

# Lipschitz Stability of Solutions to Some State-Constrained Elliptic Optimal Control Problems

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**Abstract.** In this paper, optimal control problems with pointwise state constraints for linear and semilinear elliptic partial differential equations are studied. The problems are subject to perturbations in the problem data. Lipschitz stability with respect to perturbations of the optimal control and the state and adjoint variables is established initially for linear–quadratic problems. Both the distributed and Neumann boundary control cases are treated. Based on these results, and using an implicit function theorem for generalized equations, Lipschitz stability is also shown for an optimal control problem involving a semilinear elliptic equation.

**Keywords.** Optimal control, elliptic equations, state constraints, Lipschitz stability

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## 1. Introduction

In this paper, we consider optimal control problems on bounded domains  $\Omega \subset \mathbb{R}^N$  of the form:

$$\text{Minimize } \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|u - u_d\|_{L^2(\Omega)}^2 \quad (1)$$

for the control  $u$  and state  $y$ , subject to linear or semilinear elliptic partial differential equations. For instance, in the linear case with distributed control  $u$  we have

$$-\Delta y + a_0 y = u \quad \text{on } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \quad (2a)$$

while the boundary control case reads

$$-\Delta y + a_0 y = f \quad \text{on } \Omega, \quad \frac{\partial y}{\partial n} + \beta y = u \quad \text{on } \partial\Omega. \quad (2b)$$

Instead of the Laplace operator, an elliptic operator in divergence form is also

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permitted. Moreover, the problem is subject to pointwise state constraints

$$y_a \leq y \leq y_b \quad \text{on } \Omega \text{ (or } \overline{\Omega}), \quad (3)$$

where  $y_a$  and  $y_b$  are the lower and upper bound functions, respectively. Unless otherwise specified,  $y_a$  and  $y_b$  may be arbitrary functions with values in  $\mathbb{R} \cup \{\pm\infty\}$  such that  $y_a \leq y_b$  holds everywhere. Problems of type (1)–(3) appear as subproblems after linearization of semilinear state-constrained optimal control problems, such as the example considered in Section 3, but they are also of independent interest.

Under suitable conditions, one can show the existence of an adjoint state and a Lagrange multiplier associated with the state constraint (3). We refer to [9] for distributed control of elliptic equations and [6, 10, 12, 13] for their boundary control. We also mention [7, 8, 33] and [3–5, 7, 11, 31–33] for distributed and boundary control, respectively, of parabolic equations. In the distributed case, the optimality system comprises

$$\text{the state equation} \quad -\Delta y + a_0 y = u \quad \text{on } \Omega \quad (4)$$

$$\text{the adjoint equation} \quad -\Delta \lambda = -(y - y_d) - \mu \quad \text{on } \Omega \quad (5)$$

$$\text{the optimality condition} \quad \gamma(u - u_d) - \lambda = 0 \quad \text{on } \Omega, \quad (6)$$

and a complementarity condition for the multiplier  $\mu$  associated with the state constraint (3).

In this paper, we extend the above-mentioned results by proving the Lipschitz stability of solutions for semilinear and linear elliptic state-constrained optimal control problems with respect to perturbations of the problem data. We begin by showing that the linear–quadratic problem (1)–(3) admits solutions which depend Lipschitz continuously on particular perturbations  $\delta = (\delta_1, \delta_2, \delta_3)$  of the right hand sides in the first order optimality system (4)–(6), i.e.,

$$\begin{aligned} -\Delta \lambda + (y - y_d) + \mu &= \delta_1 & \text{on } \Omega \\ \gamma(u - u_d) - \lambda &= \delta_2 & \text{on } \Omega \\ -\Delta y + a_0 y - u &= \delta_3 & \text{on } \Omega \end{aligned}$$

in the case of distributed control. The perturbations  $\delta_1$  and  $\delta_2$  generate additional linear terms in the objective (1). Our main result for the linear–quadratic cases is given in Theorems 2.3 and 4.3, for distributed and boundary control, respectively. It has numerous applications: Firstly, it may serve as a starting point to prove the convergence of numerical algorithms for nonlinear state-constrained optimal control problems. The central notion in this context is the strong regularity property of the first order necessary conditions, which precisely requires their linearization to possess the Lipschitz stability proved in this paper, compare [2]. Secondly, proofs of convergence of the discrete to

the continuous solution as the mesh size tends to zero are also based on the strong regularity property, see, e.g., [26]. Thirdly, our results ensure the well-posedness of problem (1)–(3) in the following sense: If the optimality system is solved only up to a residual  $\delta$  (for instance, when solving it numerically), our stability result implies that the approximate solution found is the exact and nearby solution of a perturbed problem. Fourthly, our results can be used to prove the Lipschitz stability for optimal control problems with *semilinear* elliptic equations and with respect to more *general perturbations* by means of Dontchev’s implicit function theorem for generalized equations, see [14]. We illustrate this technique in Section 3.

To the author’s knowledge, the Lipschitz dependence of solutions in optimal control of partial differential equations (PDEs) in the presence of pointwise state constraints has not yet been studied. Most existing results concern control-constrained problems: Malanowski and Tröltzsch [28] prove Lipschitz dependence of solutions for a control-constrained optimal control problem for a linear elliptic PDE subject to nonlinear Neumann boundary control. In the course of their proof, the authors establish the Lipschitz property also for the linear–quadratic problem obtained by linearization of the first order necessary conditions. In [36], Tröltzsch proves the Lipschitz stability for a linear–quadratic optimal control problem involving a parabolic PDE. In Malanowski and Tröltzsch [27], this result is extended to obtain Lipschitz stability in the case of a semilinear parabolic equation. In the same situation, Malanowski [25] has recently proved parameter differentiability. This result is extended in [18, 19] to an optimal control problem governed by a system of semilinear parabolic equations, and numerical results are provided there. All of the above citations cover the case of pointwise *control constraints*. Note also that the general theory developed in [23] does not apply to the problems treated in the present paper since the hypothesis of surjectivity [23, (H3)] is not satisfied for bilateral state constraints (3).

The case of state-constrained optimal control problems governed by *ordinary* differential equations was studied in [15, 24]. The analysis in these papers relies heavily on the property that the state constraint multiplier  $\mu$  is Lipschitz on the interval  $[0, T]$  of interest (see, e.g., [22]), so it cannot be applied to the present situation.

The remainder of this paper is organized as follows: In Section 2, we establish the Lipschitz continuity with respect to perturbations of optimal solutions in the linear–quadratic distributed control case, in the presence of pointwise state constraints. In Section 3, we use these results to obtain Lipschitz stability also for a problem governed by a semilinear equation with distributed control, and with respect to a wider set of perturbations. Finally, Section 4 is devoted to the case of Neumann (co-normal) boundary control in the linear–quadratic case.

