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Corrigendum to:

On Harmonic Potential Fields and the Structure of Monogenic Functions

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The assertion on page 270, Section 4.1 (Z. Anal. Anw. 22 (2003)(2)), that:
the function

$$\widetilde{h}_{k+1}(\underline{x}) = \frac{1}{2(2-m-2k)}\, \underline{x}^2\, \widetilde{h}_{k-1}(\underline{x}) \tag{4.2}$$

is the unique $\mathbb{R}_{0,m}$-valued homogeneous polynomial of degree $(k+1)$ *of this particular form, which satisfies equation* (4.1),

is erroneous, since the mentioned function $\widetilde{h}_{k+1}$ is not a solution of equation (4.1). However the Fischer decomposition of homogeneous polynomials (see, e.g., [1]) may be used to show the existence of a unique solution of equation (4.1) which is given by (4.2) in the revised Section 4 below. Accordingly the formulae (4.4) on page 270 and (4.9) on page 271, and the first formula appearing on page 272, have to be adjusted.

4. Homogeneous harmonic polynomials

The results on conjugate harmonics and on structure of monogenic functions, as established in the previous sections, is now applied to the special case of homogeneous harmonic polynomials in $\mathbb{R}^{m+1}$. As $\mathbb{R}^{m+1}$ is trivially normal with respect to each e_j-direction ($j = 0, 1, ..., m$), no supplementary geometric conditions are necessary. In the sequel $2 \le k \in \mathbb{N}$ will be fixed.

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4.1. The general case. Let $U_k(x_0, \underline{x})$ be an $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree k in $\mathbb{R}^{m+1}$. We aim at establishing the general form of an $\mathbb{R}_{0,m}$-valued homogeneous polynomial $V_k(x_0, \underline{x})$ of degree k in $\mathbb{R}^{m+1}$ which is conjugate harmonic to $U_k(x_0, \underline{x})$. In view of the results obtained in Section 2 and choosing $x_0^* = 0$, we have to solve in $\mathbb{R}^m$ the equation

$$\Delta_{\underline{x}} \tilde{h}(\underline{x}) = \tilde{h}_{k-1}(\underline{x}) \tag{4.1}$$

where $\tilde{h}_{k-1}(\underline{x}) = \partial_{x_0} U_k(0, \underline{x})$ is an $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree $(k-1)$ in the variable $\underline{x} \in \mathbb{R}^m$.

According to the Fischer decomposition of homogeneous polynomials (see, e.g., [1]), there exists a unique $\mathbb{R}_{0,m}$-valued homogeneous polynomials u'_{k-1} of degree $(k-1)$ in $\mathbb{R}^m$ such that

$$\tilde{u}_{k+1}(\underline{x}) = \underline{x}^2 u'_{k-1}(\underline{x}) \tag{4.2}$$

is the unique $\mathbb{R}_{0,m}$-valued homogeneous polynomial of degree $(k+1)$ in $\mathbb{R}^m$ which satisfies the equation (4.1). It thus follows that the function

$$\tilde{V}_k(x_0, \underline{x}) = -\partial_{\underline{x}} \tilde{H}_{k+1}(x_0, \underline{x}), \tag{4.3}$$

where

$$\tilde{H}_{k+1}(x_0, \underline{x}) = \int_0^{x_0} U_k(t, \underline{x})\, dt - \tilde{u}_{k+1}(\underline{x}) \tag{4.4}$$

is the unique $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree k of this particular form which is conjugate harmonic to U_k in $\mathbb{R}^{m+1}$.

Obviously, any $\mathbb{R}_{0,m}$-valued homogeneous conjugate harmonic of degree k to U_k is given by

$$V_k(x_0, \underline{x}) = -\partial_{\underline{x}} H_{k+1}(x_0, \underline{x}), \tag{4.5}$$

where

$$H_{k+1}(x_0, \underline{x}) = \tilde{H}_{k+1}(x_0, \underline{x}) - h_{k+1}(\underline{x}), \tag{4.6}$$

$h_{k+1}(\underline{x})$ being an arbitrary $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree $(k+1)$ in $\mathbb{R}^m$.

Finally, denoting by $M^+(k; \mathbb{R}_{0,m+1})$ the space of $\mathbb{R}_{0,m+1}$-valued homogeneous monogenic polynomials of degree k in $\mathbb{R}^{m+1}$ (see [4]), in view of Theorem 3.1 the following structure theorem is obtained.

