$, using the Lebesgue dominated convergence theorem, it follows that $\lim_{T\to\infty} J(T) = 0$.

In summary, $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |(\phi * g)(t)|\,dt = 0$, and hence $\phi * g \in AP_0(\mathbb{R})$. □

Example 3.2. Setting $f(t) = \cos^{10}(4t)\sin^4(5t)$, and $g(t) = \frac{e^{-t^2}}{4+t^2}$ it is clear that

$$\xi(t) = \int_{-\infty}^{+\infty} \frac{\cos^{10}(4\sigma)\sin^4(5\sigma)\ e^{-(t-\sigma)^2}}{4+(t-\sigma)^2}\,d\sigma$$

is pseudo almost periodic.

Similarly, for the asymptotic almost periodicity of the convolution function we have the following:

Theorem 3.3. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be an asymptotically almost periodic function and let $g \in L^1(\mathbb{R})$. Then the convolution function $f * g \in AAP(\mathbb{R})$, i.e., $g \in \mathbb{A}_f$.*

Proof. Let $f \in AAP(\mathbb{R})$ and let h and ϕ be the principal and the corrective terms of the function f, respectively. Obviously $f * g = h * g + \phi * g$. Moreover, as in the proof of Theorem 3.1 we infer that $f * g \in BC(\mathbb{R})$, and $h * g \in AP(\mathbb{R})$. Furthermore, $\phi * g \in BC(\mathbb{R})$. Since by assumption, $\lim_{t\to+\infty} |\phi(t)| = \lim_{t\to-\infty} |\phi(t)| = 0$ and that $g \in L^1(\mathbb{R})$, then,

$$\lim_{t\to+\infty} \left| \int_{-\infty}^{+\infty} \phi(t-\sigma)g(\sigma)\,d\sigma \right| = \lim_{t\to-\infty} \left| \int_{-\infty}^{+\infty} \phi(t-\sigma)g(\sigma)\,d\sigma \right| = 0\,,$$

by the Lebesgue dominated convergence theorem. And hence, $\phi * g$ is the corrective term of $f * g$, which completes the proof. □

Remark 3.4. Consider the function $F(t) = \int_{-\infty}^{t} a(t-s)f(s)ds$ given in Eq. (3), where $f \in PAP(\mathbb{R})$ and $a \in L^1(\mathbb{R})$. Setting $u = t - s$ one can rewrite it as

$$F(t) = \int_{0}^{+\infty} a(u)f(t-u)du.$$

Consequently, F can be seen as the convolution of f with $h(u) = \mathbb{I}_{[0,\infty)}\ a(u)$, where $\mathbb{I}_{[0,\infty)}$ is the characteristic function of the interval $[0,\infty)$. Using Theorem 3.1 one can readily see that F is pseudo almost periodic. Similarly, if $f : \mathbb{R} \to \mathbb{R}$ is an asymptotically almost periodic function and if a is integrable over $\mathbb{R}$, then the function F is asymptotically almost periodic.

4. Applications to differential and integral equations

4.1. First-order linear differential equations. We shall make use of Remark 3.4 to characterize asymptotically (respectively, pseudo) almost periodic solutions to the first-order linear differential equations of the form

$$u'(t) = \lambda u(t) + f(t), \quad t \in \mathbb{R}, \tag{5}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a pseudo almost periodic function and $\lambda \in \mathbb{R}$ is a (nonzero) negative real number.

Theorem 4.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a pseudo almost periodic function and let λ be a negative real number. Then Eq.* (5) *has a pseudo almost periodic solution given by*

$$u(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} f(s)ds, \quad \forall t \in \mathbb{R}. \tag{6}$$

Proof. It is clear that $u(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} f(s)ds$ is a solution to Eq. (5). Now setting $h(t) = \mathbb{I}_{[0,\infty)}\, e^{\lambda t}$ it is clear that u given above is pseudo almost periodic, by Remark 3.4. □

Similarly,

Theorem 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an asymptotically almost periodic function and let λ be a negative real number. Then Eq.* (5) *has an asymptotically almost periodic solution defined by Eq.* (6)*.*

Remark 4.3. Theorem 4.2 extends Theorem 4.1.2 from [19] in the case where f is almost automorphic.

