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Blow-up of Solutions for a Class of Nonlinear Parabolic Equations

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Abstract. In this paper, the blow up of solutions for a class of nonlinear parabolic equations

$$u_t(x,t) = \nabla_x(a(u(x,t))b(x)c(t)\nabla_x u(x,t)) + g(x, |\nabla_x u(x,t)|^2, t)f(u(x,t))$$

with mixed boundary conditions is studied. By constructing an auxiliary function and using Hopf's maximum principles, an existence theorem of blow-up solutions, upper bound of "blow-up time" and upper estimates of "blow-up rate" are given under suitable assumptions on a, b, c, f, g, initial data and suitable mixed boundary conditions. The obtained result is illustrated through an example in which a, b, c, f, g are power functions or exponential functions.

Keywords. Nonlinear parabolic equations, blow-up solutions, maximum principles
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1. Introduction

It is well known that the blow-up of solutions is very important in nonlinear partial differential equations. In recent years, many authors have studied them (see, e.g., [1 – 4, 6]). In paper [4], the following problem was discussed :

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } D \times (0,T) \\ u = 0 & \text{on } \partial D \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \bar{D}, \end{cases}$$

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where $\bar{D}$ is the closure of D. In paper [3], the following problem was studied:

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } D \times (0,T) \\ \dfrac{\partial u}{\partial n} + \sigma(x,t)u = 0 & \text{on } \partial D \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \bar{D}. \end{cases}$$

In paper [1], the following problem was investigated:

$$\begin{cases} u_t = \Delta u + f(x,u,q,t) & \text{in } D \times (0,T) \\ u = 0 & \text{on } \Gamma_1 \times (0,T) \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma_2 \times (0,T) \\ u(x,0) = u_0(x) \geq 0, \not\equiv 0 & \text{in } \bar{D}, \end{cases}$$

where $\Gamma_1 \cup \Gamma_2 = \partial D$, $q = |\nabla u|^2$.

In this paper, we shall study the following nonlinear parabolic equations:

$$\begin{cases} \text{(i)} \quad u_t = \nabla(a(u)b(x)c(t)\nabla u) + g(x,q,t)f(u) & \text{in } D \times (0,T) \\ \text{(ii)} \quad u = 0 & \text{on } \Gamma_1 \times (0,T) \\ \qquad \dfrac{\partial u}{\partial n} + \sigma(x,t)u = 0 & \text{on } \Gamma_2 \times (0,T) \\ \text{(iii)} \quad u(x,0) = u_0(x) \geq 0, \not\equiv 0 & \text{in } \bar{D}, \end{cases}$$

where $\Gamma_1 \cup \Gamma_2 = \partial D$, $q = |\nabla u|^2$.∇ denotes the gradient operator, n is the outer normal vector, $\frac{\partial u}{\partial n}$ denotes the outward normal derivative, and D is a smooth bounded domain of $R^N, N \geq 2,\ 0 < T < +\infty$.

The function a is assumed to be a positive C^2-function, the functions b and c positive C^1-functions, the function g a nonnegative C^1-function, the function f a nonnegative C^2-function, and the function σ a nonnegative C^1-function. Throughout this paper, for simplicity we denote the derivatives of $f(s)$ with respect to s by $f'(s)$, the second derivatives by $f''(s)$, the partial derivatives of $g(x,d,t)$ with respect to d by $g_d(x,d,t)$. i.e.

$$f'(s) = \frac{df(s)}{ds}, \quad f''(s) = \frac{d^2 f(s)}{ds^2}, \quad g_d(x,d,t) = \frac{\partial g(x,d,t)}{\partial d}.$$

In this paper, an existence theorem of blow-up solutions is obtained. Upper bounds of "blow-up time" and upper estimates of "blow-up rate" are given. The result extends and supplements those obtained in [1 – 4, 6]. Our approach depends heavily upon Hopf's maximum principles.

This paper is organized as follows. In Section 2 the main result and its proof are presented. In Section 3 we shall give an example to illustrate our result in this paper may be applied.

