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Best Possible Maximum Principles for Fully Nonlinear Elliptic Partial Differential Equations

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Abstract. We investigate a class of equations including generalized Monge–Ampere equations as well as Weingarten equations and prove a maximum principle for suitable functions involving the solution and its gradient. Since the functions which enjoy the maximum principles are constant for special domains, we have a so called best possible maximum principle that can be used to find accurate estimates for the solution of the corresponding Dirichlet problem. For these equations we also give a variational form which may have its own interest.

Keywords. Fully nonlinear elliptic equations, Weingarten surfaces, best possible maximum principles

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let u be a smooth function defined in Ω. If $x = (x_1 \dots, x_N) \in \Omega$ we put $u_i = \frac{\partial u}{\partial x_i}$, $i = 1, \cdots, N$, and define $A = [u_{ij}]$, the (symmetric) Hessian matrix of u. For $1 \leq k \leq N$ let

$$S^{(k)}(\lambda_1, \dots, \lambda_N)$$

be the k-th elementary symmetric function of the eigenvalues of A. Famous equations investigated by L. Caffarelli, L. Nirenberg and J. Spruck in [2] are the following:

$$S^{(k)}(\lambda_1, \dots, \lambda_N) = 1. \tag{1.1}$$

Blow-up solutions of the same equations are discussed by P. Salani in [11]. One can generalize equations (1.1) as follows. Let $g(s)$ be a positive smooth real function satisfying

$$G(s) = g(s) + 2sg'(s) > 0. \tag{1.2}$$

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Define the matrix

$$Q = I - \frac{2g'(|\nabla u|^2)}{G(|\nabla u|^2)} R, \quad R = \nabla u (\nabla u)^t, \tag{1.3}$$

where I is the $N \times N$ unit matrix and $(\nabla u)^t$ is the transposed matrix of ∇u. Note that Q is symmetric and positive definite. One finds

$$Q^{-1} = I + \frac{2g'(|\nabla u|^2)}{g(|\nabla u|^2)} R. \tag{1.4}$$

Using the Hessian matrix $A = [u_{ij}]$ of u, define the new matrix

$$E = [e_{ij}] = g(|\nabla u|^2) Q^{-1} A. \tag{1.5}$$

By standard results on matrix theory [1], E is similar to a diagonal matrix with entries $\mu_1, \ldots, \mu_N$, the (real) eigenvalues of E. For $1 \le k \le N$ let $\sigma^{(k)}(\mu_1, \ldots, \mu_N)$ be the k-th elementary symmetric function of the eigenvalues of E. In this paper we deal with the equations

$$\sigma^{(k)}(\mu_1, \ldots, \mu_N) = 1. \tag{1.6}$$

For $g = 1$ we have $E = A$ and equations (1.6) coincide with equations (1.1). For $g = (1+s)^{-\frac{1}{2}}$ we find $Q = I + R$, and we have

$$E = (1 + |\nabla u|^2)^{-\frac{1}{2}} Q^{-1} A.$$

In this situation $\mu_1, \ldots, \mu_N$ coincide with the principal curvatures of the surface $u = u(x)$ in $\mathbb{R}^{N+1}$, and the corresponding equations (1.6) describe the Weingarten surfaces investigated by L. Caffarelli, L. Nirenberg and J. Spruck in [3].

Let $\alpha \in \mathbb{R}$ with

$$0 \le \alpha \le \binom{N}{k}^{-\frac{1}{k}}. \tag{1.7}$$

If $G(s)$ is the function introduced in (1.2) and if α satisfies (1.7), we define

$$\Psi(x) = \frac{1}{2} \int_0^{|\nabla u|^2} G(s) ds - \alpha u$$

In Section 3 we prove that if u is a solution to equation (1.6), then the function Ψ cannot assume its maximum value in Ω unless it is a constant. For $g = 1$ and $g(s) = (1+s)^{-\frac{1}{2}}$ this result has been proved in [8] by G. A. Philippin and the second author. We also show that if Ω is a ball, then we have $\Psi(x) = \textit{constant}$, therefore our result is a best possible maximum principle. In Section 4 we give a variational form for the invariants $\sigma^{(k)}(\mu_1, \ldots, \mu_N)$.

