Composition Operators between H^∞ and a-Bloch Spaces on the Polydisc

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Abstract. Let U^n be the unit polydisc of $\mathbb{C}^n$ and $\varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ a holomorphic self-map of U^n. Let $H(U^n)$ denote the space of all holomorphic functions on U^n, $H^\infty(U^n)$ the space of all bounded holomorphic functions on U^n, and $\mathcal{B}^a(U^n)$, $a > 0$, the a-Bloch space, i.e.,

$$\mathcal{B}^a(U^n) = \left\{ f \in H(U^n) \,\Big|\, \|f\|_{\mathcal{B}^a} = |f(0)| + \sup_{z \in U^n} \sum_{k=1}^n \left| \frac{\partial f}{\partial z_k}(z) \right| \left(1 - |z_k|^2\right)^a < +\infty \right\}.$$

We give a necessary and sufficient condition for the composition operator C_φ induced by φ to be bounded and compact between $H^\infty(U^n)$ and a-Bloch space $\mathcal{B}^a(U^n)$, when $a \geq 1$.

Keywords. Composition operators, a-Bloch space, unit polydisc, compactness, boundedness

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1. Introduction

Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in the complex vector space $\mathbb{C}^n$. By $\langle z, \omega \rangle = \sum_{k=1}^n z_k \bar{w}_k$ we denote the inner product of z and w, $|z| = \sqrt{\langle z, z \rangle}$, U^n the unit polydisc in $\mathbb{C}^n$, and $H(U^n)$ the space of all holomorphic functions on U^n. Let φ be a holomorphic self-map of U^n. Then the composition operator C_φ induced by φ is defined by $(C_\varphi f)(z) = f(\varphi(z))$ for z in U^n and $f \in H(U^n)$.

Let $a > 0$, a function f holomorphic in U^n is said to belong to the a-Bloch space $\mathcal{B}^a(U^n)$ if

$$\|f\|_{\mathcal{B}^a} = |f(0)| + \sup_{z \in U^n} \sum_{k=1}^n \left| \frac{\partial f}{\partial z_k}(z) \right| \left(1 - |z_k|^2\right)^a < +\infty.$$

It is known that $\mathcal{B}^a(U^n)$ is a Banach space with the norm $\| \cdot \|_{\mathcal{B}^a}$.

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As usual, $H^\infty(U^n)$ denotes the space of all bounded holomorphic functions on U^n with the norm $\|f\|_\infty = \sup_{z\in U^n} |f(z)|$, i.e.,

$$H^\infty(U^n) = \left\{ f \in H(U^n) \,|\, \|f\|_\infty = \sup_{z\in U^n} |f(z)| < \infty \right\}.$$

Composition operators between Bloch spaces on the unit polydisc are investigated in [7, 8, 9] where some sufficient and necessary conditions are given so that C_φ be compact on the Bloch space. The following theorem was formulated in [7]:

Theorem A. *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of U^n. Then C_φ is compact on $\mathcal{B}^1(U^n)$ if and only if for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that*

$$\sum_{k,l=1}^n \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{1 - |z_k|^2}{1 - |\varphi_l(z)|^2} < \varepsilon,$$

whenever $\operatorname{dist}(\varphi(z), \partial U^n) < \delta$.

Unfortunately, there is a gap in the proof of the result. Though the proof of sufficiency is complete, the proof of necessity contains at least one gap. More specifically, if $(z^j)_{j\in\mathbb{N}}$ is a sequence in U^n, such that $\varphi(z^j) \to \partial U^n$ as $j \to \infty$, and if inequality (9) in [7, p. 289] holds, then one cannot omit consideration of the case when $|\varphi_1(z^j)| \not\to 1$ as $j \to \infty$.

We will not consider here composition operators between a-Bloch spaces, but we want to say that the methods we used here can be used to correct their proof. Motivated by Theorem A, in this paper we investigate compactness of composition operators between $H^\infty(U^n)$ and a-Bloch space $\mathcal{B}^a(U^n)$, with $a \geq 1$. We would like to point out that unlike the case of the unit disk, the proofs of the corresponding results on the polydisc are more involved and they are not their easy generalizations. This paper can also be considered as a continuation of our investigations devoted to spaces and operators of analytic functions on the polydisc, see [2, 3, 4]. Some closely related results can be found in [1, 5, 6].

