Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 25 (2006), 411–420

What do we Learn from the Discrepancy Principle?

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Abstract. The author analyzes the discrepancy principle when smoothness is given in terms of general source conditions. As it turns out, this framework is particularly well suited to reveal the mechanism under which this principle works. For general source conditions there is no explicit way to compute rates of convergence. Instead arguments must be based on geometric properties. Still this approach allows to generalize previous results. The analysis is accomplished with a result showing why this discrepancy principle inherently has the early saturation for a large class of regularization methods of bounded qualification.

Keywords. Regularization, general source condition, saturation, variable Hilbert scale

Mathematics Subject Classification (2000). Primary 65Q05, secondary 65R22

1. Introduction

We shall study linear operator equations

$$y_\delta = Ax + \delta\xi \tag{1}$$

for a compact and injective operator acting between Hilbert spaces X and Y and with bounded deterministic noise ξ, i.e., $\|\xi\| \le 1$. The analysis can easily be extended to non-injective operators, in which case the projection onto the closure of the range of A will appear, as well as to non-compact ones, when a more thorough spectral calculus must be applied. This can be easily seen in the respective places. We aim at solving such equations by means of regularization, which is controlled by some parameter α.

If the smoothness of the true solution x is known, then theoretical results tell us how to choose the parameter α. Otherwise a *data-driven* choice of the parameter is necessary. Several strategies for choosing the parameter are known. The most classical one is the *discrepancy principle*. It is often called *Morozov's* discrepancy principle, see [6], although it can already be found in Phillips' original paper [8].

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Previous analysis of the discrepancy principle under general source conditions was carried out in [7]. Theorem 3.8 below corresponds to Theorem 3.1, and the results in § 4 in that study. It is the intention of the author to derive this result from a unified point of view, especially from some recent interpolation inequality and the notion of qualification of the chosen regularization. By doing this, the impact of the discrepancy principle to different components in the error decomposition becomes transparent. We give some additional discussion in Remarks 3.5 and 3.9.

We accomplish the study by discussing the issue of saturation of the discrepancy principle. Again we may extend previous results, as e.g. in [1], by geometric reasoning.

2. Regularization under known smoothness

We will analyze regularization by means of operator families in the form of $g_\alpha\colon (0, \|A^*A\|] \to \mathbb{R}^+$, i.e., the regularized solution based on data y_δ is given as

$$x_{\alpha,\delta} := g_\alpha(A^*A)A^*y_\delta, \quad \alpha > 0, \tag{2}$$

for the operator A from equation (1). For technical reasons we shall assume that for each t the mapping $\alpha \mapsto g_\alpha(t)$ is continuous from the left, throughout. We recall the following

Definition 2.1 (see [5]). A family $g_\alpha,\ 0 < \alpha \le \|A^*A\|$, is called regularization, if there are constants γ_* and γ for which

$$\sup_{0<\lambda\le\|A^*A\|} |1 - \lambda g_\alpha(\lambda)| \le \gamma, \qquad 0 < \alpha \le \|A^*A\|$$

$$\sup_{0<\lambda\le\|A^*A\|} \sqrt{\lambda}\, |g_\alpha(\lambda)| \le \frac{\gamma_*}{\sqrt{\alpha}}, \qquad 0 < \alpha \le \|A^*A\|.$$

The regularization g_α is said to have *qualification* ρ, for an increasing function $\rho : (0, \|A^*A\|] \to \mathbb{R}_+$, if

$$\sup_{0<\lambda\le\|A^*A\|} |1 - \lambda g_\alpha(\lambda)|\, \rho(\lambda) \le \gamma\rho(\alpha), \quad 0 < \alpha \le \|A^*A\|.$$

Notice that the mapping $r_\alpha(A^*A) := \mathrm{I} - A^*Ag_\alpha(A^*A)$ is norm bounded by γ.