2. Newton tensor

Let E be a matrix similar to the diagonal matrix $\operatorname{diag}\{\mu_1, \dots, \mu_N\}$. For $k = 1, \dots, N-1$, the k-th Newton tensor $T^{(k)}$ is defined as

$$T^{(k)} = \sigma^{(k)} I - T^{(k-1)} E^t, \qquad T^{(0)} = I, \tag{2.1}$$

where I is the $N \times N$ unit matrix, E^t is the transposed of E and $\sigma^{(k)}$ denotes the k-th elementary symmetric function of the eigenvalues of E. By (2.1) one finds

$$T^{(k)} = \sigma^{(k)} I - \sigma^{(k-1)} E^t + \cdots + (-1)^k (E^t)^k.$$

Since $T^{(k)}$ is polynomial in E^t, it is clear that

$$T^{(k)} E^t = E^t T^{(k)}. \tag{2.2}$$

A representation of the invariant $\sigma^{(k)}$ in terms of the entries e_{ij} of the matrix E has been given by R. C. Reilly [9]:

$$\sigma^{(k)} = \frac{1}{k!} \binom{i_1 \cdots i_k}{j_1 \cdots j_k} e_{i_1 j_1} \cdots e_{i_k j_k}, \tag{2.3}$$

where the generalized Kronecker symbol $\binom{i_1 \cdots i_k}{j_1 \cdots j_k}$ has the value 1 (respectively -1) if the indexes $i_1, \dots, i_k$ are distinct and $(j_1, \dots, j_k)$ is an even (respectively odd) permutation of $(i_1, \dots, i_k)$, otherwise it has value zero. Moreover, the summation convention over repeated indexes from 1 to N is in effect.

We also have a formula for the entries $T^{(k)}_{ij}$ of the Newton tensor [9]

$$T^{(k)}_{ij} = \frac{1}{k!} \binom{i_1 \cdots i_k i}{j_1 \cdots j_k j} e_{i_1 j_1} \cdots e_{i_k j_k}. \tag{2.4}$$

By (2.3) and (2.4) we find

$$\operatorname{tr}\{T^{(k)} E^t\} = (k+1)\sigma^{(k+1)}, \tag{2.5}$$

where $\operatorname{tr}\{M\}$ denotes the trace of the matrix M. We shall use the equation

$$T^{(k-1)}_{ij} = \frac{\partial \sigma^{(k)}}{\partial e_{ij}}, \tag{2.6}$$

which follows by (2.3) and (2.4).

3. Maximum principles

Let k be a fixed integer with $1 \le k \le N$, and let $g(s)$ be a smooth positive function satisfying

$$G(s) = g(s) + 2s\, g'(s) > 0 \tag{3.1}$$

for $s \ge 0$. Typical examples are $g(s) = (1+\gamma s)^{-\frac{1}{2}}$, $\gamma \ge 0$. For $p \in \mathbb{R}^N$ define

$$Q = I - \frac{2g'(|p|^2)}{G(|p|^2)} R, \quad R = p\, p^t,$$

where I is the $N \times N$ unit matrix, p is used as a column matrix, p^t is its transposed. Note that Q is symmetric and positive definite. Indeed, the eigenvalues of Q are 1 (counted $N-1$ times) and $\frac{g}{G}$. We have

$$Q^{-1} = I + \frac{2g'(|p|^2)}{g(|p|^2)} R.$$

Let $u = u(x)$ be a smooth function defined in a bounded domain $\Omega \subset \mathbb{R}^N$. If $A = [u_{ij}]$ is the Hessian matrix of u, define the new matrix

$$E = E(A,p) = g(|p|^2) Q^{-1} A. \tag{3.2}$$

Although E is not, in general, symmetric, it is similar to a diagonal matrix with entries $\mu_1, \dots, \mu_N$, the (real) eigenvalues of E. If $p = \nabla u$ and $g(s) = (1+s)^{-\frac{1}{2}}$ then $\mu_1, \dots, \mu_N$ are the principal curvatures of the surface $u = u(x)$ contained in $\mathbb{R}^{N+1}$.