Note that the boundedness of the operator $C_\varphi : H^\infty(U^n) \to \mathcal{B}^a(U^n)$ follows from Lemma 3, since for $a \geq 1$, we have

$$\|C_\varphi(f)\|_{\mathcal{B}^a} \leq (4n+1)\|C_\varphi(f)\|_\infty \leq (4n+1)\|f\|_\infty.$$

The main result in the paper is the following:

Theorem 1. *Let $a \geq 1$, and $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of U^n. Then $C_\varphi : H^\infty(U^n) \to \mathcal{B}^a(U^n)$ is compact if and only if for every $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that*

$$\sum_{k,l=1}^n \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^a}{(1 - |\varphi_l(z)|^2)} < \varepsilon, \tag{1}$$

whenever $\operatorname{dist}(\varphi(z), \partial U^n) < \delta$.

The Bergman metric $H_z(u, \bar{u})$ in U^n is

$$H_z(u, \bar{u}) = \sum_{k=1}^{n} \frac{|u_k|^2}{(1 - |z_k|^2)^2}, \tag{2}$$

where $z \in U^n$ and $u = (u_1, \ldots, u_n) \in \mathbb{C}^n$.

Corollary. *Let* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *be a holomorphic self-map of* U^n, *satisfying the condition*

$$H_z(u, \bar{u}) \leq C H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u}),$$

for some $C > 0$ *and for each* $z \in U^n, u \in \mathbb{C}^n$, *where* $J\varphi(z) = \left(\frac{\partial \varphi_j}{\partial z_k}(z)\right)_{1 \leq j,k \leq n}$
denotes the Jacobian matrix of φ, *and* $J\varphi(z)u$ *denotes a vector, which* j-*th component is* $(J\varphi(z)u)_j = \sum_{k=1}^{n} \frac{\partial \varphi_j}{\partial z_k}(z)u_k$. *Then* $C_\varphi : H^\infty(U^n) \to \mathcal{B}^1(U^n)$ *is not compact.*

Throughout the rest of the paper C denotes a positive constant, which may vary from line to line.

2. Auxiliary results

In order to prove the main result we need several lemmas. Some of them seem to be known, however we prove them in order to make the paper selfcontained.

Lemma 1. *Let* $f \in \mathcal{B}^a(U^n), 0 < a < \infty$. *Then*

$$|f(z)| \leq \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^a} \sum_{j=1}^{n} \frac{1 - (1 - |z_j|)^{1-a}}{1-a}, & a \neq 1 \\ |f(0)| + \|f\|_{\mathcal{B}^a} \sum_{j=1}^{n} \ln \frac{1}{1 - |z_j|}, & a = 1. \end{cases}$$

Proof. We have

$$|f(z)| = \left| f(0) + \int_0^1 \langle \nabla f(tz), \bar{z} \rangle \, dt \right|$$

$$\leq |f(0)| + \sum_{j=1}^{n} \int_0^1 |z_j| \left| \frac{\partial f}{\partial z_j}(tz) \right| dt$$

$$\leq |f(0)| + \|f\|_{\mathcal{B}^a} \sum_{j=1}^{n} \int_0^1 \frac{|z_j|}{(1 - t|z_j|)^a} \, dt.$$

Calculating the last integral we obtain the result. $\qquad \square$

Lemma 2. *Let* $a > 0$ *and* φ *be a holomorphic self-map of* U^n, *then* $C_\varphi :$ $H^\infty(U^n) \to \mathcal{B}^a(U^n)$ *is compact if and only if for any bounded sequence* $(f_k)_{k \in \mathbb{N}}$ *in* $H^\infty(U^n)$ *which converges to zero uniformly on compact subsets of* U^n, *we have* $\|C_\varphi f_k\|_{\mathcal{B}^a} \to 0$, *as* $k \to \infty$.