2. The main result and its proof

The main result is stated in the following theorem:

Theorem 2.1. *Let u be a $C^3(D \times (0,T)) \cap C^2(\bar{D} \times [0,T))$-solution of* (a) – (c). *Suppose that the following conditions* (H_1) – (H_5) *hold:*

(H_1) *For $s \in R$,* $\quad a(s) > 0, f(s) \geq 0, f(0) = 0,$

and for $s \in R^+$, $\quad a'(s) \geq 0, f(s) > 0, \left(\frac{f'(s)}{a(s)}\right)' \geq 0, \left(\frac{f(s)}{a(s)}\right)' \geq 0, \left(\frac{sa(s)}{f(s)}\right)' \leq 0;$

(H_2) *for $(x, d, t) \in D \times R^+ \times R^+$,*

$$b(x) > 0,\ c(t) > 0,\ c'(t) > 0$$
$$g(x,d,t) \geq 0,\ g_d(x,d,t) \geq 0,\ g_t(x,d,t) \geq 0,\ \left(\frac{g}{c}\right)_t (x,d,t) \geq 0,$$

and for $(x,t) \in \Gamma_2 \times R^+$, $\sigma(x,t) \geq 0$, $\sigma_t(x,t) \leq 0$;

(H_3) *at $x \in \bar{D}$ where $f(u_0(x)) = 0$,* $\quad \nabla\left(a(u_0)b(x)c(0)\nabla u_0\right) \geq 0;$

(H_4) $\beta := \min\limits_{D_1} \left\{ \dfrac{a(u_0)}{f(u_0)} \left[\nabla\left(a(u_0)b(x)c(0)\nabla u_0\right) + g(x,q_0,0)f(u_0)\right] \right\} > 0,$

where $D_1 = \{\ x \ \big|\ x \in \bar{D},\ f(u_0(x) \neq 0\} \neq \emptyset,\ q_0 = |\nabla u_0|^2$;

(H_5) $\displaystyle\int_{M_0}^{+\infty} \frac{a(s)}{f(s)}\,ds < +\infty$, *where $M_0 := \max\limits_{\bar{D}} u_0(x)$.*

Then $u(x,t)$ must blow-up in finite time T satisfying

$$T \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{a(s)}{f(s)}\,ds$$

and

$$u(x,t) \leq H^{-1}\big(\beta\,(T-t)\,\big),$$

where H^{-1} is the inverse function of $H(z) := \int_z^{+\infty} \frac{a(s)}{f(s)}\,ds$, $z > 0$.

Proof. By (i) we know that

$$abc\Delta u + (a'bc\nabla u + ac\nabla b) \cdot \nabla u - u_t = -fg \leq 0. \tag{1}$$

From (1), (ii), (iii) and (H_2), it is easy to check that the solution $u(x,t)$ is nonnegative. Construct an auxiliary function as follows:

$$P(x,t) = -a(u)u_t + \beta f(u). \tag{2}$$

Then

$$\nabla P = -a'u_t\nabla u - a\nabla u_t + \beta f'\nabla u \tag{3}$$
$$\Delta P = -a'u_t\Delta u - a''u_t q - 2a'\nabla u \cdot \nabla u_t - a\Delta u_t + \beta f'\Delta u + \beta f'' q \tag{4}$$

and

$$\begin{aligned} P_t &= -a'(u_t)^2 - a(u_t)_t + \beta f' u_t \\ &= -a'(u_t)^2 - a(abc\Delta u + a'bcq + ac\nabla b \cdot \nabla u + gf)_t + \beta f' u_t \\ &= -a'(u_t)^2 - a^2 bc\Delta u_t - aa'bcu_t\Delta u - aa''bcu_t q \\ &\quad - 2aa'bc\nabla u \cdot \nabla u_t - aa'cu_t\nabla b \cdot \nabla u - a^2bc'\Delta u \\ &\quad - aa'bc'q - a^2c'\nabla b \cdot \nabla u - a^2c\nabla b \cdot \nabla u_t - ag_t f \\ &\quad - 2ag_q f\nabla u \cdot \nabla u_t - agf'u_t + \beta f' u_t. \end{aligned} \tag{5}$$

In order to prove the theorem by using Hopf's maximum principles, firstly we prove that

$$abc\Delta P + (ac\nabla b + 2g_q f\nabla u) \cdot \nabla P + \left(gf' - \tfrac{a'}{a}2g_q fq + \tfrac{c'}{c}\right) P - P_t \geq 0.$$

So we do some relevant calculation. From (4) and (5), it follows that

$$\begin{aligned} abc\Delta P - P_t = {} & \beta abcf'\Delta u + \beta abcf''q + a'(u_t)^2 + aa'cu_t\nabla b \cdot \nabla u \\ & + a^2c\nabla b \cdot \nabla u_t + ag_t f + 2ag_q f\nabla u \cdot \nabla u_t \\ & + a^2bc'\Delta u + aa'bc'q + a^2c'\nabla b \cdot \nabla u + agf'u_t - \beta f'u_t. \end{aligned} \tag{6}$$