Let $F(E) = \sigma^{(k)}(\mu_1, \dots, \mu_N)$, $E = [e_{ij}]$, the k-th elementary symmetric function of the eigenvalues of E. Since E depends on A and p, we introduce the notation $Z(A,p) = F(E)$, and define the associated matrices

$$A^* = \left[\frac{\partial Z}{\partial u_{ij}}\right], \qquad E^* = \left[\frac{\partial F}{\partial e_{ij}}\right].$$

Since F is homogeneous with respect to e_{ij}, also Z is homogeneous with respect to u_{ij} and

$$\operatorname{tr}\{A^* A\} = \operatorname{tr}\{E^* E^t\} = k\, F(E). \tag{3.3}$$

By using the chain rule one finds

$$A^* = g(|p|^2) Q^{-1} E^*. \tag{3.4}$$

By (2.6) we have

$$E^* = \left[\frac{\partial \sigma^{(k)}}{\partial e_{ij}}\right] = T^{(k-1)},$$

where $T^{(k-1)}=T^{(k-1)}(E^t)$ is the $(k-1)$-th Newton tensor associated with the matrix E^t. By (2.2) we have

$$E^*E^t = E^tE^*. \tag{3.5}$$

We have the following

Lemma 3.1. *Let $h \in \mathbb{R}^N$ be an arbitrary column vector and let $W = ph^t + hp^t$. Then,*

$$\frac{\partial Z(A,p)}{\partial p}h = \mathrm{tr}\left\{E^*E^t\left[\frac{2g'}{G}WQ^{-1} + 4\left(\frac{g'}{G}\right)'p^thRQ^{-1} + \frac{2g'}{g}p^thI\right]\right\}.$$

Proof. Since

$$Q(p) = I - \frac{2g'(|p|^2)}{G(|p|^2)}R, \quad R = p\,p^t,$$

we have

$$\begin{aligned} Q(p+h) &= I - \frac{2g'(|p+h|^2)}{G(|p+h|^2)}(p+h)(p+h)^t \\ &= I - \left(\frac{2g'}{G} + 4\left(\frac{g'}{G}\right)'p^th + \cdots\right)(R + W + \cdots) \\ &= Q(p) - \frac{2g'}{G}W - 4\left(\frac{g'}{G}\right)'p^thR + \cdots \\ &= Q\left[I - \frac{2g'}{G}Q^{-1}W - 4\left(\frac{g'}{G}\right)'p^thQ^{-1}R + \cdots\right], \end{aligned}$$

where dots stand for terms of higher order with respect to $|h|$. Hence,

$$\begin{aligned} Q^{-1}(p+h) &= \left[I + \frac{2g'}{G}Q^{-1}W + 4\left(\frac{g'}{G}\right)'p^thQ^{-1}R + \cdots\right]Q^{-1} \\ &= Q^{-1} + \frac{2g'}{G}Q^{-1}WQ^{-1} + 4\left(\frac{g'}{G}\right)'p^thQ^{-1}RQ^{-1} + \cdots \end{aligned}$$

Using this expansion we find

$$\begin{aligned} &E(A, p+h) \\ &= g(|p+h|^2)Q^{-1}(p+h)A \\ &= \left(g + 2g'p^th + \cdots\right)\left(Q^{-1}A + \frac{2g'}{G}Q^{-1}WQ^{-1}A + 4\left(\frac{g'}{G}\right)'p^thQ^{-1}RQ^{-1}A + \cdots\right) \\ &= E + \frac{2g'}{G}Q^{-1}WE + 4\left(\frac{g'}{G}\right)'p^thQ^{-1}RE + \frac{2g'}{g}p^thE + \cdots . \end{aligned}$$

Using the previous formula we find

$$\begin{aligned}
&Z(A,p+h)\\
&\quad = F\big(E(A,p+h)\big)\\
&\quad = F\left(E + \frac{2g'}{G}Q^{-1}WE + 4\Big(\frac{g'}{G}\Big)'p^t hQ^{-1}RE + \frac{2g'}{g}p^t hE + \cdots\right)\\
&\quad = Z(A,p) + \operatorname{tr}\left\{E^*E^t\left[\frac{2g'}{G}WQ^{-1} + 4\Big(\frac{g'}{G}\Big)'p^t hRQ^{-1} + \frac{2g'}{g}p^t hI + \cdots\right]\right\}.
\end{aligned}$$

The lemma follows. □

For $p = \nabla u$ and $h = A\nabla u$ we have $R = \nabla u(\nabla u)^t$, $W = RA + AR$. Using Lemma 3.1 we find

$$\begin{aligned}
\frac{\partial Z(A,\nabla u)}{\partial(\nabla u)}A\nabla u &= \operatorname{tr}\Big\{E^*E^t\frac{2g'}{G}(RA+AR)Q^{-1}\Big\}\\
&\qquad + (\nabla u)^t A\nabla u\ \operatorname{tr}\Big\{E^*E^t\Big[4\Big(\frac{g'}{G}\Big)'RQ^{-1} + \frac{2g'}{g}I\Big]\Big\}.
\end{aligned}$$