Throughout, let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  for some  $N \in \mathbb{N}$ , and let  $\overline{\Omega}$  denote its closure. By  $C(\overline{\Omega})$  we denote the space of continuous functions on  $\overline{\Omega}$ , endowed with the norm of uniform convergence.  $C_0(\Omega)$  is the subspace of  $C(\overline{\Omega})$  of functions with zero trace on the boundary. The dual spaces of  $C(\overline{\Omega})$  and  $C_0(\Omega)$  are known to be  $\mathcal{M}(\overline{\Omega})$  and  $\mathcal{M}(\Omega)$ , the spaces of finite signed regular measures with the total variation norm, see for instance [17, Proposition 7.16] or [35, Theorem 6.19]. Finally, we denote by  $W^{m,p}(\Omega)$  the Sobolev space of functions on  $\Omega$  whose distributional derivatives up to order  $m$  are in  $L^p(\Omega)$ , see Adams [1]. In particular, we write  $H^m(\Omega)$  instead of  $W^{m,2}(\Omega)$ . The space  $W_0^{m,p}(\Omega)$  is the closure of  $C_c^\infty(\Omega)$  (the space of infinitely differentiable functions on  $\Omega$  with compact support) in  $W^{m,p}(\Omega)$ .

## 2. Linear–quadratic distributed control

Throughout this section, we are concerned with optimal control problems governed by a state equation with an elliptic operator in divergence form and distributed control. As delineated in the introduction, the problem depends on perturbation parameters  $\delta = (\delta_1, \delta_2, \delta_3)$ :

$$\text{Minimize } \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|u - u_d\|_{L^2(\Omega)}^2 - \langle y, \delta_1 \rangle_{W, W'} - \int_{\Omega} u \delta_2 \quad (7)$$

$$\text{over } u \in L^2(\Omega)$$

$$\text{s.t. } -\operatorname{div}(A\nabla y) + a_0 y = u + \delta_3 \quad \text{on } \Omega \quad (8)$$

$$y = 0 \quad \text{on } \partial\Omega \quad (9)$$

$$\text{and } y_a \leq y \leq y_b \quad \text{on } \Omega. \quad (10)$$

We work with the state space  $W = H^2(\Omega) \cap H_0^1(\Omega)$  so that the pointwise state constraint (10) is meaningful. The perturbations are introduced below. Let us fix the standing assumption for this section:

*Assumption 2.1.* Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \in \{1, 2, 3\}$ ) with  $C^{1,1}$  boundary  $\partial\Omega$ , see [20, p. 5]. The state equation is governed by an operator with  $N \times N$  symmetric coefficient matrix  $A$  with entries  $a_{ij}$  which are Lipschitz continuous on  $\overline{\Omega}$ . We assume the condition of uniform ellipticity: There exists  $m_0 > 0$  such that

$$\xi^\top A \xi \geq m_0 |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and almost all } x \in \overline{\Omega}.$$

The coefficient  $a_0 \in L^\infty(\Omega)$  is assumed to be nonnegative a.e. on  $\Omega$ . Moreover,  $y_d$  and  $u_d$  denote desired states and controls in  $L^2(\Omega)$ , respectively, while  $\gamma$  is a positive number. The bounds  $y_a$  and  $y_b$  may be arbitrary functions on  $\Omega$  such that the admissible set  $K_W = \{y \in W : y_a \leq y \leq y_b \text{ on } \Omega\}$  is nonempty.

The following result allows us to define the solution operator

$$T_\delta : L^2(\Omega) \rightarrow W$$

such that  $y = T_\delta(u)$  satisfies (8)–(9) for given  $\delta$  and  $u$ . For the proof we refer to [20, Theorems 2.4.2.5 and 2.3.3.2]:

**Proposition 2.2** (The State Equation). *Given  $u$  and  $\delta_3$  in  $L^2(\Omega)$ , the state equation (8)–(9) has a unique solution  $y \in W$  in the sense that (8) is satisfied almost everywhere on  $\Omega$ . The solution verifies the a priori estimate*

$$\|y\|_{H^2(\Omega)} \leq c_A \|u + \delta_3\|_{L^2(\Omega)}. \tag{11}$$

In order to apply the results of this section to prove the Lipschitz stability of solutions in the semilinear case in Section 3, we consider here very general perturbations

$$(\delta_1, \delta_2, \delta_3) \in W' \times L^2(\Omega) \times L^2(\Omega),$$

where  $W'$  is the dual of the state space  $W$ . Of course, this comprises more regular perturbations. In particular, (7) includes perturbations of the desired state in view of

$$\frac{1}{2} \|y - (y_d + \delta_1)\|_{L^2(\Omega)}^2 = \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 - \int_\Omega y \delta_1 + c$$

where  $c$  is a constant. Likewise,  $\delta_2$  covers perturbations in the desired control  $u_d$ , and  $\delta_3$  accounts for perturbations in the right hand side of the PDE.

We can now state the main result of this section which proves the Lipschitz stability of the optimal state and control with respect to perturbations. It relies on a variational argument and does not invoke any dual variables.

**Theorem 2.3** (Lipschitz Continuity). *For any  $\delta = (\delta_1, \delta_2, \delta_3) \in W' \times L^2(\Omega) \times L^2(\Omega)$ , problem (7)–(10) has a unique solution. Moreover, there exists a constant  $L > 0$  such that for any two perturbations  $(\delta'_1, \delta'_2, \delta'_3)$  and  $(\delta''_1, \delta''_2, \delta''_3)$ , the corresponding solutions of (7)–(10) satisfy*

$$\begin{aligned} & \|y' - y''\|_{H^2(\Omega)} + \|u' - u''\|_{L^2(\Omega)} \\ & \leq L \left( \|\delta'_1 - \delta''_1\|_{W'} + \|\delta'_2 - \delta''_2\|_{L^2(\Omega)} + \|\delta'_3 - \delta''_3\|_{L^2(\Omega)} \right). \end{aligned}$$

*Proof.* Let  $\delta \in W' \times L^2(\Omega) \times L^2(\Omega)$  be arbitrary. We introduce the shifted control variable  $v := u + \delta_3$  and define

$$\begin{aligned} \tilde{f}(y, v, \delta) &= \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|v - u_d - \delta_3\|_{L^2(\Omega)}^2 \\ & \quad - \langle y, \delta_1 \rangle_{W, W'} - \int_\Omega (v - \delta_3) \delta_2. \end{aligned}$$

Obviously, our problem is now to

$$\text{minimize } \tilde{f}(y, v, \delta) \quad \text{subject to } (y, v) \in M$$

where  $M = \{(y, v) \in K_W \times L^2(\Omega) : -\text{div}(A\nabla y) + a_0 y = v \text{ on } \Omega\}$ . Due to Assumption 2.1, the feasible set  $M$  is nonempty, closed and convex and also independent of  $\delta$ . In view of  $\gamma > 0$  and the a priori estimate (11), the objective is strictly convex. It is also weakly lower semicontinuous and radially unbounded, hence it is a standard result from convex analysis [16, Chapter II, Proposition 1.2] that (7)–(10) has a unique solution  $(y, u) \in W \times L^2(\Omega)$  for any  $\delta$ .