By (3), we have

$$\begin{aligned} a^2c\nabla b \cdot \nabla u_t &= ac\nabla b \cdot (-\nabla P - a'u_t\nabla u + \beta f'\nabla u) \\ &= -ac\nabla b \cdot \nabla P - aa'cu_t\nabla b \cdot \nabla u + \beta acf'\nabla b \cdot \nabla u \end{aligned} \tag{7}$$

and

$$\begin{aligned} 2ag_q f\nabla u \cdot \nabla u_t &= 2g_q f\nabla u \cdot (-\nabla P - a'u_t\nabla u + \beta f'\nabla u) \\ &= -2g_q f\nabla u \cdot \nabla P - 2a'g_q fu_t q + 2\beta g_q ff'q. \end{aligned} \tag{8}$$

It follows from (3), (6) – (8) that

$$\begin{aligned} abc\Delta P + {} & (ac\nabla b + 2g_q f\nabla u) \cdot \nabla P - P_t \\ & = \beta abcf'\Delta u + \beta abcf''q + a'(u_t)^2 + \beta acf'\nabla b \cdot \nabla u \\ & \quad + a^2bc'\Delta u + aa'bc'q + a^2c'\nabla b \cdot \nabla u + ag_t f - 2a'g_q fu_t q \\ & \quad + 2\beta g_q ff'q + agf'u_t - \beta f'u_t. \end{aligned} \tag{9}$$

By (1), we have

$$\begin{aligned} \beta abcf'\Delta u &= \beta f'(u_t - gf - a'bcq - ac\nabla b \cdot \nabla u) \\ &= \beta f'u_t - \beta a'bcf'q - \beta acf'\nabla b \cdot \nabla u - \beta gff' \end{aligned} \tag{10}$$

$$\begin{aligned} a^2bc'\Delta u &= a\tfrac{c'}{c}(u_t - gf - a'bcq - ac\nabla b \cdot \nabla u) \\ &= a\tfrac{c'}{c}u_t - aa'bc'q - a^2c'\nabla b \cdot \nabla u - a\tfrac{c'}{c}gf. \end{aligned} \tag{11}$$

From (9) – (11) it follows that

$$\begin{aligned}
abc\Delta P+ & (ac\nabla b + 2g_q f\nabla u)\cdot\nabla P - P_t \\
&= \beta abcf''q - \beta a'bcf'q + a\tfrac{c'}{c}u_t - a\tfrac{c'}{c}gf - \beta gff' + a'(u_t)^2 + ag_tf \\
&\quad - 2a'g_qfu_tq + 2\beta g_qff'q + agf'u_t \\
&= \beta a^2bc\left(\tfrac{f'}{a}\right)'q - \beta gff' + a'(u_t)^2 + ag_tf \\
&\quad + a\tfrac{c'}{c}u_t - a\tfrac{c'}{c}gf - 2a'g_qfu_tq + 2\beta g_qff'q + agf'u_t.
\end{aligned} \tag{12}$$

From (2), it is easy to get that

$$agf'u_t = agf'\tfrac{1}{a}(\beta f - P) = -gf'P + \beta gff' \tag{13}$$

$$a\tfrac{c'}{c}u_t = a\,\tfrac{c'}{c}\,\tfrac{1}{a}\,(\beta f - P) = -\tfrac{c'}{c}P + \tfrac{c'}{c}\beta f \tag{14}$$

$$-2a'g_qfu_tq = \tfrac{a'}{a}2g_qfqP - \tfrac{a'}{a}2\beta g_qf^2q. \tag{15}$$

It follows from (12) – (15) that

$$\begin{aligned}
abc\Delta P+ & (ac\nabla b + 2g_q f\nabla u)\cdot\nabla P - P_t \\
&= \beta a^2bc\left(\tfrac{f'}{a}\right)'q + a'(u_t)^2 + ag_tf + \tfrac{c'}{c}\beta f - a\tfrac{c'}{c}gf \\
&\quad + 2\beta ag_qf\left(\tfrac{f}{a}\right)'q + \left(-gf' + \tfrac{a'}{a}2g_qfq - \tfrac{c'}{c}\right)P,
\end{aligned}$$