Using equation (1.3) we find $\frac{2g'}{G}(RA+AR) = 2A - QA - AQ$. Hence, recalling that $A = \frac{1}{g}QE = \frac{1}{g}E^tQ$, we have

$$\begin{aligned}
\operatorname{tr}\Big\{E^*E^t\frac{2g'}{G}(RA+AR)Q^{-1}\Big\} &= \operatorname{tr}\big\{E^*E^t(2AQ^{-1} - QAQ^{-1} - A)\big\}\\
&= \frac{1}{g}\operatorname{tr}\big\{E^*E^t(2E^t - QE^t - E^tQ)\big\}\\
&= \frac{2}{g}\operatorname{tr}\big\{E^*(E^t)2(I-Q)\big\}\\
&= \frac{4g'}{g\,G}\operatorname{tr}\big\{E^*(E^t)^2R\big\}.
\end{aligned}$$

We have used (1.3) and the equation $\operatorname{tr}\big\{E^*E^tQE^t\big\} = \operatorname{tr}\big\{E^*(E^t)^2Q\big\}$, true because $E^*E^t = E^tE^*$. Therefore,

$$\begin{aligned}
&\frac{\partial Z(A,\nabla u)}{\partial(\nabla u)}A\nabla u\\
&= \frac{4g'}{g\,G}\operatorname{tr}\big\{E^*(E^t)^2R\big\} + (\nabla u)^t A\nabla u\ \operatorname{tr}\Big\{E^*E^t\Big[4\Big(\frac{g'}{G}\Big)'RQ^{-1} + \frac{2g'}{g}I\Big]\Big\}.
\end{aligned} \tag{3.6}$$

Our main result deals with the following equation:

$$Z(A,\nabla u) = F(E) = \sigma^{(k)}(\mu_1,\ldots,\mu_N) = 1. \tag{3.7}$$

The Dirichlet problem for equation (3.7) in case of $g = 1$ or $g = (1+s)^{-\frac{1}{2}}$ has been investigated in [2] and [3]. A solution to (3.7) is called *admissible* if the corresponding matrix $A^* = \left[\frac{\partial Z}{\partial u_{ij}}\right]$ is positive definite. We refer to [2, 3, 8, 11] for existence, uniqueness and regularity results.

Theorem 3.2. *Let α satisfy* (1.7)*, let g satisfy* (3.1) *and let $u(x)$ be an admissible solution of the nonlinear elliptic equation* (3.7) *in a bounded domain $\Omega \subset \mathbb{R}^N$. Then the function*

$$\Psi(x) = \frac{1}{2}\int_0^{|\nabla u|^2} G(s)\,ds - \alpha u \tag{3.8}$$

cannot assume its maximum value in Ω unless it is a constant.

Proof. Differentiating Ψ we find

$$\begin{aligned} \Psi_i &= G u_{i\ell} u_\ell - \alpha\, u_i \\ \Psi_{ij} &= 2G' u_{i\ell} u_\ell u_{js} u_s + G u_{ij\ell} u_\ell + G u_{i\ell} u_{j\ell} - \alpha u_{ij}. \end{aligned} \tag{3.9}$$

With $B = [\Psi_{ij}]$ we have

$$\operatorname{tr}\{A^*B\} = 2G' \operatorname{tr}\{A^*ARA\} + G \operatorname{tr}\{A^*A_\ell\}u_\ell + G \operatorname{tr}\{A^*A2\} - \alpha \operatorname{tr}\{A^*A\}. \tag{3.10}$$

Using the equations (3.2), (3.4) with $p = \nabla u$, and recalling that $QE = E^tQ$ (by (3.2)) and that $E^*E^t = E^tE^*$ (by (3.5)) we find

$$\operatorname{tr}\{A^*A2\} = \operatorname{tr}\{A^2A^*\} = \frac{1}{g}\operatorname{tr}\{E^tQE^tE^*\} = \frac{1}{g}\operatorname{tr}\{E^*(E^t)^2Q\}. \tag{3.11}$$