Theorem 4.1. *Let $P_k \in M^+(k; \mathbb{R}_{0,m+1})$. Then there exists an $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial H_{k+1} of degree $(k+1)$ in $\mathbb{R}^{m+1}$ such that $P_k = \overline{D}_x H_{k+1}$.*

4.2. The scalar case. Now assume that $u_k(x_0, \underline{x})$ is a real-valued homogeneous harmonic polynomial of degree k in $\mathbb{R}^{m+1}$, i.e., u_k is a solid harmonic of degree k. Then, according to (4.2) and (4.4), $\tilde{u}_{k+1}(\underline{x})$ and $\tilde{H}_{k+1}(x_0, \underline{x})$ are also real-valued, whence, in view of (4.3), $\tilde{v}_k(x_0, \underline{x})$ is $\mathbb{R}^{0,m}$-valued. So we have at once

Theorem 4.2. *Given a real-valued solid harmonic $u_k(x_0, \underline{x})$ of degree k in $\mathbb{R}^{m+1}$, there exists a unique $\mathbb{R}^{0,m}$-valued homogeneous harmonic polynomial $\tilde{v}_k(x_0, \underline{x})$ of degree k conjugate to $u_k(x_0, \underline{x})$ of the particular form*

$$\tilde{v}_k(x_0, \underline{x}) = -\partial_{\underline{x}} \tilde{H}_{k+1}(x_0, \underline{x}), \tag{4.7}$$

where

$$\tilde{H}_{k+1}(x_0, \underline{x}) = \int_0^{x_0} u_k(t, \underline{x})\, dt - \tilde{u}_{k+1}(\underline{x}) \tag{4.8}$$

is a real-valued homogeneous harmonic polynomial of degree $(k+1)$ in $\mathbb{R}^{m+1}$.

Remark 4.3. Taking in (4.6) $h_{k+1}(\underline{x})$ real-valued, then (4.5) – (4.6) express the general form of an $\mathbb{R}^{0,m}$-valued homogeneous polynomial $v_k(x_0, \underline{x})$ of degree k conjugate to the real-valued $u_k(x_0, \underline{x})$.

Remark 4.4. Writing $\tilde{v}_k$, defined by (4.7) as $\tilde{v}_k = \sum_{j=1}^m e_j \tilde{v}_{k,j}$, the, according to the general theory outlined in the previous sections, $P_k = u_k + \overline{e}_0 \tilde{v}_k$ is an $\mathbb{R} \oplus \overline{e}_0 \mathbb{R}^{0,m}$-valued homogeneous monogenic polynomial of degree k in $\mathbb{R}^{m+1}$. Equivalently, the $(m+1)$-tuple $F_k^* = (u_k, -v_{k,1}, \ldots, -v_{k,m})$ is a real-valued Stein-Weißfield, which can be realized as the gradient of a real-valued solid harmonic of degree $(k+1)$, which is precisely the function $\tilde{H}_{k+1}$ given by (4.8).

Remark 4.5. The conjugate harmonic $\tilde{v}_k$ to u_k given by (4.7) – (4.8) may be rewritten as

$$\tilde{v}_k(x_0, \underline{x}) = \tilde{v}_k^{(1)}(x_0, \underline{x}) + \underline{x} w_{k-1}^{(1)}(\underline{x}) + \underline{x}^2 w_{k-2}^{(2)}(\underline{x}), \tag{4.9}$$

where

(i) $\tilde{v}_k^{(1)}(x_0, \underline{x}) = \int_0^{x_0} \partial_{\underline{x}} u_k(t, \underline{x})\, dt$ is a homogeneous polynomial of degree k in $\mathbb{R}^{m+1}$

(ii) $w_{k-1}^{(1)}(\underline{x}) = 2u'_{k-1}(\underline{x})$ is a homogeneous polynomial of degree $(k-1)$ in $\mathbb{R}^m$

(iii) $w_{k-2}^{(2)}(\underline{x}) = -\partial_{\underline{x}} u'_{k-1}(\underline{x})$ is a homogeneous polynomial of degree $(k-2)$ in $\mathbb{R}^m$.

Remark 4.6. When applied to the specific case where dimension $m = 1$, Theorem 4.2 yields the existence of a unique homogeneous conjugate harmonic polynomial $\tilde{v}_k(x_0, x_1)$ to the real-valued homogeneous polynomial $u_k(x_0, x_1)$ in $\mathbb{R}^2$. This conjugate harmonic is $\mathbb{R}^{0,1}$-valued and of the particular form

$$\tilde{v}_k(x_0, x_1) = -e_1 \partial_{x_1} \tilde{H}_{k+1}(x_0, x_1),$$

where

$$\tilde{H}_{k+1}(x_0, x_1) = \int_0^{x_0} u_k(t, x_1)\, dt - \tilde{u}_{k+1}(x_1). \tag{4.10}$$

The corresponding homogeneous monogenic polynomial in $\mathbb{R}^2$ takes the form

$$P_k(x_0, x_1) = (\partial_{x_0} - \varepsilon_1 \partial_{x_1})\tilde{H}_{k+1}(x_0, x_1)$$

which is nothing else but a homogeneous holomorphic polynomial in the plane, where, as already mentioned in Remark 2.4, $\varepsilon_1 = \overline{e}_0 e_1$ takes over the role of the imaginary unit i in $\mathbb{C}$.

References

[1] Brackx, F., Constales, D., Ronveaux, A. and Serras,H., On the harmonic and monogenic decomposition of polynomials. *J. Symbolic Computation* 8 (1989), 297 – 304.

[4] Delanghe, R., Sommen, F. and Souček, V., *Clifford Algebra and Spinor-Valued Functions.* Dordrecht: Kluwer 1992.

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