Throughout we shall measure smoothness relative to the operator A in terms of the following type of conditions:

$$A_\varphi(R) := \{x,\ \ x = \varphi(A^*A)v \ \text{ for some } v \in X \text{ with } \|v\| \le R\}\,.$$

Here $\varphi\colon (0, \|A^*A\|] \to \mathbb{R}^+$ is continuous increasing and $\varphi(0+) = 0$. Such functions are called *index functions*. These give rise to weighted Hilbert spaces X_φ as follows:

As mentioned earlier we restrict the construction to compact operators A. In this case A^*A admits a (monotonic) singular value decomposition for an orthonormal system $u_1, u_1, \dots,$ given by

$$A^*Ax = \sum_{j=1}^{\infty} s_j \langle x, u_j\rangle u_j, \quad x \in X.$$

Then the weighted Hilbert space X_φ is the completion of finite expansions $x = \sum_{j=1}^{n} \langle x, u_j\rangle u_j$ with respect to the scalar product

$$\langle x, y\rangle_\varphi := \sum_{j=1}^{\infty} \frac{\langle x, u_j\rangle \langle y, u_j\rangle}{\varphi^2(s_j)}.$$

In this case we have $A_\varphi(R) = \{x,\ \|x\|_\varphi \le R\}$.

For the regularizing properties of g_α the interplay between the qualification ρ and the actual smoothness φ of the solution, in particular properties of the quotient $\Phi(t) := \varphi(t)/\rho(t),\ 0 < t \le \|A^*A\|$, are relevant. The approach below is equivalent, though different from the one in [5].

We agree to denote by $\bar\Phi(t) := \sup_{s>t} \Phi(s),\ t > 0$, the decreasing majorant of Φ, possible equal to ∞ throughout.

Definition 2.2. The *qualification* ρ *covers* φ *with constant* C, if

$$\bar\Phi(t) \le C\Phi(t), \quad 0 < t \le \|A^*A\|.$$

The basic implication of this definition is captured in

Proposition 2.3. *Suppose* $x \in A_\varphi(R)$. *If the qualification* ρ *of some regularization* g_α *covers* φ *with constant* C, *then*

$$\|r_\alpha(A^*A)x\| \le C\gamma R\varphi(\alpha).$$

Proof. Observe that under $x \in A_\varphi(R)$ we have

$$\|r_\alpha(A^*A)x\| \le R \sup_{0<t\le\|A^*A\|} |r_\alpha(t)|\,\varphi(t).$$

For $t \le \alpha$ we obtain from monotonicity that $|r_\alpha(t)|\,\varphi(t) \le \gamma\varphi(\alpha)$. Otherwise we can bound

$$\sup_{t>\alpha} |r_\alpha(t)|\,\varphi(t) = \sup_{t>\alpha} |r_\alpha(t)|\,\rho(t)\Phi(t) \le \gamma\rho(\alpha)\bar\Phi(\alpha) \le C\gamma\rho(\alpha)\Phi(\alpha) = C\gamma\varphi(\alpha).$$

In both cases we obtain the required upper bound, because $C \ge 1$. □

As an important consequence we recall the following result from [5]. It will be convenient to assign every index function φ the related index function

$$\Theta(t) := \sqrt{t}\varphi(t), \quad 0 < t \le \|A^*A\|. \tag{3}$$

Theorem 2.4. *Let φ be any index function, and let $\bar{\alpha}$ be chosen to satisfy $\Theta(\bar{\alpha}) = \frac{\delta}{R}$. If the qualification of g_α covers φ with constant C, then*

$$e\big(A_\varphi(R), g_{\bar{\alpha}}, \delta\big) \leq R\,(C\gamma + \gamma_*)\,\varphi\big(\Theta^{-1}\big(\tfrac{\delta}{R}\big)\big), \quad 0 < \delta \leq R\|A^*A\|.$$

3. The discrepancy principle

The classical discrepancy principle can be phrased as follows: Let g_α be a regularization scheme and $x_{\alpha,\delta}$ as in (2). Determine α_* by

$$\alpha_* := \sup\left\{\alpha \leq \|A^*A\|,\ \|Ax_{\alpha,\delta} - y_\delta\| \leq \tau\delta\right\}. \tag{4}$$

By left continuity of $\alpha \mapsto g_\alpha$, the supremum is attained by α_*. For this choice of regularization parameter we consider $x_{\alpha_*,\delta}$ as final approximation to the exact solution x. Notice, that by construction of $x_{\alpha,\delta}$ it holds true that $\|Ax_{\alpha,\delta} - y_\delta\| = \|r_\alpha(AA^*)y_\delta\|$.