Consider first $g' \neq 0$. Using (1.3) and (3.11) we find

$$\begin{aligned} \operatorname{tr}\{A^*ARA\} &= \frac{G}{2g'}\operatorname{tr}\{A^*A(I-Q)A\} \\ &= \frac{G}{2g'}\operatorname{tr}\{A^*A2\} - \frac{G}{2g'}\operatorname{tr}\{A^*AQA\} \\ &= \frac{G}{2gg'}\Big[\operatorname{tr}\{E^*(E^t)^2Q\} - \operatorname{tr}\{Q^{-1}E^*E^tQ^2E^tQ\}\Big] \\ &= \frac{G}{2gg'}\operatorname{tr}\{E^*(E^t)2(Q - Q2)\}. \end{aligned}$$

Using the last equation and the formula $Q - Q2 = \frac{2g'g}{G2}R$ we find

$$\operatorname{tr}\{A^*ARA\} = \frac{1}{G}\operatorname{tr}\{E^*(E^t)^2R\}. \tag{3.12}$$

If $g' = 0$ then $G = g$, $Q = I$ and $E = gA$, therefore (3.12) holds trivially in this situation.

Differentiating the equation $Z(A, \nabla u) = 1$ with respect to x_ℓ and multiplying by u_ℓ we find

$$\frac{\partial Z}{\partial u_{ij}} u_{ij\ell} u_\ell + \frac{\partial Z}{\partial u_i} u_{i\ell} u_\ell = 0,$$

or, using matrix notation,

$$\operatorname{tr}\{A^* A_\ell\} u_\ell + \frac{\partial Z(A, \nabla u)}{\partial(\nabla u)} A \nabla u = 0.$$

The latter equation and (3.6) yield

$$\begin{aligned} \operatorname{tr}\{A^* A_\ell\} u_\ell = & -\frac{4g'}{g\,G} \operatorname{tr}\{E^*(E^t)^2 R\} \\ & - (\nabla u)^t A \nabla u \ \operatorname{tr}\left\{E^* E^t \left[4\Big(\frac{g'}{G}\Big)' R Q^{-1} + \frac{2g'}{g} I\right]\right\}. \end{aligned} \tag{3.13}$$

Since $F(E) = F(e_{ij})$ is homogeneous of degree k with respect to e_{ij}, also $Z(A, \nabla u) = Z(u_{ij}, u_i)$ is homogeneous of degree k with respect to u_{ij} and, using equation (3.7) we find

$$\operatorname{tr}\{A^* A\} = \operatorname{tr}\{E^* E^t\} = k\,\sigma^{(k)} = k. \tag{3.14}$$

Insertion of (3.11), (3.12), (3.13) and (3.14) into (3.10) leads to

$$\begin{aligned} \operatorname{tr}\{A^* B\} = & \frac{2G'}{G} \operatorname{tr}\{E^*(E^t)^2 R\} - \frac{4g'}{g} \operatorname{tr}\{E^*(E^t)^2 R\} \\ & - G(\nabla u)^t A \nabla u \ \operatorname{tr}\left\{E^* E^t \left[4\Big(\frac{g'}{G}\Big)' R Q^{-1} + \frac{2g'}{g} I\right]\right\} \\ & + \frac{G}{g} \operatorname{tr}\{E^*(E^t)^2 Q\} - \alpha k. \end{aligned} \tag{3.15}$$

By (3.9) we get

$$G(\nabla u)^t A \nabla u = \alpha |\nabla u|^2 + (\nabla u)^t \nabla \Psi \tag{3.16}$$

and

$$GAR = \alpha R + \nabla \Psi (\nabla u)^t. \tag{3.17}$$

We claim that

$$\begin{aligned} & \frac{2G'}{G} \operatorname{tr}\{E^*(E^t)^2 R\} - G(\nabla u)^t A \nabla u \ \operatorname{tr}\left\{E^* E^t 4\Big(\frac{g'}{G}\Big)' R Q^{-1}\right\} \\ & = \frac{6g'}{g} \operatorname{tr}\{E^*(E^t)^2 R\} + C^t \nabla \Psi, \end{aligned} \tag{3.18}$$

where C is a regular vector field. Note that $E^tR = gAQ^{-1}R = GAR$, $RQ^{-1} = \frac{G}{g}R$. Using the latter equations and (3.16) we have

$$\begin{aligned}
&\Big(\frac{2G'}{G}-\frac{6g'}{g}\Big)\operatorname{tr}\{E^*(E^t)^2R\} - G(\nabla u)^tA\nabla u\operatorname{tr}\Big\{E^*E^t4\Big(\frac{g'}{G}\Big)'RQ^{-1}\Big\}\\
&\quad = 2\frac{G'g-3g'G}{G\,g}\operatorname{tr}\{E^*E^tGAR\} - G(\nabla u)^tA\nabla u\,4\Big(\frac{g'}{G}\Big)'\frac{G}{g}\operatorname{tr}\{E^*E^tR\}\\
&\quad = 4|\nabla u|^2\frac{g''g-3(g')2}{G\,g}\operatorname{tr}\{E^*E^tGAR\}\\
&\qquad - 4\alpha|\nabla u|^2\frac{g''g-3(g')2}{G\,g}\operatorname{tr}\{E^*E^tR\} + C^t\nabla\Psi.
\end{aligned}$$