i.e.,

$$\begin{aligned}
abc\Delta P &+ (ac\nabla b + 2g_q f\nabla u)\cdot\nabla P + \left(gf' - \tfrac{a'}{a}2g_qfq + \tfrac{c'}{c}\right)P - P_t \\
&= \beta a^2bc\left(\tfrac{f'}{a}\right)'q + a'(u_t)^2 + ag_tf + 2\beta ag_qf\left(\tfrac{f}{a}\right)'q + \beta\tfrac{c'}{c}f - a\tfrac{c'}{c}gf \\
&= \beta a^2bc\left(\tfrac{f'}{a}\right)'q + a'(u_t)^2 + 2\beta ag_qf\left(\tfrac{f}{a}\right)'q + \beta\tfrac{c'}{c}f + afc(\tfrac{g}{c})_t.
\end{aligned} \tag{16}$$

The conditions (H_1), (H_2) and (1) – (3) guarantee that the right side in the equality (16) is nonnegative, i.e.,

$$abc\Delta P + (ac\nabla b + 2g_q f\nabla u)\cdot\nabla P + \left(gf' - \tfrac{a'}{a}2g_qfq + \tfrac{c'}{c}\right)P - P_t \geq 0. \tag{17}$$

From (H_3) and (H_4), it is easy to see that

$$\begin{aligned}
&\max_{\bar{D}} P(x,0) \\
&= \max_{\bar{D}}\left\{-a(u_0)\left[\nabla\left(a(u_0)b(x)c(0)\nabla u_0\right) + g(x,q_0,0)f(u_0)\right] + \beta f(u_0)\right\} = 0.
\end{aligned} \tag{18}$$

On $\Gamma_1 \times (0,T)$ we have $u_t = 0$ and thus

$$P(x,t) = a(0)u_t + \beta f(0) = 0. \tag{19}$$

On $\Gamma_2 \times (0,T)$ we have

$$\begin{aligned}\frac{\partial P}{\partial n} &= -a' u_t \frac{\partial u}{\partial n} - a\frac{\partial u_t}{\partial n} + \beta f' \frac{\partial u}{\partial n} \\ &= a'\sigma u u_t - a\left(\frac{\partial u}{\partial n}\right)_t - \beta f'\sigma u \\ &= a'\sigma u u_t + a\left(\sigma u\right)_t - \beta f'\sigma u \\ &= \sigma(a'u + a)u_t + a\sigma_t u - \beta f'\sigma u \\ &= \sigma(a'u+a)\left(-\frac{P}{a} + \frac{\beta f}{a}\right) + a\sigma_t u - \beta f'\sigma u \\ &= -\frac{\sigma}{a}(a'u+a)P + a\sigma_t u + \frac{\beta f^2\sigma}{a}\left(\frac{au}{f}\right)'.\end{aligned} \tag{20}$$

Combining (17) – (20) and Hopf's Maximum Principle [5, 7], it follows that P cannot assume its maximum on $\Gamma_2 \times (0,T)$, and in $\bar{D} \times [0,T)$ the maximum of P is 0. Hence we have in $\bar{D} \times [0,T)$, $P \le 0$ and

$$\frac{a(u)}{f(u)} u_t \ge \beta. \tag{21}$$

At the point $x_0 \in \bar{D}$ where $u_0(x_0) = M_0$, we get by integration

$$\frac{1}{\beta}\int_{M_0}^{u(x_0,t)} \frac{a(s)}{f(s)}\, ds \ge t.$$

By using condition (H_5), it follows that $u(x,t)$ must blow-up for a finite time $t = T$. Further the following inequality must hold $T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{a(s)}{f(s)}\, ds$. By integrating the inequality (21) over $[t,s]$ $(0 < t < s < T)$, for each fixed x, one gets

$$H(u(x,t)) \ge H(u(x,t)) - H(u(x,s)) = \int_{u(x,t)}^{u(x,s)} \frac{a(s)}{f(s)}\, ds = \int_t^s \frac{a(u)}{f(u)} u_t\, dt \ge \beta(s-t),$$

so that $u(x,t) \le H^{-1}\left(\beta(s-t)\right)$. Hence, by letting $s \to T$, we have $u(x,t) \le H^{-1}\left(\beta(T-s)\right)$. The proof of the theorem is completed. □