A necessary and sufficient condition for optimality is

$$\tilde{f}_y(y, v, \delta)(\bar{y} - y) + \tilde{f}_v(y, v, \delta)(\bar{v} - v) \geq 0 \quad \text{for all } (\bar{y}, \bar{v}) \in M. \quad (12)$$

Now let  $\delta'$  and  $\delta''$  be two perturbations with corresponding solutions  $(y', v')$  and  $(y'', v'')$ . From the variational inequality (12), evaluated at  $(y', v')$  and with  $(\bar{y}, \bar{v}) = (y'', v'')$  we obtain

$$\begin{aligned} \int_{\Omega} (y - y_d)(y'' - y') + \gamma \int_{\Omega} (v' - u_d - \delta'_3)(v'' - v') \\ - \langle y'' - y', \delta'_1 \rangle_{W, W'} - \int_{\Omega} (v'' - v') \delta'_2 \geq 0 \end{aligned}$$

By interchanging the roles of  $(y', v')$  and  $(y'', v'')$  and adding the inequalities, we obtain

$$\begin{aligned} \|y' - y''\|_{L^2(\Omega)}^2 + \gamma \|v' - v''\|_{L^2(\Omega)}^2 \\ \leq \langle y' - y'', \delta'_1 - \delta''_1 \rangle_{W, W'} + \gamma \int_{\Omega} (v' - v'')(\delta'_3 - \delta''_3) + \int_{\Omega} (v' - v'')(\delta'_2 - \delta''_2) \\ \leq \|y' - y''\|_{H^2(\Omega)} \|\delta'_1 - \delta''_1\|_{W'} \\ + \|v' - v''\|_{L^2(\Omega)} \left( \gamma \|\delta'_3 - \delta''_3\|_{L^2(\Omega)} + \|\delta'_2 - \delta''_2\|_{L^2(\Omega)} \right). \end{aligned}$$

Using the a priori estimate (11), the left hand side can be replaced by

$$\frac{\gamma}{2} \|v' - v''\|_{L^2(\Omega)}^2 + \frac{\gamma}{2c_A^2} \|y' - y''\|_{H^2(\Omega)}^2.$$

Now we apply Young's inequality to the right hand side and absorb the terms involving the state and control into the left hand side, which yields the Lipschitz stability of  $y$  and  $v$ , hence also of  $u$ .  $\square$

As a precursor for the semilinear case in Section 3, we recall in Proposition 2.4 a known result concerning the adjoint state and the Lagrange multiplier associated with problem (7)–(10).

**Proposition 2.4.** *Let  $\delta \in W' \times L^2(\Omega) \times L^2(\Omega)$  be a given perturbation and let  $(y, u)$  be the corresponding unique solution of (7)–(10). If  $K_W$  has nonempty interior, then there exists a unique adjoint variable  $\lambda \in L^2(\Omega)$  and unique Lagrange multiplier  $\mu \in W'$  such that the following holds:*

$$-\int_{\Omega} \lambda \operatorname{div}(A\nabla \bar{y}) + \int_{\Omega} a_0 \lambda \bar{y} = -\int_{\Omega} (y - y_d) \bar{y} + \langle \bar{y}, \delta_1 - \mu \rangle_{W, W'} \quad \forall \bar{y} \in W \quad (13)$$

$$\langle \bar{y}, \mu \rangle_{W, W'} \leq \langle y, \mu \rangle_{W, W'} \quad \forall \bar{y} \in K_W \quad (14)$$

$$\gamma(u - u_d) - \lambda = \delta_2 \quad \text{on } \Omega. \quad (15)$$

*Proof.* Let  $\tilde{y}$  be an interior point of  $K_W$ . Since  $T'_\delta(u)$  is an isomorphism from  $L^2(\Omega) \rightarrow W$ ,  $\tilde{u}$  can be chosen such that  $\tilde{y} = T_\delta(u) + T'_\delta(u)(\tilde{u} - u)$ , hence a Slater condition is satisfied. The rest of the proof can be carried out along the lines of Casas [9], or using the abstract multiplier theorem [10, Theorem 5.2].  $\square$

In the proposition above, we have assumed that  $K_W$  has nonempty interior. This is not a very restrictive assumption, as any  $\tilde{y} \in K_W$  satisfying  $\tilde{y} - y_a \geq \varepsilon$  and  $y_b - \tilde{y} \geq \varepsilon$  on  $\Omega$  for some  $\varepsilon > 0$  is an interior point of  $K_W$ .

**Remark 2.5.**

1. In [9], it was shown that the state constraint multiplier  $\mu$  is indeed a measure in  $\mathcal{M}(\Omega)$ , i.e.,  $\mu$  has better regularity than just  $W'$ . However, in the following section we will not be able to use this extra regularity.

2. In view of the previous statement, if  $\delta_1 \in \mathcal{M}(\Omega)$ , then so is the right hand side  $-(y - y_d) + \delta_1 - \mu$  of the adjoint equation (13) and thus the adjoint state  $\lambda$  is an element of  $W_0^{1,s}(\Omega)$  for all  $s \in [1, \frac{N}{N-1})$ , see [9].

3. Note that we do not have a stability result for the Lagrange multiplier  $\mu$  so that we cannot use (13) to derive a stability result for the adjoint state  $\lambda$  even in the presence of regular perturbations. This observation is very much in contrast with the control-constrained case, where the control-constraint multiplier does not appear in the adjoint equation’s right hand side and hence the stability of  $\lambda$  can be obtained using an a priori estimate for the adjoint PDE.

4. Nevertheless, from the optimality condition (15) we can derive the Lipschitz estimate

$$\|\lambda' - \lambda''\|_{L^2(\Omega)} \leq (\gamma L + 1) \|\delta' - \delta''\| \quad (16)$$

for the adjoint states belonging to two perturbations  $\delta'$  and  $\delta''$ . However, we use here that the control is distributed on all of  $\Omega$ .

We close this section by another observation: Let  $\delta'$  and  $\delta''$  be two perturbations with associated optimal states  $y'$  and  $y''$  and Lagrange multipliers  $\mu'$  and  $\mu''$ . Then

$$\langle y' - y'', \mu' - \mu'' \rangle_{W, W'} \leq 0$$

holds, as can be inferred directly from (14).

### 3. A semilinear distributed control problem

In this section we show how the Lipschitz stability results for state-constrained linear–quadratic optimal control problems can be transferred to semilinear problems using an appropriate implicit function theorem for generalized equations, see Dontchev [14] and also Robinson [34]. To illustrate this technique, we consider the following parameter-dependent problem  $P(p)$ :

$$\text{Minimize } \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|u - u_d\|_{L^2(\Omega)}^2 \tag{17}$$

$$\text{over } u \in L^2(\Omega)$$

$$\text{s.t. } -D\Delta y + \beta y^3 + \alpha y = u + f \quad \text{on } \Omega \tag{18}$$

$$y = 0 \quad \text{on } \partial\Omega \tag{19}$$

$$\text{and } y_a \leq y \leq y_b \quad \text{on } \Omega. \tag{20}$$

The semilinear state equation is a stationary Ginzburg–Landau model, see [21]. We work again with the state space  $W = H^2(\Omega) \cap H_0^1(\Omega)$ . Throughout this section, we make the following standing assumption:

*Assumption 3.1.* Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \in \{1, 2, 3\}$ ) with  $C^{1,1}$  boundary. Let  $D, \alpha$  and  $\beta$  be positive numbers, and let  $f \in L^2(\Omega)$ . Moreover, let  $y_d$  and  $u_d$  be in  $L^2(\Omega)$  and  $\gamma > 0$ . The bounds  $y_a$  and  $y_b$  may be arbitrary functions on  $\Omega$  such that the admissible set  $K_W = \{y \in W : y_a \leq y \leq y_b \text{ on } \Omega\}$  has nonempty interior.  $\square$

The results obtained in this section can immediately be generalized to the state equation

$$-\text{div} (A\nabla y) + \phi(y) = u + f$$

with appropriate assumptions on the semilinear term  $\phi(y)$ . However, we prefer to consider an example which explicitly contains a number of parameters which otherwise would be hidden in the nonlinearity. In the example above, we can take  $p = (y_d, u_d, f, D, \alpha, \beta, \gamma) \in \Pi = [L^2(\Omega)]^3 \times \mathbb{R}^4$  as the perturbation parameter and we introduce

$$\Pi^+ = \{p \in P : D > 0, \alpha > 0, \beta > 0, \gamma > 0\}.$$

In the sequel, we refer to problem (17)–(20) as  $P(p)$  when we wish to emphasize its dependence on the parameter  $p$ . Note that in contrast to the previous section, the parameter  $p$  now appears in a more complicated fashion which cannot be expressed solely as right hand side perturbations of the optimality system.



**Proposition 3.2** (The State Equation). *For fixed parameter  $p \in \Pi^+$  and for any given  $u$  in  $L^2(\Omega)$ , the state equation (18)–(19) has a unique solution  $y \in W$  in the sense that  $y$  satisfies (18) almost everywhere on  $\Omega$ . The solution depends Lipschitz continuously on the data, i.e., there exists  $c > 0$  such that*

$$\|y - y'\|_{H_0^1(\Omega)} \leq c \|u - u'\|_{L^2(\Omega)}$$

holds for all  $u, u'$  in  $L^2(\Omega)$ . Moreover, the nonlinear solution map

$$T_p : L^2(\Omega) \rightarrow H^2(\Omega) \cap H_0^1(\Omega)$$

defined by  $u \mapsto y$  is Fréchet differentiable. Its derivative  $T'_p(u)\delta u$  at  $u$  in the direction of  $\delta u$  is given by the unique solution  $\delta y$  of

$$\begin{aligned} -D\Delta\delta y + (3\beta y^2 + \alpha)\delta y &= \delta u && \text{on } \Omega \\ \delta y &= 0 && \text{on } \partial\Omega \end{aligned}$$

where  $y = T_p(u)$ . Moreover,  $T'_p(u)$  is an isomorphism from  $L^2(\Omega) \rightarrow W$ .

*Proof.* Existence and uniqueness in  $H_0^1(\Omega)$  of the solution for (18)–(19) and the assertion of Lipschitz continuity follow from the theory of monotone operators, see [37, p. 557], applied to

$$A : H_0^1(\Omega) \ni y \mapsto -D\Delta y + \beta y^3 + \alpha y - f \in H^{-1}(\Omega).$$

Note that  $A$  is strongly monotone, coercive, and hemicontinuous. The solution's  $H^2(\Omega)$  regularity now follows from considering  $\beta y^3$  an additional source term, which is in  $L^2(\Omega)$  due to the Sobolev Embedding Theorem (see [1, p. 97]). Fréchet differentiability of the solution map is a consequence of the implicit function theorem, see, e.g., [38, p. 250]. The isomorphism property of  $T'_p(u)$  follows from Proposition 2.2. Note that  $3\beta y^2 + \alpha \in L^\infty(\Omega)$  since  $y \in L^\infty(\Omega)$ .  $\square$

Before we turn to the main discussion, we state the following existence result for global minimizers:

**Lemma 3.3.** *For any given parameter  $p \in \Pi^+$ ,  $P(p)$  has a global optimal solution.*

*Proof.* The proof follows a standard argument and is therefore only sketched. Let  $\{(y_n, u_n)\}$  be a feasible minimizing sequence for the objective (17). Then  $\{u_n\}$  is bounded in  $L^2(\Omega)$  and, by Lipschitz continuity of the solution map,  $\{y_n\}$  is bounded in  $H_0^1(\Omega)$ . Extracting weakly convergent subsequences, one shows that the weak limit satisfies the state equation (18)–(19). By compactness of the embedding  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  (see [1, p. 144]) and extracting a pointwise a.e. convergent subsequence of  $\{y_n\}$ , one sees that the limit satisfies the state constraint (20). Weak lower semicontinuity of the objective (17) completes the proof.  $\square$

For the remainder of this section, let  $p^* = (y_d^*, u_d^*, f^*, \alpha^*, \beta^*, \gamma^*) \in \Pi^+$  denote a fixed reference parameter. Our strategy for proving the Lipschitz dependence of solutions for  $P(p)$  near  $p^*$  with respect to changes in the parameter  $p$  is as follows:

1. We verify a Slater condition and show that for every local optimal solution of  $P(p^*)$ , there exists an adjoint state and a Lagrange multiplier satisfying a certain first order necessary optimality system (Proposition 3.5).
2. We pick a solution  $(y^*, u^*, \lambda^*)$  of the first order optimality system (for instance the global minimizer) and rewrite the optimality system as a generalized equation.
3. We linearize this generalized equation and introduce new perturbations  $\delta$  which correspond to right hand side perturbations of the optimality system. We *identify* this generalized equation with the optimality system of an auxiliary linear-quadratic optimal control problem  $AQP(\delta)$ , see Lemma 3.7.
4. We assume a coercivity condition (AC) for the Hessian of the Lagrangian at  $(y^*, u^*, \lambda^*)$  and use the results obtained in Section 2 to prove the existence and uniqueness of solutions to  $AQP(\delta)$  and their Lipschitz continuity with respect to  $\delta$ . Consequently, the solutions to the linearized generalized equation from Step 3 are unique and depend Lipschitz continuously on  $\delta$  (Proposition 3.9).
5. In virtue of an implicit function theorem for generalized equations [14], the solutions of the optimality system for  $P(p)$  near  $p^*$  are shown to be locally unique and to depend Lipschitz continuously on the perturbation  $p$  (Theorem 3.10).
6. We verify that the coercivity condition (AC) implies second order sufficient conditions, which are then shown to be stable under perturbations, to the effect that solutions of the optimality system are indeed local optimal solutions of the perturbed problem (Theorem 3.11).