Remark 3.1. In practice, we start with large α_0, e.g. $\alpha_0 := \|A^*A\|$, and decrease stepwise $\alpha_{n+1} := \frac{\alpha_n}{q}$, for some $q > 1$. Thus, we may find the "optimal" regularization parameter only up to some bandwidth.

Let us introduce the auxiliary $x_\alpha := g_\alpha(A^*A)A^*y = g_\alpha(A^*A)A^*Ax$ with x being the true solution to (1).

Lemma 3.2. *Let α_* be chosen according to* (4) *for $\tau > \gamma$. At $y = Ax$ the following assertions are valid:*

$$\|r_{\alpha_*}(AA^*)y\| \leq (\tau + \gamma)\delta. \tag{5}$$

For any $\alpha > \alpha_*$ it holds true that

$$\|r_\alpha(AA^*)y\| \geq (\tau - \gamma)\delta. \tag{6}$$

Proof. Using the triangle inequality we deduce

$$\|r_{\alpha_*}(AA^*)y\| \leq \|r_{\alpha_*}(AA^*)(y - y_\delta)\| + \|r_{\alpha_*}(AA^*)y_\delta\|.$$

For α_* the second term is bounded by $\tau\delta$. Plainly, the first one is bounded by $\gamma\delta$, which proves (5). By reverting the inequality and using $\alpha > \alpha_*$ the bound (6) can be proved similarly. □

The error analysis will use the following obvious error decomposition:

$$\|x - x_{\alpha,\delta}\| \leq \|x - x_\alpha\| + \|x_\alpha - x_{\alpha,\delta}\|, \tag{7}$$

where the first summand is noise-free and the second one is the (pure) noise term. As can be seen below, the above choice from (4) has implications to both, the noise term and the noise-free term in the error decomposition.

Below we shall frequently need properties of concave index functions, and we find it convenient to recall some of their properties.

Lemma 3.3. *The following properties hold true for concave index functions φ:*

1. *For $0 < \alpha < 1$ we have $\varphi(\alpha t) \geq \alpha\varphi(t),\ 0 < t \leq \|A^*A\|$.*
2. *The mapping $t \mapsto \frac{\varphi(t)}{t}$ is non-increasing.*
3. *For each t the mapping $r \mapsto r\varphi\left(\frac{t}{r}\right)$ is increasing.*

3.1. Bounding the noise term. By Definition 2.1 the noise term allows for the bound $\|x_\alpha - x_{\alpha,\delta}\| \leq \gamma_* \frac{\delta}{\sqrt{\alpha}}$, and lower bounds for α_* yield upper bounds for it.

Lemma 3.4. *Suppose $x \in A_\varphi(R)$. Let $\Theta(t)$ be as in (3). If the qualification of g_α covers Θ with constant C, then for $q > 1$ we have $\Theta(q\alpha_*) \geq \frac{\tau-\gamma}{C\gamma}\frac{\delta}{R}$. Consequently, under (4) and for $\delta \leq C\gamma\frac{R}{\tau-\gamma}\|A^*A\|$ it holds true that*

$$\frac{\delta}{\sqrt{\alpha_*}} \leq \frac{C\gamma}{\tau-\gamma} R\varphi\left(\Theta^{-1}\left(\frac{\tau-\gamma}{C\gamma}\frac{\delta}{R}\right)\right).$$