The latter equation and (3.17) yield (3.18).

Insertion of (3.18) into (3.15) and use of (3.16) leads to

$$\begin{aligned}
\operatorname{tr}\{A^*B\} &= \frac{6g'}{g}\operatorname{tr}\{E^*(E^t)^2R\} - \frac{4g'}{g}\operatorname{tr}\{E^*(E^t)^2R\} + \frac{G}{g}\operatorname{tr}\{E^*(E^t)^2Q\}\\
&\quad - \alpha|\nabla u|^2\frac{2g'}{g}\operatorname{tr}\{E^*E^t\} - \alpha k + C^t\nabla\Psi.
\end{aligned}$$

Since $\frac{2g'}{g}R + \frac{G}{g}Q = \frac{G}{g}I$ and $\operatorname{tr}\{E^*E^t\} = k$ we find

$$\begin{aligned}
\operatorname{tr}\{A^*B\} + C^t\nabla\Psi &= \frac{G}{g}\operatorname{tr}\{E^*(E^t)2\} - \alpha|\nabla u|^2\frac{2g'}{g}k - \alpha k\\
&= \frac{G}{g}\Big[\operatorname{tr}\{E^*(E^t)2\} - \alpha\,k\Big].
\end{aligned} \tag{3.19}$$

To evaluate the quantity in the square brackets in (3.19) we recall the Newton tensor relative to E^t

$$T^{(k)} = \sigma^{(k)}I - T^{(k-1)}E^t, \qquad T_0 = I, \tag{3.20}$$

introduced in Section 2. Since $E^* = T^{(k-1)}$, by (3.20) we find

$$E^*(E^t)2 = T^{(k-1)}(E^t)2 = \sigma^{(k)}E^t - T^{(k)}E^t.$$

By (2.5) we have $\operatorname{tr}\{T^{(k)}E^t\} = (k+1)\sigma^{(k+1)}$. Hence,

$$\operatorname{tr}\{E^*(E^t)2\} = \sigma^{(k)}\sigma^{(1)} - (k+1)\sigma^{(k+1)}. \tag{3.21}$$

Define $\alpha_k = \binom{N}{k}^{-\frac{1}{k}}$. By standard inequalities (see [11]) we have

$$\sigma^{(k+1)} \le \binom{N}{k+1}\alpha_k^{k+1}\big(\sigma^{(k)}\big)^{\frac{k+1}{k}}$$

and

$$\sigma^{(1)} \geq N\alpha_k (\sigma^{(k)})^{\frac{1}{k}}.$$

Hence, by (3.21) we find

$$\begin{aligned} \mathrm{tr}\{E^*(E^t)2\} &\geq N\alpha_k (\sigma^{(k)})^{\frac{k+1}{k}} - (k+1)\binom{N}{k+1}\alpha_k^{k+1}(\sigma^{(k)})^{\frac{k+1}{k}} \\ &= k\alpha_k (\sigma^{(k)})^{\frac{k+1}{k}} \\ &= k\alpha_k, \end{aligned}$$

where the equation $\sigma^{(k)} = 1$ has been used in the last step. Taking into account this result, from (3.19) we find

$$A^*_{ij}\Psi_{ij} + C^t\nabla\Psi = \mathrm{tr}\{A^*B\} + C^t\nabla\Psi \geq 0.$$

Since the matrix A^* is positive definite, the theorem follows by Hopf's first maximum principle. □