We refer to the individual steps as Step 1–Step 6 and begin with **Step 1**. For the proof of adjoint states and Lagrange multipliers, we verify the following Slater condition:

**Lemma 3.4** (Slater Condition). *Let  $p \in \Pi^+$  and let  $u$  be a local optimal solution for problem  $P(p)$  with optimal state  $y = T_p(u)$ . Then there exists  $\tilde{u}_p \in L^2(\Omega)$  such that*

$$\tilde{y} := T_p(u) + T'_p(u)(\tilde{u}_p - u) \tag{21}$$

*lies in the interior of the set of admissible states  $K_W$ .*

*Proof.* By Assumption 3.1 there exists an interior point  $\tilde{y}$  of  $K_W$ . Since  $T'_p(u)$  is an isomorphism,  $\tilde{u}$  can be chosen such that (21) is satisfied.  $\square$

Using this Slater condition, the following result follows directly from the abstract multiplier theorem in [10, Theorem 5.2]:

**Proposition 3.5** (Lagrange Multipliers). *Let  $p \in \Pi^+$  and let  $(y, u) \in W \times L^2(\Omega)$  be a local optimal solution for problem  $P(p)$ . Then there exists a unique adjoint variable  $\lambda \in L^2(\Omega)$  and unique Lagrange multiplier  $\mu \in W'$  such that*

$$-D \int_{\Omega} \lambda \Delta \bar{y} + \int_{\Omega} (3\beta|y|^2 + \alpha)\lambda \bar{y} = - \int_{\Omega} (y - y_d) \bar{y} - \langle \bar{y}, \mu \rangle_{W, W'} \quad \forall \bar{y} \in W \quad (22)$$

$$\langle \bar{y}, \mu \rangle_{W, W'} \leq \langle y, \mu \rangle_{W, W'} \quad \forall \bar{y} \in K_W \quad (23)$$

$$\gamma(u - u_d) - \lambda = 0 \quad \text{on } \Omega. \quad (24)$$

From now on, we denote by  $(y^*, u^*, \lambda^*)$  a local optimal solution of (17)–(20) for the parameter  $p^*$  with corresponding adjoint state  $\lambda^*$  and multiplier  $\mu^*$ .

Our next **Step 2** is to rewrite the optimality system as a generalized equation in the form  $0 \in F(y, u, \lambda; p) + N(y)$  where  $N$  is a set-valued operator which represents the variational inequality (23) using the dual cone of the admissible set  $K_W$ . We define

$$F : W \times L^2(\Omega) \times L^2(\Omega) \times \Pi \rightarrow W' \times L^2(\Omega) \times L^2(\Omega)$$

$$F(y, u, \lambda; p) = \begin{pmatrix} -D\Delta\lambda + (3\beta y^2 + \alpha)\lambda + (y - y_d) \\ \gamma(u - u_d) - \lambda \\ -D\Delta y + \beta y^3 + \alpha y - u - f \end{pmatrix}$$

and

$$N(y) = \{\bar{\mu} \in W' : \langle \bar{y} - y, \bar{\mu} \rangle_{\Omega} \leq 0 \text{ for all } \bar{y} \in K_W\} \times \{0\} \times \{0\} \subset Z$$

if  $y \in K_W$ , and  $N(y) = \emptyset$  else. The term  $\Delta\lambda$  is understood in the sense of distributions, i.e.,  $\langle \Delta\lambda, \phi \rangle_{W', W} = \int_{\Omega} \lambda \Delta\phi$  for all  $\phi \in W$ .

It is now easy to check that the optimality system (18)–(19), (22)–(23) is *equivalent* to the generalized equation

$$0 \in F(y, u, \lambda; p) + N(y). \quad (25)$$

Hence a solution  $(y, u, \lambda)$  of (25) for given  $p \in \Pi^+$  will be called a *critical point*. For future reference, we summarize the following evident properties of the operator  $F$ :

**Lemma 3.6** (Properties of  $F$ ).

- (a)  $F$  is partially Fréchet differentiable with respect to  $(y, u, \lambda)$  in a neighborhood of  $(y^*, u^*, \lambda^*; p^*)$ . (This partial derivative is denoted by  $F'$ .)
- (b) The map  $(y, u, \lambda; p) \mapsto F'(y, u, \lambda; p)$  is continuous at  $(y^*, u^*, \lambda^*; p^*)$ .

- (c)  $F$  is Lipschitz in  $p$ , uniformly in  $(y, u, \lambda)$  at  $(y^*, u^*, \lambda^*)$ , i.e., there exist  $L > 0$  and neighborhoods  $U$  of  $(y^*, u^*, \lambda^*)$  in  $W \times L^2(\Omega) \times L^2(\Omega)$  and  $V$  of  $p^*$  in  $P$  such that

$$\|F(y, u, \lambda; p_1) - F(y, u, \lambda; p_2)\| \leq L \|p_1 - p_2\|_P$$

for all  $(y, u, \lambda) \in U$  and all  $p_1, p_2 \in V$ .

In Step 3 we set up the following linearization:

$$\delta \in F(y^*, u^*, \lambda^*; p^*) + F'(y^*, u^*, \lambda^*; p^*) \begin{pmatrix} y - y^* \\ u - u^* \\ \lambda - \lambda^* \end{pmatrix} + N(y). \tag{26}$$

For the present example, (26) reads

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} \in \begin{pmatrix} -D^* \Delta \lambda + (3\beta^* |y^*|^2 + \alpha^*) \lambda + 6\beta^* y^* \lambda^* (y - y^*) + y - y_d^* \\ \gamma^* (u - u_d^*) - \lambda \\ -D^* \Delta y + (3\beta^* |y^*|^2 + \alpha^*) y - 2\beta^* (y^*)^3 - u - f^* \end{pmatrix} + N(y). \tag{27}$$

We confirm in Lemma 3.7 below that (27) is exactly the first order optimality system for the following auxiliary linear–quadratic optimal control problem, termed AQP( $\delta$ ):

$$\text{Minimize } \left\{ \frac{1}{2} \|y - y_d^*\|_{L^2(\Omega)}^2 + 3\beta^* \int_{\Omega} y^* \lambda^* (y - y^*)^2 + \frac{\gamma^*}{2} \|u - u_d^*\|_{L^2(\Omega)}^2 - \langle y, \delta_1 \rangle_{W, W'} - \int_{\Omega} u \delta_2 \right\} \tag{28}$$

$$\text{over } u \in L^2(\Omega)$$

$$\text{s.t. } -D^* \Delta y + (3\beta^* |y^*|^2 + \alpha^*) y = u + f^* + 2\beta^* (y^*)^3 + \delta_3 \quad \text{on } \Omega \tag{29}$$

$$y = 0 \quad \text{on } \partial\Omega \tag{30}$$

$$\text{and } y_a \leq y \leq y_b \quad \text{on } \Omega. \tag{31}$$

**Lemma 3.7.** *Let  $\delta \in W' \times L^2(\Omega) \times L^2(\Omega)$  be arbitrary. If  $(y, u) \in W \times L^2(\Omega)$  is a local optimal solution for AQP( $\delta$ ), then there exists a unique adjoint variable  $\lambda \in L^2(\Omega)$  and unique Lagrange multiplier  $\mu \in W'$  such that (27) is satisfied with  $\mu \in N(y)$ .*

*Proof.* We note that the state equation (29)–(30) defines an affine solution operator  $\bar{T} : L^2(\Omega) \rightarrow W$  which turns out to satisfy

$$\bar{T}(u) = T_{p^*}(u^*) + T'_{p^*}(u^*)(u - u^* + \delta_3).$$

Hence if  $u$  is a local optimal solution of (28)–(31) with optimal state  $y = \bar{T}(u)$ , then  $\tilde{y}$  and  $\tilde{u}_{p^*} - \delta_3$ , taken from Lemma 3.4, satisfy the Slater condition