Proof. Let $\alpha := q\alpha_*$ and $x = \varphi(A^*A)v$ with $\|v\| \leq R$. If the qualification of g_α covers Θ with constant C, then by Lemma 3.2 we obtain

$$(\tau-\gamma)\delta \leq \|r_\alpha(AA^*)y\| = \|r_\alpha(A^*A)(A^*A)^{\frac{1}{2}}\varphi(A^*A)v\| \leq C\gamma R\Theta(\alpha),$$

which proves the first statement. By definition of Θ we have for any $0 < t \leq R\|A^*A\|$ that $\frac{t}{\sqrt{\Theta^{-1}(\frac{t}{R})}} = R\varphi(\Theta^{-1}(\frac{t}{R}))$. The previous estimate yields

$$\frac{\delta}{\sqrt{q\alpha_*}} \leq \frac{\delta}{\sqrt{\Theta^{-1}(\frac{\tau-\gamma}{C\gamma}\frac{\delta}{R})}} = \frac{C\gamma}{\tau-\gamma}\frac{\frac{\tau-\gamma}{C\gamma}\delta}{\sqrt{\Theta^{-1}(\frac{\tau-\gamma}{C\gamma}\frac{\delta}{R})}} = \frac{C\gamma}{\tau-\gamma}R\varphi\big(\Theta^{-1}(\tfrac{\tau-\gamma}{C\gamma}\tfrac{\delta}{R})\big).$$

Letting $q \to 1$ allows to complete the proof. □

Remark 3.5. For classical Hilbert scales, e. g., when $\varphi(t) := t^\mu$ for some $\mu > 0$, this is well known and can be derived from [1, Chapter 4.3].

We emphasize that for the bound to be proved, the chosen regularization must cover the smoothness Θ, which is a stronger assumption than needed for known smoothness, see [5].

Notice that by Theorem 2.4, under known smoothness the optimal parameter $\bar{\alpha}$ must satisfy $\Theta(\bar{\alpha}) = \frac{\delta}{R}$. Thus the discrepancy principle tends to choose $\alpha_* \geq \bar{\alpha}$, provided the chosen regularization g_α covers Θ. To put it differently, either g_α covers Θ, then $\alpha_* \geq \bar{\alpha}$, or it does not, in the sense of Theorem 4.3, below. If this is the case, then the optimal order cannot be obtained. The case $\alpha_* \leq \bar{\alpha}$, as considered in the proof of [7, Theorem 4.3] may hardly be met.

3.2. Bounding the noise-free term. Recall the auxiliary quantities

$$x_\alpha := g_\alpha(A^*A)A^*y \quad \text{and} \quad y_\alpha := Ax_\alpha.$$

Lemma 3.2 also implies a bound for the noise free term.

Lemma 3.6. *Let α_* be chosen according to the discrepancy principle* (4). *If the function $t \mapsto \varphi^2((\Theta^2)^{-1}(t))$ is concave, then we obtain*

$$\|x - x_{\alpha_*}\| \leq (\tau + \gamma)\, R\varphi\big(\Theta^{-1}\big(\tfrac{\delta}{R}\big)\big). \tag{8}$$

Proof. Firstly, the noise free term rewrites as $\|x_\alpha - x\| = \|r_\alpha(A^*A)x\|$. This will be bounded by means of the following interpolation inequality, which holds under the above concavity assumption, we refer to [4, Theorem 4]:

$$\varphi^{-1}\left(\frac{\|r_\alpha(A^*A)x\|_{\varphi/\varphi}}{\|r_\alpha(A^*A)x\|_\varphi}\right) \leq \Theta^{-1}\left(\frac{\|r_\alpha(A^*A)x\|_{\varphi/\Theta}}{\|r_\alpha(A^*A)x\|_\varphi}\right).$$

After rewriting this we arrive at

$$\|r_\alpha(A^*A)x\| \leq \|r_\alpha(A^*A)x\|_\varphi \varphi\left(\Theta^{-1}\left(\frac{\|r_\alpha(A^*A)x\|_{1/\sqrt{t}}}{\|r_\alpha(A^*A)x\|_\varphi}\right)\right).$$