4.2. First-order semilinear equations. In what follows, we consider the existence of asymptotically (respectively, pseudo) almost periodic solutions to semilinear differential equations of the form

$$u'(t) = \lambda u(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{7}$$

where $F : \mathbb{R} \times \Omega \mapsto \mathbb{R}$, $(t, s) \mapsto F(t, s)$ is pseudo almost periodic in $t \in \mathbb{R}$ for each $s \in \Omega$ and λ is a (nonzero) negative real number. Throughout this subsection, we set $\Omega = \mathbb{R}$.

To prove the existence of pseudo almost periodic solutions to Eq. (7) we use the classical Banach fixed-point principle. The following assumptions will be made:

(H.1) $F : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}, (t, s) \mapsto F(t, s)$ is pseudo almost periodic in $t \in \mathbb{R}$ uniformly in $s \in \mathbb{R}$ ($F \in PAP(\mathbb{R} \times \mathbb{R})$);

(H.2) $F : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, $(t, s) \mapsto F(t, s)$ is Lipschitz in $s \in \mathbb{R}$ uniformly in $t \in \mathbb{R}$, i.e., there exists $M > 0$ such that

$$|F(t, s) - F(t, r)| \leq \; M|s - r|,$$

for all $t \in \mathbb{R}$ and $s, r \in \mathbb{R}$.

Theorem 4.4. *Let λ be a nonzero negative real number. Under assumptions* (H.1)–(H.2)*, Eq.* (7) *has a unique pseudo almost periodic solution whenever* $\frac{M}{|\lambda|} < 1$.

Proof. It is well-known (see [4]) that functions of the form

$$u(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} F(s, u(s))\, ds, \quad t \in \mathbb{R},$$

are solutions to Eq. (7). Define the nonlinear operator $\Gamma(u) : \mathbb{R} \mapsto \mathbb{R}$ by

$$\Gamma(u)(t) := \int_{-\infty}^{t} e^{\lambda(t-s)} F(s, u(s))\, ds.$$

Let $u \in PAP(\mathbb{R})$. From (H.1)–(H.2), it is clear that $t \mapsto F(t, u(t))$ is pseudo almost periodic, by [4]. From Remark 3.4 it follows that Γ maps $PAP(\mathbb{R})$ into itself. It remains to show that Γ has a unique fixed-point which is a pseudo almost periodic solution to Eq. (7).

Let $u, v \in PAP(\mathbb{R})$, then

$$\begin{aligned} |\Gamma u(t) - \Gamma v(t)| &\leq \int_{-\infty}^{t} |e^{\lambda(t-s)}(F(s, u(s)) - F(s, v(s)))|\, ds \\ &\leq M \int_{-\infty}^{t} e^{\lambda(t-s)} |u(s) - v(s)|\, ds \qquad \text{(by (H.2))} \\ &\leq M\, \|u - v\|_{\infty} \int_{-\infty}^{t} e^{\lambda(t-s)}\, ds \qquad (\text{set } u = t - s) \\ &= \frac{M}{|\lambda|} \|u - v\|_{\infty}\,, \end{aligned}$$

for each $t \in \mathbb{R}$. And hence, $\|\Gamma u - \Gamma v\|_{\infty} \leq \frac{M}{|\lambda|} \|u - v\|_{\infty}$. If $\frac{M}{|\lambda|} < 1$, then $\Gamma : (PAP(\mathbb{R}), \| \cdot \|_{\infty}) \mapsto (PAP(\mathbb{R}), \| \cdot \|_{\infty})$ is a strict contraction, and hence there exists a unique $u_0 \in PAP(\mathbb{R})$ satisfying

$$u_0(t) := \int_{-\infty}^{t} e^{\lambda(t-s)} F(s, u_0(s)) ds,$$

by the Banach fixed-point principle. □

Now let pass to asymptotically almost periodic solutions to Eq. (7). Assume that:

(H.3) $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(t, s) \to F(t, s)$ is asymptotically almost periodic in $t \in \mathbb{R}$, uniformly in $s \in \mathbb{R}$ ($F \in AAP(\mathbb{R} \times \mathbb{R})$).