$\tilde{y} = \bar{T}(u) + \bar{T}'(u)(\tilde{u}_{p^*} - \delta_3 - u)$  with  $\tilde{y}$  in the interior of  $K_W$ . Along the lines of Casas [9], or using the abstract multiplier theorem [10, Theorem 5.2], one proves as in Proposition 2.4 that there exist  $\lambda \in L^2(\Omega)$  and  $\mu \in W'$  such that

$$\begin{aligned}
 -D^* \int_{\Omega} \lambda \Delta \bar{y} + \int_{\Omega} [(3\beta^* |y^*|^2 + \alpha^*) \lambda \\
 + 6\beta^* y^* \lambda^* (y - y^*) + y - y_d^*] \bar{y} &= \langle \bar{y}, \delta_1 - \mu \rangle_{W, W'} \quad \forall \bar{y} \in W \\
 \gamma^*(u - u_d^*) - \lambda &= \delta_2 \quad \text{on } \Omega \\
 \langle \mu, \bar{y} - y \rangle_{W', W} &\leq 0 \quad \forall \bar{y} \in K_W
 \end{aligned}$$

hold. That is,

$$-D^* \Delta \lambda + (3\beta^* |y^*|^2 + \alpha^*) \lambda + 6\beta^* y^* \lambda^* (y - y^*) + y - y_d^* - \delta_1 + \mu = 0,$$

and  $\mu \in N(y)$  holds. Hence, (27) is satisfied.  $\square$

In order that AQP( $\delta$ ) has a unique global solution, we assume the following coercivity property:

*Assumption 3.8.* Suppose that at the reference solution  $(y^*, u^*)$  with corresponding adjoint state  $\lambda^*$ , there exists  $\rho > 0$  such that

$$\frac{1}{2} \|y\|_{L^2(\Omega)}^2 + 3\beta^* \int_{\Omega} y^* \lambda^* |y|^2 + \frac{\gamma^*}{2} \|u\|_{L^2(\Omega)}^2 \geq \rho \left( \|y\|_{H^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right) \quad (\text{AC})$$

holds for all  $(y, u) \in W \times L^2(\Omega)$  which obey

$$-D^* \Delta y + (3\beta^* |y^*|^2 + \alpha^*) y = u \quad \text{on } \Omega \tag{32a}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{32b}$$

Note that Assumption 3.8 is satisfied if  $\beta^* \|y^* \lambda^*\|_{L^2(\Omega)}$  is sufficiently small, since then the second term in (AC) can be absorbed into the third.

**Proposition 3.9.** *Suppose that Assumption 3.8 holds and let  $\delta \in W' \times L^2(\Omega) \times L^2(\Omega)$  be given. Then AQP( $\delta$ ) is strictly convex and thus it has a unique global solution. The generalized equation (27) is a necessary and sufficient condition for local optimality, hence (27) is also uniquely solvable. Moreover, the solution depends Lipschitz continuously on  $\delta$ .*

*Proof.* Due to (AC), the quadratic part of the objective (28) is strictly convex, independent of  $\delta$ . Hence we may repeat the proof of Theorem 2.3 with only minor modifications due to the now different objective (28). The existence of a unique adjoint state follows as in Proposition 2.4 and it is Lipschitz in  $\delta$  by (16). We conclude that for any given  $\delta$ , AQP( $\delta$ ) has a unique solution  $(y, u)$  and adjoint state  $\lambda$  which depend Lipschitz continuously on  $\delta$ . In addition, the necessary conditions (27) are sufficient, hence the generalized equation (26) is uniquely solvable and its solution depends Lipschitz continuously on  $\delta$ .  $\square$

We note in passing that the property assured by Proposition 3.9 is called strong regularity of the generalized equation (25). We are now in the position to give our main theorem (Step 5):

**Theorem 3.10** (Lipschitz Stability for  $P(p)$ ). *Let Assumption 3.8 be satisfied. Then there are numbers  $\varepsilon, \varepsilon' > 0$  such that for any two parameter vectors  $(y'_d, u'_d, f', D', \alpha', \beta', \gamma')$  and  $(y''_d, u''_d, f'', D'', \alpha'', \beta'', \gamma'')$  in the  $\varepsilon$ -ball around  $p^*$  in  $\Pi$ , there are critical points  $(y', u', \lambda')$  and  $(y'', u'', \lambda'')$ , i.e., solutions of (25), which are unique in the  $\varepsilon'$ -ball of  $(y^*, u^*, \lambda^*)$ . These solutions depend Lipschitz continuously on the parameter perturbation, i.e., there exists  $L > 0$  such that*

$$\begin{aligned} & \|y' - y''\|_{H^2(\Omega)} + \|u' - u''\|_{L^2(\Omega)} + \|\lambda' - \lambda''\|_{L^2(\Omega)} \\ & \leq L \left( \|y'_d - y''_d\|_{L^2(\Omega)}^2 + \|u'_d - u''_d\|_{L^2(\Omega)}^2 + \|f' - f''\|_{L^2(\Omega)} \right. \\ & \quad \left. + |D' - D''| + |\alpha' - \alpha''| + |\beta' - \beta''| + |\gamma' - \gamma''| \right). \end{aligned}$$

*Proof.* Using the properties of  $F$  (Lemma 3.6) and the strong regularity of the first order necessary optimality conditions (25) (Proposition 3.9), the claim follows directly from the implicit function theorem for generalized equations [14, Theorem 2.4 and Corollary 2.5].  $\square$

In the sequel, we denote these critical points by  $(y_p, u_p, \lambda_p)$ . Finally, in Step 6 we are concerned with second order sufficient conditions:

**Theorem 3.11** (Second Order Sufficient Conditions). *Suppose that Assumption 3.8 holds and that  $y_a, y_b \in H^2(\Omega)$ . Then second order sufficient conditions are satisfied at  $(y^*, u^*)$ . Moreover, there exists  $\varepsilon > 0$  (possibly smaller than above) such that second order sufficient conditions hold also at the perturbed critical points in the  $\varepsilon$ -ball around  $p^*$ . Hence they are indeed local minimizers of the perturbed problems  $P(p)$ .*

*Proof.* In order to apply the theory of Maurer [29], we make the following identifications:

$$\begin{aligned} G_1(y, u) &= \Delta y - \beta y^3 - \alpha y + u + f \\ K_1 &= \{0\} \subset Y_1 = L^2(\Omega) \\ G_2(y, u) &= (y - y_a, y_b - y)^\top \\ K_2 &= [\{\varphi \in H^2(\Omega) : \varphi \geq 0 \text{ on } \Omega\}]^2 \subset Y_2 = [H^2(\Omega)]^2. \end{aligned}$$

Note that  $K_2$  is a convex closed cone of  $Y_2$  with nonempty interior. For instance,  $\varphi \equiv 1$  is an interior point. Since  $\Pi^+$  is open, one has  $p \in \Pi^+$  for all  $p$  such that  $\|p - p^*\| < \varepsilon$  for sufficiently small  $\varepsilon$ . Consequently, the Slater condition (Lemma 3.4) is satisfied also at the perturbed critical points. That is, there

exists  $\tilde{u}_p$  such that  $\tilde{y} = T_p(u_p) + T'_p(u_p)(\tilde{u}_p - u_p)$  holds. This entails that  $(y_p, u_p)$  is a regular point in the sense of [29, equation (2.3)] with the choice

$$h = \begin{pmatrix} T'_p(u_p)(\tilde{u}_p - u_p) \\ \tilde{u}_p - u_p \end{pmatrix}.$$

The multiplier theorem [29, Theorem 2.1] yields the existence of  $\lambda_p$  and nonnegative  $\mu_p^+, \mu_p^- \in W'$  which coincide with our adjoint variable and state constraint multiplier via  $\mu_p = \mu_p^+ - \mu_p^-$ .