Since $x \in A_\varphi(R)$ implies $\|r_\alpha(A^*A)x\|_\varphi \leq \gamma R$ and $r \mapsto r\varphi(\Theta^{-1}(\frac{t}{r})))$ is increasing for each t this yields

$$\|r_\alpha(A^*A)x\| \leq \gamma R\varphi\left(\Theta^{-1}\left(\frac{\|r_\alpha(A^*A)x\|_{1/\sqrt{t}}}{\gamma R}\right)\right). \tag{9}$$

Using Lemma 3.2, under the discrepancy principle it holds true that

$$\|r_{\alpha_*}(A^*A)x\|_{1/\sqrt{t}} = \|Ar_{\alpha_*}(A^*A)x\| = \|r_\alpha(AA^*)y\| \leq (\tau + \gamma)\delta.$$

Inserting this into (9) and using concavity once more, the proof of (8) is complete. ☐

Remark 3.7. It is important to notice that the above bound in Lemma 3.6 does not use any assumption on the regularization. Thus, under the discrepancy principle and for smooth x the noise free term can be made small even if the chosen regularization does not cover the smoothness of x.

3.3. The error under the discrepancy principle. As an immediate consequence of the above bounds we may formulate the main result. Let us recall that the function φ is said to obey a Δ_2-condition, if there is $0 < C_2 < \infty$ for which $\varphi(2t) \leq C_2\varphi(t),\ t > 0$.

Theorem 3.8. *Suppose that $x \in A_\varphi(R)$ for an index function φ which obeys a Δ_2-condition and that the qualification of g_α covers Θ with constant C. Moreover we assume that $t \mapsto \varphi^2((\Theta^2)^{-1}(t))$ is concave. Under the discrepancy principle* (4) *there is a constant $M = M(\tau, \gamma, \gamma_*, C, C_2)$ such that*

$$\|x_{\alpha_*,\delta} - x\| \leq MR\varphi\big(\Theta^{-1}\big(\tfrac{\delta}{R}\big)\big), \quad \text{as } \delta \to 0.$$

Remark 3.9. In [7] the authors prove similar bounds. It is interesting to discuss the relation for two typical cases. First, for functions φ which have a concave square φ^2, the optimal order is proved in both studies. However, if φ^2 is convex which relates to higher smoothness, different requirements on the qualification of the chosen regularization are made. In [7] a bound is proved under the assumption that g_α covers φ^2, whereas here we require g_α to cover Θ. Indeed, for smoothness φ with convex square, the function $\frac{\varphi(t)}{\sqrt{t}}$ is increasing, such that the function $\Phi(t) = \frac{\Theta(t)}{\varphi^2(t)}$ is decreasing, and according to Definition 2.2, any regularization which covers φ^2 necessarily covers Θ. Thus the optimal order of reconstruction can be proved for a larger class of regularization schemes. As Theorem 4.3 will indicate, the minimal gap of $\sqrt{t}$ between smoothness and qualification is natural.

We add that for Tikhonov regularization, i.e., when $\gamma = 1$ and concave index functions φ, in which case the qualification constant $C = 1$, the best choice for τ is $\tau = 2$, which results in an error bound $\|x_{\alpha_*,\delta} - x\| \leq 4R\varphi(\Theta^{-1}(\frac{\delta}{R}))$.

4. Saturation of the discrepancy principle

As can be drawn from the above result, we needed the regularization g_α to cover Θ instead of the true φ in order to achieve the best possible rate of approximation. This is not a lack of the proof, but reveals an intrinsic lack of the discrepancy principle. For specific regularization methods, in particular for Tikhonov regularization the following is well known, see [2]: The best possible rate for the discrepancy principle applied to Tikhonov regularization is $\delta \mapsto \sqrt{\delta}$ as $\delta \to 0$, see [1, Proposition 4.20] for a proof. This rate corresponds to the optimal rate for $\varphi(t) := \sqrt{t}$, although Tikhonov regularization covers smoothness up to $\rho(t) = t$. Thus we loose smoothness by a factor of $\sqrt{t}$.