Applications. If u is an admissible solution of equation (3.7) in a bounded domain Ω, then some estimates follow from Theorem 3.1. First of all, taking $\alpha = 0$ it follows that $|\nabla u|$ takes its maximum value on the boundary $\partial\Omega$. Moreover, if $u = 0$ on $\partial\Omega$, then $u(x) < 0$ in Ω and, by Theorem 3.1 with $\alpha = \alpha_k = \binom{N}{k}^{-\frac{1}{k}}$ we have

$$\frac{1}{2}\int_0^{|\nabla u|^2} G(s)\,ds - \alpha_k u \leq \frac{1}{2}\int_0^{|\nabla u|_M 2} G(s)\,ds,$$

where $|\nabla u|_M = \sup_{x\in\partial\Omega} |\nabla u|$. In particular, if $u_m = \min_{x\in\Omega} u(x)$, then

$$-\alpha_k u_m \leq \frac{1}{2}\int_0^{|\nabla u|_M 2} G(s)\,ds. \tag{3.22}$$

Inequality (3.22) becomes an equality if Ω is a ball. If Ω is strictly convex, a bound for $|\nabla u|_M$ can be found in terms of the geometry of $\partial\Omega$. We refer to [8] for details.

The radial case. If $u = u(r)$ with $r = |x|$ we have

$$u_i = u'\,\frac{x_i}{r}, \qquad u_{ij} = u''\,\frac{x_i x_j}{r2} + u'\frac{r2\delta_{ij} - x_i x_j}{r3},$$

where δ_{ij} is the familiar Kroneker delta. At the point $(r, 0, \cdots, 0)$ we have $A = \mathrm{diag}\{u'', \frac{u'}{r}, \ldots, \frac{u'}{r}\}$, $Q^{-1} = \mathrm{diag}\{\frac{G}{g}, 1, \ldots, 1\}$. Hence, $E = g\,\mathrm{diag}\{\frac{G}{g}u'', \frac{u'}{r}, \ldots, \frac{u'}{r}\}$. Since $\sigma^{(k)}$ is invariant under rotations, at a point x with $|x| = r$ we find

$$\sigma^{(k)} = g^k\left[\binom{N-1}{k-1}\frac{G}{g}u''\left(\frac{u'}{r}\right)^{k-1} + \binom{N-1}{k}\left(\frac{u'}{r}\right)^k\right]. \tag{3.23}$$

If $u = u(r)$ is a solution of the ordinary differential equation

$$g((u')2)u' = r\alpha_k, \quad \alpha_k = \binom{N}{k}^{-\frac{1}{k}},$$

we have

$$G\big((u')2\big)u'' = \alpha_k. \tag{3.24}$$

By (3.23) we find $\sigma^{(k)} = 1$. The corresponding function Ψ with $\alpha = \alpha_k$ reads as

$$\Psi(r) = \frac{1}{2}\int_0^{(u')2} G(s)\,ds - \alpha_k u.$$

Using (3.24) we find

$$\frac{d\Psi}{dr} = G\big((u')2\big)u''u' - \alpha_k u' = 0.$$

We conclude that Theorem 3.1 yields a best possible maximum principle.

4. Variational equations

In this section we present a variational form of the invariants $\sigma^{(k)}$. By equation (1.5), for $g = 1$ we have $E = A$, and the equation

$$\frac{1}{k}\Big(T_{ij}^{(k-1)}(A)u_j\Big)_i = S^{(k)}(A)$$

is well known. In case of $g(s) = (1+s)^{-\frac{1}{2}}$ the following formula is proved in [10, p. 381]:

$$\frac{1}{k}\left(\frac{1}{(1+|\nabla u|^2)^{\frac{1}{2}}}T_{ij}^{(k-1)}(E)u_j\right)_i = \sigma^{(k)}(E). \tag{4.1}$$

We prove a similar formula for a general g by using the Newton tensor $T^{(k-1)}(A)$ instead of $T^{(k-1)}(E)$.

Proposition 4.1. *We have*

$$\frac{1}{k}\Big(g^k T_{ij}^{(k-1)}(A)u_j\Big)_i = \sigma^{(k)}(E), \tag{4.2}$$

where $g = g(|\nabla u|^2)$.

Proof. In this proof the Newton tensors $T^{(k)}(A)$ are relative to A and we write $T^{(k)}$ instead of $T^{(k)}(A)$. As usual we denote by $S^{(k)}$ the k-th elementary symmetric function of the eigenvalues of the Hessian matrix A.