We continue by defining the Lagrangian

$$\begin{aligned} L(y, u, \lambda, \mu^+, \mu^-; p) &= \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|u - u_d\|_{L^2(\Omega)}^2 \\ &\quad + \int_{\Omega} (-\Delta y + \beta y^3 + \alpha y - u - f) \lambda \\ &\quad + \langle y_a - y, \mu^- \rangle_{W, W'} + \langle y - y_b, \mu^+ \rangle_{W, W'}. \end{aligned}$$

By coercivity assumption (AC), abbreviating  $x = (y, u)$ , we find that the Lagrangian's second derivative with respect to  $x$ ,

$$L_{xx}(y^*, u^*, \lambda^*; p^*)(x, x) = \frac{1}{2} \|y\|_{L^2(\Omega)}^2 + 3\beta^* \int_{\Omega} y^* \lambda^* |y|^2 + \frac{\gamma^*}{2} \|u\|_{L^2(\Omega)}^2$$

(which no longer depends on  $\mu$ ) is coercive on the space of all  $(y, u)$  satisfying (32), thus, in particular, the second order sufficient conditions [29, Theorem 2.3] are satisfied at the nominal critical point  $(y^*, u^*, \lambda^*)$ .

We now show that (AC) continues to hold at the perturbed Kuhn–Tucker points. The technique of proof is inspired by [27, Lemma 5.2]. For a parameter  $p$  from the  $\varepsilon$ -ball around  $p^*$ , we denote by  $(y_p, u_p, \lambda_p)$  the corresponding solution of the first order necessary conditions (25). One easily sees that

$$|L_{xx}(y_p, u_p, \lambda_p; p)(\bar{x}, \bar{x}) - L_{xx}(y^*, u^*, \lambda^*; p^*)(\bar{x}, \bar{x})| \leq c_1 \varepsilon' \|\bar{x}\|^2 \quad (33)$$

holds for some  $c_1 > 0$  and for all  $\bar{x} = (\bar{y}, \bar{u}) \in W \times L^2(\Omega)$ , the norm being the usual norm of the product space. For arbitrary  $\bar{u} \in L^2(\Omega)$ , let  $\bar{y}$  satisfy the linear PDE

$$-D\Delta\bar{y} + (3\beta y_p^2 + \alpha)\bar{y} = \bar{u} \quad \text{on } \Omega \quad (34a)$$

$$\bar{y} = 0 \quad \text{on } \partial\Omega. \quad (34b)$$

Let  $\bar{y}$  be the solution to (32) corresponding to the control  $\bar{u}$ , then  $\bar{y} - \bar{y}$  satisfies

$$-D^* \Delta y + (3\beta^* |y^*|^2 + \alpha^*) y = [(3\beta^* |y^*|^2 + \alpha^*) - (3\beta y_p^2 + \alpha)] \bar{y} + (D - D^*) \Delta \bar{y} \quad \text{on } \Omega$$

and  $y = 0$  on  $\partial\Omega$ , i.e., by the standard a priori estimate and boundedness of  $\|3\beta y_p^2 + \alpha\|_{L^\infty(\Omega)}$  near  $p^*$ ,

$$\|\bar{y} - \bar{y}\|_{H^2(\Omega)} \leq c_2 \varepsilon' \|\bar{y}\|_{H^2(\Omega)} \tag{36}$$

holds with some  $c_2 > 0$ . Using the triangle inequality, we obtain from (36)

$$\|\bar{y} - \bar{y}\|_{H^2(\Omega)} \leq \frac{c_2 \varepsilon'}{1 - c_2 \varepsilon'} \|\bar{y}\|_{H^2(\Omega)}.$$

We have thus proved that for any  $\bar{x} = (\bar{y}, \bar{u})$  which satisfies (34), there exists  $\bar{x} = (\bar{y}, \bar{u})$  which satisfies (32) such that

$$\|\bar{x} - \bar{x}\| \leq \frac{c_2 \varepsilon'}{1 - c_2 \varepsilon'} \|\bar{x}\|. \tag{37}$$

Using the estimate from Maurer and Zowe [30, Lemma 5.5], it follows from (37) that

$$L_{xx}(y^*, u^*, \lambda^*; p^*)(\bar{x}, \bar{x}) \geq \rho' \|\bar{x}\|^2 \tag{38}$$

holds with some  $\rho' > 0$ . Combining (33) and (38) finally yields

$$\begin{aligned} L_{xx}(y_p, u_p, \lambda_p; p)(\bar{x}, \bar{x}) &\geq L_{xx}(y^*, u^*, \lambda^*; p^*)(\bar{x}, \bar{x}) - c_1 \varepsilon' \|\bar{x}\|^2 \\ &\geq (\rho' - c_1 \varepsilon') \|\bar{x}\|^2 \end{aligned}$$

which proves that (AC) holds at the perturbed Kuhn–Tucker points, possibly after further reducing  $\varepsilon'$ . Concluding as above for the nominal solution, the second order sufficient conditions in [29, Theorem 2.3] imply that  $(y_p, u_p)$  is in fact a local optimal solution for our problem (17)–(20).  $\square$

### 4. Linear–quadratic boundary control

In this section, we briefly cover the case of optimal boundary control of a linear elliptic equation with quadratic objective. Due to the similarity of the arguments to the ones used in Section 2, they are kept short. We consider the optimal control problem, subject to perturbations  $\delta = (\delta_1, \delta_2, \delta_3)$ :

$$\text{Minimize } \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|u - u_d\|_{L^2(\partial\Omega)}^2 - \int_{\bar{\Omega}} y \, d\delta_1 - \int_{\partial\Omega} u \, \delta_2 \tag{39}$$

$$\text{over } u \in L^2(\partial\Omega)$$

$$\text{s.t. } -\text{div}(A\nabla y) + a_0 y = f \quad \text{on } \Omega \tag{40}$$

$$\partial y / \partial n_A + \beta y = u + \delta_3 \quad \text{on } \partial\Omega \tag{41}$$

$$\text{and } y_a \leq y \leq y_b \quad \text{on } \bar{\Omega}. \tag{42}$$



where  $\partial/\partial n_A$  denotes the co-normal derivative of  $y$  corresponding to  $A$ , i.e.,  $\partial y/\partial n_A = n^\top A \nabla y$ . The standing assumption for this section is the following one:

*Assumption 4.1.* Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  ( $N \in \{1, 2\}$ ) with  $C^{1,1}$  boundary  $\partial\Omega$ , see [20, p. 5]. The state equation is governed by an operator with  $N \times N$  symmetric coefficient matrix  $A$  with entries  $a_{ij}$  which are Lipschitz continuous on  $\overline{\Omega}$ . We assume the condition of uniform ellipticity: There exists  $m_0 > 0$  such that

$$\xi^\top A \xi \geq m_0 |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and almost all } x \in \overline{\Omega}.$$

The coefficient  $a_0 \in L^\infty(\Omega)$  is assumed to satisfy  $\text{ess inf } a_0 > 0$ , while  $\beta \in L^\infty(\partial\Omega)$  is nonnegative. Finally, the source term  $f$  is an element of  $L^2(\Omega)$ . Again,  $y_d \in L^2(\Omega)$  and  $u_d \in L^2(\partial\Omega)$  denote desired states and controls, while  $\gamma$  is a positive number. The bounds  $y_a$  and  $y_b$  may be arbitrary functions on  $\overline{\Omega}$  such that the admissible set  $K_{C(\overline{\Omega})} = \{y \in C(\overline{\Omega}) : y_a \leq y \leq y_b \text{ on } \overline{\Omega}\}$  is nonempty.

Note that we restrict ourselves to one- and two-dimensional domains, as in three dimensions we would need the control  $u \in L^s(\partial\Omega)$  for some  $s > 2$  to obtain solutions in  $C(\overline{\Omega})$  for which a pointwise state constraint is meaningful.

**Proposition 4.2** (The State Equation). *Under Assumption 4.1, and given  $u$  and  $\delta_3$  in  $L^2(\partial\Omega)$ , the state equation (40)–(41) has a unique solution  $y \in H^1(\Omega) \cap C(\overline{\Omega})$  in the weak sense:*

$$\int_{\Omega} A \nabla y \cdot \nabla \bar{y} + \int_{\Omega} a_0 y \bar{y} + \int_{\partial\Omega} \beta y \bar{y} = \int_{\Omega} f \bar{y} + \int_{\partial\Omega} u \bar{y} \quad \text{for all } \bar{y} \in H^1(\Omega). \quad (43)$$

*The solution verifies the a priori estimate*

$$\|y\|_{H^1(\Omega)} + \|y\|_{C(\overline{\Omega})} \leq c_A (\|u\|_{L^2(\partial\Omega)} + \|\delta_3\|_{L^2(\partial\Omega)} + \|f\|_{L^2(\Omega)}).$$

*Proof.* Uniqueness and existence of the solution in  $H^1(\Omega)$  and the a priori bound in  $H^1(\Omega)$  follow directly from the Lax–Milgram Theorem applied to the variational equation (43). The proof of  $C(\overline{\Omega})$  regularity and the corresponding a priori estimate follow from Casas [10, Theorem 3.1] if  $\beta y$  is considered a right hand side term.  $\square$

The perturbations are taken as  $(\delta_1, \delta_2, \delta_3) \in \mathcal{M}(\overline{\Omega}) \times L^2(\partial\Omega) \times L^2(\partial\Omega)$ . They comprise in particular perturbations of the desired state  $y_d$  and control  $u_d$ . Notice that  $\delta_3$  affects only the boundary data so that, as in the proof of Theorem 2.3, we can absorb this perturbation into the control and obtain an admissible set independent of  $\delta$ .

**Theorem 4.3** (Lipschitz Continuity). *For any  $\delta = (\delta_1, \delta_2, \delta_3) \in \mathcal{M}(\overline{\Omega}) \times L^2(\partial\Omega) \times L^2(\partial\Omega)$ , problem (40)–(42) has a unique solution. Moreover, there exists a constant  $L > 0$  such that for any two  $(\delta'_1, \delta'_2, \delta'_3)$  and  $(\delta''_1, \delta''_2, \delta''_3)$ , the corresponding solutions of (40)–(42) satisfy*

$$\begin{aligned} & \|y' - y''\|_{H^1(\Omega)} + \|y' - y''\|_{C(\overline{\Omega})} + \|u' - u''\|_{L^2(\partial\Omega)} \\ & \leq L \left( \|\delta'_1 - \delta''_1\|_{\mathcal{M}(\overline{\Omega})} + \|\delta'_2 - \delta''_2\|_{L^2(\partial\Omega)} + \|\delta'_3 - \delta''_3\|_{L^2(\partial\Omega)} \right). \end{aligned}$$

Similar to the distributed control case, if  $K_{C(\overline{\Omega})}$  has nonempty interior, one can prove the existence of an adjoint state  $\lambda \in W^{1,s}(\Omega)$  for all  $s \in [1, \frac{N}{N-1})$  and Lagrange multiplier  $\mu \in \mathcal{M}(\overline{\Omega})$  such that

$$\langle \mu, \bar{y} - y \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \leq 0 \qquad \forall \bar{y} \in K_{C(\overline{\Omega})} \qquad (44a)$$

$$\gamma(u - u_d) - \lambda = \delta_2 \qquad \text{on } \partial\Omega \qquad (44b)$$

$$-\operatorname{div}(A\nabla\lambda) + a_0\lambda = -(y - y_d) - \mu_\Omega + \delta_{1\Omega} \qquad \text{on } \Omega \qquad (44c)$$

$$\frac{\partial\lambda}{\partial n_A} + \beta\lambda = -\mu_{\partial\Omega} + \delta_{1\partial\Omega} \qquad \text{on } \partial\Omega \qquad (44d)$$

where (44c) is understood in the sense of distributions, and (44d) holds in the sense of traces (see Casas [10]). The measures  $\mu_\Omega$  and  $\mu_{\partial\Omega}$  are obtained by restricting  $\mu$  to  $\Omega$  and  $\partial\Omega$ , respectively, and the same splitting applies to  $\delta_1$ .

Note that again, we have no stability result for the Lagrange multiplier  $\mu$ , and hence we cannot derive a stability result for the adjoint state  $\lambda$  from (44c)–(44d). We merely obtain from (44b) that on the boundary  $\partial\Omega$ ,

$$\|\lambda' - \lambda''\|_{L^2(\partial\Omega)} \leq (\gamma L + 1)\|\delta' - \delta''\|$$

holds. Unless the state constraint is restricted to the boundary  $\partial\Omega$ , this difficulty prevents the treatment of a semilinear boundary control case along the lines of Section 3.

## 5. Conclusion

In this paper, we have proved the Lipschitz stability with respect to perturbations of solutions to pointwise state-constrained optimal control problems for elliptic equations. For distributed control, it was shown how the stability result for linear state equations can be extended to the semilinear case, using an implicit function theorem for generalized equations. In the boundary control case, this method seems not applicable since we are lacking a stability estimate for the state constraint multiplier and thus for the adjoint state on the domain  $\Omega$ . This is due to the fact that the control variable and the state constraint act on different parts of the domain  $\overline{\Omega}$ .

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