Next we will show that this is a rather general phenomenon. To this end we shall impose the following restrictions on the regularization method g_α. As in [3] we suppose that

1. For some $c > 0$ the following lower bound is valid:

$$\sup_{0<\lambda\leq a} \sqrt{\lambda}\, |g_\alpha(\lambda)| \geq \frac{c}{\sqrt{\alpha}}, \quad 0 < \alpha \leq a. \tag{10}$$

2. For some increasing function ρ, the regularization has *maximal qualification* ρ, i.e., for all $0 < \lambda \leq a$ there is $c := c(\lambda) > 0$, for which
$$\inf_{0<\alpha\leq a} \frac{|r_\alpha(\lambda)|}{\rho(\alpha)} \geq c. \tag{11}$$
3. For all $0 < \alpha \leq a$ the functions
$$\lambda \longrightarrow |r_\alpha(\lambda)|^2, \quad 0 < \lambda \leq a, \tag{12}$$
are convex.

These assumptions are shown to be fulfilled for a variety of regularization methods, see [3].

To the maximal qualification ρ as in (11) we assign $\bar{\varphi}(t) := \rho(t)/\sqrt{t}$. Moreover, we shall assume that $\bar{\varphi}$ is an index function. The latter is certainly true, if ρ^2 was convex. Under all these assumptions the *early saturation effect* as seen for Tikhonov regularization used with the discrepancy principle can be generalized.

Lemma 4.1. *Let α_* be chosen according to the discrepancy principle* (4). *Under assumptions* (11) *and* (12) *there is a constant $C < \infty$ such that*
$$\rho(\alpha_*) \leq \frac{C\delta}{\|y\|R}. \tag{13}$$
Consequently we obtain $\frac{\delta}{\sqrt{\alpha_*}} \geq \frac{\|y\|}{C} R\bar{\varphi}\big(\rho^{-1}(\frac{\delta}{R})\big)$.

Proof. For any $x \neq 0$ let y and y_α as before. As in [3] we use a variant of Peierls–Bogolyubov inequality, see e.g. [3, Lemma 2.5], to deduce that under assumption (12) it holds true that
$$\|r_{\alpha_*}(AA^*)y\| \geq \|y\| r_{\alpha_*}\left(\frac{\|A^*y\|^2}{\|y\|^2}\right).$$
Thus, using Lemma 3.2 and assumption (11) we find $c > 0$ for which
$$(\tau + \gamma)\delta \geq \|r_{\alpha_*}(AA^*)y\| \geq c\|y\|\rho(\alpha_*),$$
which proves (13) with $C = \frac{\tau+\gamma}{c}$. The remaining assertion follows by the same arguments as used in the proof of Lemma 3.4. □

We add some technical

Lemma 4.2. *Let f, g be two index functions. Assume that g obeys a Δ_2-condition, i.e., there is $C_2 > 0$ such that $g(2t) \leq C_2 g(t)$, $0 < t \leq \frac{1}{2}\|A^*A\|$. The following assertion holds true:*
$$\frac{f(t)}{g(t)} \to 0 \quad \textit{implies} \quad \frac{g^{-1}(t)}{f^{-1}(t)} \to 0, \quad \textit{as } t \to 0.$$

Proof. Let $n \geq 1$ be any integer. By iterating the Δ_2-condition we obtain for $t > 0$ small enough the estimate $g(2^n t) \leq C_2^n g(t)$ and as a consequence $g^{-1}(C_2^{-n} g(t)) \leq 2^{-n} t$. For $t > 0$ small enough let $t = f(s)$. If $\frac{f(s)}{g(s)} \leq C_2^{-n}$, then

$$\frac{g^{-1}(t)}{f^{-1}(t)} = \frac{g^{-1}(f(s))}{s} \leq \frac{g^{-1}(C_2^{-n} g(s))}{s} \leq \frac{2^{-n} s}{s} = 2^{-n}.$$

Because $n \geq 1$ was arbitrary the lemma is proved. □

With this preparation we can formulate the main result in this section.