Similarly as above one can prove

Theorem 4.5. *Let λ be a nonzero negative real number. Under assumptions* (H.2) *and* (H.3)*, Eq.* (7) *has a unique asymptotically almost periodic solution whenever* $\frac{M}{|\lambda|} < 1$.

Proof. Let Γ be the operator defined in the proof of Theorem 4.4 and let $u \in AAP(\mathbb{R})$. From (H.1) and (H.3), it is clear that $t \to F(t, u(t))$ is asymptotically almost periodic, by [16, Lemma 2.7 and 2.8]. Now, using Remark 3.4 it follows that Γ maps $AAP(\mathbb{R})$ into itself. Further, we argue in a similar way as in the proof of Theorem 4.4. □

4.3. First-order semilinear functional-differential equations. In this subsection we extend Theorem 4.4 and Theorem 4.5 to the case of the functional-differential equations given by

$$u'(t) = \lambda u(t) + F(t, u(h(t))), \quad t \in \mathbb{R}, \tag{8}$$

where f and λ denote the same as in the previous subsection and $h : \mathbb{R} \to \mathbb{R}$ is a given function satisfying some additional conditions.

To prove the existence of asymptotically (respectively, pseudo) almost periodic solutions to Eq. (8) we combine Remark 3.4 and an extension of the Banach contraction principle, the so-called Zima's fixed-point theorem. We recall this theorem [7, 21] in an arbitrary Banach space.

Let $(\mathbb{Y}, \|\cdot\|)$ be a Banach space equipped with a binary relation $\prec$ and a mapping $m : \mathbb{Y} \mapsto \mathbb{Y}$. Assume that:

(i) the relation $\prec$ is transitive;

(ii) the norm $\|\cdot\|$ is monotonic, i.e., $\theta \prec u \prec v$, then $\|u\| \leq \|v\|$, for all $u, v \in \mathbb{Y}$;

(iii) $\theta \prec m(u)$ and $\|m(u)\| = \|u\|$ for each $u \in \mathbb{Y}$.

We have

Theorem 4.6 ([21]). *In the space Banach space $(\mathbb{Y}, \|\cdot\|, \prec, m)$ above, let $\Gamma : \mathbb{Y} \mapsto \mathbb{Y}$, $A : \mathbb{Y} \mapsto \mathbb{Y}$ be mappings such that:*

(iv) *A is a bounded linear operator with spectral radius $r(A) < 1$;*

(v) *if $\theta \prec u \prec v$, then $Au \prec Av$ for all $u, v \in \mathbb{Y}$;*

(vi) *$m(\Gamma u - \Gamma v) \prec Am(u - v)$ for all $u, v \in \mathbb{Y}$.*

Then the equation $\Gamma u = u$ has a unique solution in $\mathbb{Y}$.

In what follows, one supposes that $\mathbb{Y} = \mathbb{R}$. Next, one requires the following assumptions:

(H.4) $(t,s) \to F(t,s)$ is Lipschitz in $s \in \mathbb{R}$ in the following sense:

$$|F(t,s) - F(t,r)| \le M(t)|s-r|$$

for all $t,s,r \in \mathbb{R}$, where $M : \mathbb{R} \to \mathbb{R}_+$ is a continuous function satisfying $\int_{-\infty}^{+\infty} M(s)\,ds < +\infty$;

(H.5) h is a continuous function such that $h(\mathbb{R}) = \mathbb{R}$ and $u \circ h$ is pseudo almost periodic for each $u \in PAP(\mathbb{R})$;

(H.6) the spectral radius of the bounded linear operator A:

$$(Au)(t) = \int_{-\infty}^{t} M(s)\, u(h(s))\,ds, \quad t \in \mathbb{R},\ u \in PAP(\mathbb{R}), \tag{9}$$

is less then 1.