Let us develop the left hand side of (4.2). Since the Newton tensor $T^{(k-1)}$ is divergence free we have

$$\frac{1}{k}\left(g^k T_{ij}^{(k-1)} u_j\right)_i = g^{k-1} 2g' T_{ij}^{(k-1)} u_{i\ell} u_\ell u_j + \frac{1}{k} g^k T_{ij}^{(k-1)} u_{ij}. \tag{4.3}$$

Recall the Newton equation $T^{(k)} = S^{(k)} I - T^{(k-1)} A$, or

$$T^{(k-1)} A = S^{(k)} I - T^{(k)}. \tag{4.4}$$

Insertion of (4.4) into (4.3) and use of the equation $T_{ij}^{(k-1)} u_{ij} = kS^{(k)}$ yields

$$\frac{1}{k}\left(g^k T_{ij}^{(k-1)} u_j\right)_i = g^{k-1} 2g' \left[S^{(k)} |\nabla u|^2 - T_{ij}^{(k)} u_i u_j\right] + g^k S^{(k)}. \tag{4.5}$$

Now we develop the right hand side of (4.2). We have

$$\sigma^{(k)}(E) = \sigma^{(k)}(gQ^{-1}A) = g^k \sigma^{(k)}\left[\left(I + \frac{2g'}{g} R\right) A\right]. \tag{4.6}$$

The (ij) entry of the matrix $\big(I + \frac{2g'}{g} R\big) A$ is $u_{ij} + \frac{2g'}{g} u_{j\ell} u_\ell u_i$. Therefore, using Reilly's formula (2.3) we have

$$\begin{aligned}\sigma^{(k)}&\left[\left(I + \frac{2g'}{g} R\right) A\right] \\ &= \frac{1}{k!}\begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix}\left(u_{i_1 j_1} + \frac{2g'}{g} u_{j_1 \ell_1} u_{\ell_1} u_{i_1}\right) \cdots \left(u_{i_k j_k} + \frac{2g'}{g} u_{j_k \ell_k} u_{\ell_k} u_{i_k}\right).\end{aligned} \tag{4.7}$$

When we develop the product in above we find the term

$$\frac{1}{k!}\begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix} u_{i_1 j_1} \cdots u_{i_k j_k} = S^{(k)}. \tag{4.8}$$

Moreover we find k terms of the kind

$$\frac{2g'}{g} \frac{1}{k!}\begin{pmatrix} i_1 \cdots i_{k-1} i \\ j_1 \cdots j_{k-1} j \end{pmatrix} u_{i_1 j_1} \cdots u_{i_{k-1} j_{k-1}} u_{j\ell} u_\ell u_i.$$

We can write the above k terms globally as $\frac{2g'}{g} T_{ij}^{(k-1)} u_{j\ell} u_\ell u_i$. Using (4.4), the above expression reads as

$$\frac{2g'}{g}\left[S^{(k)} |\nabla u|^2 - T_{ij}^{(k)} u_i u_j\right]. \tag{4.9}$$

All the remaining terms appearing in the product (4.7) vanish because of the skew-symmetry of the generalized Kronecker symbols. Let us consider in detail the expression

$$\frac{(2g')2}{g2}\frac{1}{k!}\binom{i_1 \cdots i_{k-2} i\, m}{j_1 \cdots j_{k-2} j\, n} u_{i_1 j_1} \cdots u_{i_{k-2} j_{k-2}} u_{j\ell} u_\ell u_i u_{ns} u_s u_m.$$

The term corresponding to $\binom{i_1 \cdots i_{k-2} i\, m}{j_1 \cdots j_{k-2} j\, n} u_i u_m$ cancels the term corresponding to $\binom{i_1 \cdots i_{k-2} i\, m}{j_1 \cdots j_{k-2} n\, j} u_m u_i$. Hence, the insertion of (4.8) and (4.9) into (4.7) yields

$$\sigma^{(k)}\left[\left(I + \frac{2g'}{g} R\right) A\right] = S^{(k)} + \frac{2g'}{g}\left[S^{(k)}|\nabla u|^2 - T^{(k)}_{ij} u_i u_j\right]. \tag{4.10}$$

Finally, insertion of (4.10) into (4.6) leads to

$$\begin{aligned} \sigma^{(k)}(E) &= g^k \left[S^{(k)} + \frac{2g'}{g}\left[S^{(k)}|\nabla u|^2 - T^{(k)}_{ij} u_i u_j\right]\right] \\ &= g^k S^{(k)} + g^{k-1} 2g' \left[S^{(k)}|\nabla u|^2 - T^{(k)}_{ij} u_i u_j\right]. \end{aligned} \tag{4.11}$$

Comparing (4.11) with (4.5), the proposition follows. □

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