Theorem 4.3. *Assume that the regularization g_α obeys* (10)–(12)*. Suppose in addition that ρ from* (11) *covers at least $t \mapsto \sqrt{t}$ and obeys a Δ_2-condition. If the true solution x belongs to $A_\varphi(R)$ for some φ smoother than $\bar{\varphi}$, i.e., $\frac{\varphi(t)}{\bar{\varphi}(t)} \to 0$ as $t \to 0$, and if $t \mapsto \varphi^2((\Theta^2)^{-1}(t))$ is concave, then there is $c > 0$ such that for α_* chosen according to* (4) *it holds true that for $x \neq 0$ we have*

$$\sup_{\|\xi\| \leq 1} \|x - x_{\alpha_*,\delta}\| \geq cR\bar{\varphi}\big(\rho^{-1}\big(\tfrac{\delta}{R}\big)\big), \quad \text{as } \delta \to 0.$$

Proof. Rewriting the error decomposition (7) and using (10) as well as the estimate in Lemma 3.6 we can find constants $c, \bar{c} > 0$ for which

$$\sup_{\|\xi\| \leq 1} \|x - x_{\alpha_*,\delta}\| \geq c\delta/\sqrt{\alpha_*} - \bar{c}\varphi\big(\Theta^{-1}\big(\tfrac{\delta}{R}\big)\big).$$

By Lemma 4.1 this implies

$$\begin{aligned} \sup_{\|\xi\| \leq 1} \|x - x_{\alpha_*,\delta}\| &\geq \tilde{c}\left(\bar{\varphi}\big(\rho^{-1}\big(\tfrac{\delta}{R}\big) - \varphi(\Theta^{-1}\big(\tfrac{\delta}{R}\big)\big)\big)\right) \\ &= \tilde{c}\bar{\varphi}(\rho^{-1}\big(\tfrac{\delta}{R}\big)\left(1 - \frac{\varphi\big(\Theta^{-1}\big(\tfrac{\delta}{R}\big)\big)}{\bar{\varphi}\big(\rho^{-1}\big(\tfrac{\delta}{R}\big)\big)}\right). \end{aligned}$$

By Lemma 4.2 the quotient $\varphi(\Theta^{-1}(\frac{\delta}{R}))/\bar{\varphi}(\rho^{-1}(\frac{\delta}{R}))$ is small for $\frac{\delta}{R}$ small enough, which allows to complete the proof. □

Example 4.4. Tikhonov regularization has maximal qualification $\rho(t) = t$ and consequently $\bar{\varphi}(t) = \sqrt{t}$, such that for smooth x the bound in Theorem 4.3 provides

$$\sup_{\|\xi\| \leq 1} \|x - x_{\alpha_*,\delta}\| \geq c\sqrt{\delta}, \quad \delta \to 0$$

and we recover the result from [1, Proposition 4.20].

Remark 4.5. The above result should be compared with the saturation phenomenon for regularization. Under the same assumptions (10)–(12) it was proved in [3] that

$$\sup_{\|\xi\|\leq 1} \|x - x_{\alpha_*,\delta}\| \geq cR\rho\big(\psi^{-1}\big(\tfrac{\delta}{R}\big)\big), \quad \delta \to 0.$$

where $\psi(t) := \sqrt{t}\rho(t)$. So, saturation under the discrepancy principle occurs exactly $\sqrt{t}$ earlier than without discrepancy principle. For smoothness in terms of powers $t \mapsto t^\mu$ this can be seen throughout, we refer to [1]: For methods of qualification t^μ optimal performance under the discrepancy principle can be proved for smoothness t^ν up to $0 < \nu \leq \mu - \frac{1}{2}$.

Acknowledgment. Part of this work was done while the author visited the Johann Radon Institute, Linz. The author expresses his gratitude for the hospitality and interest in this work.

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Received January 18, 2005; revised May 10, 2005