Now, we prove

Theorem 4.7. *Let λ be a nonzero negative real number. Under assumptions* (H.1),(H.4), (H.5), *and* (H.6), *Eq.* (8) *has a unique pseudo almost periodic solution.*

Proof. Define the nonlinear operator

$$\Gamma(u)(t) := \int_{-\infty}^{t} e^{\lambda(t-s)}\, f(s, u(h(s)))\,ds, \quad u \in PAP(\mathbb{R}),\ t \in \mathbb{R}.$$

Arguing similarly as in the proof of Theorem 4.4 we deduce that Γ maps $PAP(\mathbb{R})$ into itself.

For $u, v \in PAP(\mathbb{R})$ let us define the relation $\prec$ by

$$u \prec v \ \text{ if and only if } \ u(t) \le v(t),\ \forall t \in \mathbb{R}.$$

Moreover, let $m(u) = |u|$, that is, $(m(u))(t) = |u(t)|$ for each $t \in \mathbb{R}$. It is then easy to check that:

(α) the relation is transitive;

(β) $\theta \prec m(u)$ and $\|m(u)\|_\infty = \|u\|_\infty$ for each $u \in PAP(\mathbb{R})$;

(γ) the norm $\|\cdot\|_\infty$ is monotonic, that is, if $\theta \prec u \prec v$, then $\|u\|_\infty \le \|v\|_\infty$ for all $u, v \in PAP(\mathbb{R})$;

(δ) For $u, v \in PAP(\mathbb{R})$ we have

$$\begin{aligned}
\big|\Gamma(u)(t) - \Gamma(v)(t)\big| &\le \int_{-\infty}^{t} \Big| e^{\lambda(t-s)}\big(f(s,u(h(s))) - f(s,v(h(s)))\big)\Big|\,ds \\
&\le \int_{-\infty}^{t} e^{\lambda(t-s)} M(s)\big|u(h(s)) - v(h(s))\big|\,ds \\
&\le \int_{-\infty}^{t} M(s)\big|u(h(s)) - v(h(s))\big|\,ds,
\end{aligned}$$

hence $m(\Gamma(u) - \Gamma(v)) \prec Am(u - v)$;

(ε) A is increasing, that is, if $\theta \prec u \prec v$, then $Au \prec Av$ for $u, v \in PAP(\mathbb{R})$.

Thus by Theorem 4.6 above, we infer that the operator Γ has a unique fixed point in $PAP(\mathbb{R})$ which obviously is the unique pseudo almost periodic solution to Eq. (8). □

Remark 4.8. (1) As an example of a function h which satisfies (H.5) one can take a composition of a symmetry of center 0 and a translation.

(2) Some examples of calculation of the spectral radius of the operator A in Eq. (9) within the framework of the space of continuous functions with the norm sup can be found in [21, pp. 181–183].

(3) It is worth to mention that in many situations, Zima's fixed-point theorem gives better results than a direct application of the Banach fixed-point principle, see, e.g., [7] and [8].

For the case of asymptotically almost periodic functions we have

Theorem 4.9. *Let λ be a nonzero negative real number. Under assumptions* (H.3), (H.4), (H.5), *and* (H.6) *(in which we replace consequently $PAP(\mathbb{R})$ by $AAP(\mathbb{R})$), Eq.* (8) *has a unique asymptotically almost periodic solution.*

Proof. Let us consider again the nonlinear operator Γ defined in the proof of Theorem 4.7, now in the space $AAP(\mathbb{R})$. Arguing similarly as in the proof of Theorem 4.5 we deduce that Γ maps $AAP(\mathbb{R})$ into itself. To complete the proof, one follows along the same lines as in the proof of Theorem 4.7. □

4.4. A Volterra-type equation. Consider the Volterra-type equation given by

$$u(t) = h(t) + \int_{-\infty}^{+\infty} a(t-s)Au(s)ds, \quad \forall t \in \mathbb{R}, \tag{10}$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is pseudo almost periodic, $a : \mathbb{R} \mapsto \mathbb{R}$ is in $L^1(\mathbb{R})$ and A is a (nonzero) bounded linear operator which maps the space $PAP(\mathbb{R})$ into itself (resp, A maps $AAP(\mathbb{R})$ into itself).