3. An example

Let u be a $C^3(D \times (0,T)) \cap C^2(\bar{D} \times [0,T))$-solution of the following problem:

$$\begin{cases} u_t = \nabla\left(e^{u+t}\left(1 + \sum_{i=1}^{3} x_i^2\right)\nabla u\right) + e^t\left(24 + q\sum_{i=1}^{3} x_i^2\right)u^2 e^u & \text{in } D \times (0,T) \\ u = 0 & \text{on } \Gamma_1 \times (0,T) \\ \dfrac{\partial u}{\partial n} = -ue^{-t}\sum_{i=1}^{3} x_i^4 & \text{on } \Gamma_2 \times (0,T) \\ u(x,0) = u_0(x) = \left(1 - \sum_{i=1}^{3} x_i^2\right)^2 & \text{in } \bar{D}, \end{cases}$$

where $\Gamma_1 \cup \Gamma_2 = \partial D$, $D = \left\{x = (x_1, x_2, x_3) \;\middle|\; \sum_{i=1}^{3} x_i^2 < 1\right\}$, $0 < T < +\infty$. In this example,

$$a(u) = e^u,\ b(x) = 1 + \textstyle\sum_{i=1}^{3} x_i^2,\ c(t) = e^t$$
$$g(x, q, t) = e^t\big(24 + q\textstyle\sum_{i=1}^{3} x_i^2\big),\ f(u) = u^2 e^u,\ \sigma(x, t) = e^{-t}\textstyle\sum_{i=1}^{3} x_i^4.$$

It is easy to check that (H_1) – (H_5) hold. In addition,

$$\begin{aligned}
&\nabla\big(a(u_0)b(x)c(0)\nabla u_0\big)\\
&= \nabla\left(e^{u_0}\left(1 + \sum_{i=1}^{3} x_i^2\right)\nabla u_0\right)\\
&= e^{u_0}\nabla u_0 \cdot \nabla u_0\left(1 + \sum_{i=1}^{3} x_i^2\right) + e^{u_0}\nabla\left(1 + \sum_{i=1}^{3} x_i^2\right) \cdot \nabla u_0\\
&\quad + e^{u_0}\left(1 + \sum_{i=1}^{3} x_i^2\right)\triangle u_0\\
&= e^{u_0}\left[q_0\left(1 + \sum_{i=1}^{3} x_i^2\right) + \nabla\left(1 + \sum_{i=1}^{3} x_i^2\right) \cdot \nabla u_0 + \left(1 + \sum_{i=1}^{3} x_i^2\right)\triangle u_0\right]\\
&= e^{u_0}\left[16\sum_{i=1}^{3} x_i^2\left(1 - \sum_{i=1}^{3} x_i^2\right)^2\left(1 + \sum_{i=1}^{3} x_i^2\right) - 8\sum_{i=1}^{3} x_i^2\left(1 - \sum_{i=1}^{3} x_i^2\right)\right]\\
&\quad + e^{u_0}\left[8\sum_{i=1}^{3} x_i^2\left(1 + \sum_{i=1}^{3} x_i^2\right) - 12\left(1 + \sum_{i=1}^{3} x_i^2\right)\left(1 - \sum_{i=1}^{3} x_i^2\right)\right]
\end{aligned} \tag{22}$$

From (22), it follows that

$$\begin{aligned}
\beta &= \min_{D_1}\left\{\frac{a(u_0)}{f(u_0)}\left[\nabla\left(a(u_0)b(x)c(0)\nabla u_0\right) + g(x, q_0, 0)f(u_0)\right]\right\}\\
&= \min_{D_1}\left\{\frac{e^{u_0}}{e^{u_0}u_0^2}\left[\nabla\left(e^{u_0}\left(1 + \sum_{i=1}^{3} x_i^2\right)\nabla u_0\right) + \left(24 + q_0\sum_{i=1}^{3} x_i^2\right)e^{u_0}u_0^2\right]\right\}\\
&= \min_{0 \le y < 1}\left\{\frac{4e^{(1-y)^2}}{(1-y)^4}\left[4y^8 - 24y^7 + 60y^6 - 80y^5 + 70y^4 - 52y^3 + 43y^2 - 20y + 3\right]\right\}\\
&= 3.1302
\end{aligned}$$

According to Theorem 2.1 $u(x, t)$ must blow-up in finite time T, and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{a(s)}{f(s)}\,ds = \frac{1}{3.1302}\int_{1}^{+\infty} \frac{1}{s^2}\,ds = 0.3195$$

as well as

$$u(x, t) \le H^{-1}(\beta(T - t)) = \frac{1}{3.1302(T - t)}.$$

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