Let us require the following assumption:

(H.7) $\displaystyle\int_{-\infty}^{+\infty} |a(s)|ds < \frac{1}{\|A\|}$.

Theorem 4.10. *Under assumption* (H.7)*, then the Volterra-type equation, Eq.* (10)*, has a unique pseudo almost periodic solution.*

Proof. It is clear that $Au \in PAP(\mathbb{R})$ for $u \in PAP(\mathbb{R})$. Then under (H.7) it is clear that the function $t \mapsto \int_{-\infty}^{+\infty} a(t-s)Au(s)ds$ is pseudo almost periodic whenever $t \mapsto u(t)$ is (see Theorem 3.1). Now, since h is pseudo almost periodic it follows that the operator defined by

$$\Phi(u)(t) = h(t) + \int_{-\infty}^{+\infty} a(t-s)Au(s)ds, \quad \forall t \in \mathbb{R}$$

maps $PAP(\mathbb{R})$ into itself. Moreover, for all $u, v \in PAP(\mathbb{R})$,

$$\|\Phi(u) - \Phi(v)\|_\infty \leq C\|u - v\|_\infty,$$

where $C = \|A\| \int_{-\infty}^{+\infty} |a(s)|ds$.

Clearly $\Phi : (PAP(\mathbb{R}), \|\cdot\|_\infty) \mapsto (PAP(\mathbb{R}), \|\cdot\|_\infty)$ is a strict contraction, by (H.7). Using the Banach fixed-point principle it follows that there exists a unique pseudo almost periodic function $u_0 : \mathbb{R} \mapsto \mathbb{R}$ such that $\Phi(u_0) = u_0$, i.e.,

$$u_0(t) = h(t) + \int_{-\infty}^{+\infty} a(t-s)Au_0(s)ds.$$

□

Similarly, for asymptotically almost periodic solutions we have

Theorem 4.11. *Let $h : \mathbb{R} \to \mathbb{R}$ be asymptotically almost periodic and let $a : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Then under assumption* (H.7)*, the Volterra-type equation, Eq.* (10)*, has a unique asymptotically almost periodic solution.*

Proof. It is enough to apply the same arguments as in the proof of Theorem 4.10 with Theorem 3.3 instead of Theorem 3.1. □

Remark 4.12. As examples of bounded operators A which map $PAP(\mathbb{R})$ into itself, one may consider the projections $\pi, \pi' : PAP(\mathbb{R}) \mapsto PAP(\mathbb{R})$ defined by $\pi(f) = g$ and $\pi'(f) = \phi$ whenever $f = g + \phi$ with $g \in AP(\mathbb{R})$ and $\phi \in AP_0(\mathbb{R})$.

5. Conclusion

We end this paper with a note on further applications, and some perspectives.

(i) One can also make use of Theorem 3.1 to characterize pseudo almost periodic solutions to several other types of differential equations through the Laplace transform (e.g., second-order differential equations with constant coefficients).

(ii) Fix $f = h + \phi$, where $h \in AP(\mathbb{R})$ and $\phi \in AP_0(\mathbb{R})$. Consider the convolution operator defined by $\kappa_\phi : L^1(\mathbb{R}) \mapsto PAP(\mathbb{R})$, $g \mapsto \phi * g$. Clearly $\|\kappa_\phi\| \leq \|\phi\|_\infty$, hence κ_ϕ is a bounded linear operator. Thus one may ask the following questions: Is κ_ϕ compact? What is the spectrum $\sigma(\kappa_\phi)$ of the operator κ_ϕ? Is $0 \in \rho(\kappa_\phi)$, where $\rho(\kappa_\phi)$ is the resolvent of κ_ϕ?

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