Proof. Assume that C_φ is compact, and let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H^\infty(U^n)$ with $f_k \to 0$ uniformly on compact subsets of U^n, as $k \to \infty$. By the compactness of C_φ we have that $C_\varphi(f_k) = f_k \circ \varphi$ has a subsequence f_{k_m} which converges in $\mathcal{B}^a(U^n)$. Let $\lim_{m\to\infty} f_{k_m} = g$. By Lemma 1 and since $|f(0)| \le \|f\|_{\mathcal{B}^a}$ for every $f \in H(U^n)$, we have that for any compact $K \subset U^n$ there is a positive constant C_K independent of f such that

$$|f_{k_m}(\varphi(z)) - g(z)| \le C_K \|f_{k_m} \circ \varphi - g\|_{\mathcal{B}^a}, \quad \text{for all } z \in K.$$

It follows that $f_{k_m}(\varphi(z)) - g(z) \to 0$ uniformly on compacts of U^n. Since $\{\varphi(z)\}$ is a compact set, $f_k(\varphi(z)) \to 0$ for each $z \in U^n$. Hence the limit function g is equal to 0. Since this is true for arbitrary subsequence of $(f_k)_{k\in\mathbb{N}}$ we see that $f_k \circ \varphi \to 0$ in $\mathcal{B}^a(U^n)$, as $k \to \infty$.

Conversely, let $(g_k)_{k\in\mathbb{N}}$ be any sequence in the ball $\mathcal{K}_M = B_{H^\infty}(0, M)$ of the space $H^\infty(U^n)$. Since $\|g_k\|_\infty \le M < \infty$, the sequence $(g_k)_{k\in\mathbb{N}}$ is uniformly bounded on U^n and hence normal by Montel's theorem. Thus we may extract a subsequence $(g_{k_j})_{j\in\mathbb{N}}$ which converges uniformly on compact subsets of U^n to some $g \in H(U^n)$, moreover $\|g\|_\infty \le M$. Hence the sequence $(g_{k_j} - g)_{j\in\mathbb{N}}$ is such that $\|g_{k_j} - g\|_\infty \le 2M < \infty$, $j \in \mathbb{N}$, and converges to 0 on compact subsets of U^n. By the hypothesis we have that $g_{k_j} \circ \varphi \to g \circ \varphi$ in $\mathcal{B}^a(U^n)$. Thus the set $C_\varphi(\mathcal{K}_M)$ is relatively compact, finishing the proof. $\square$

Lemma 3. *The inclusions $H^\infty(U^n) \subset \mathcal{B}^1 \subset \mathcal{B}^a$ hold, where $a > 1$. Moreover, there is a positive constant C independent of f such that*

$$\|f\|_{\mathcal{B}^a} \le \|f\|_{\mathcal{B}^1} \le C\|f\|_\infty.$$

Proof. Since the inclusion $\mathcal{B}^1 \subset \mathcal{B}^a$ is obvious we only need to prove $H^\infty(U^n) \subset \mathcal{B}^1$. Let $k \in \{1, \ldots, n\}$. Then, by Cauchy's integral formula we have

$$\frac{\partial f}{\partial z_k}(z) = \frac{1}{(2\pi i)^n} \int_{\prod_{j=1}^n \partial B(z_j, (1-|z_j|)/2)} \frac{f(\zeta) d\zeta}{(\zeta_k - z_k) \prod_{j=1}^n (\zeta_j - z_j)}. \tag{3}$$

From (3) we obtain

$$\left| \frac{\partial f}{\partial z_k}(z) \right| \le \frac{2}{\pi^n (1-|z_k|) \prod_{j=1}^n (1-|z_j|)} \int_{\prod_{j=1}^n \partial B(z_j, (1-|z_j|)/2)} |f(\zeta)| \prod_{j=1}^n |d\zeta_j|$$

$$\le \frac{2\|f\|_\infty}{\pi^n (1-|z_k|) \prod_{j=1}^n (1-|z_j|)} \int_{\prod_{j=1}^n \partial B(z_j, (1-|z_j|)/2)} \prod_{j=1}^n |d\zeta_j|$$

$$= \frac{2\|f\|_\infty}{(1-|z_k|)}.$$

Hence

$$\sup_{z \in U^n} (1 - |z_k|^2) \left| \frac{\partial f}{\partial z_k}(z) \right| \leq 4\|f\|_\infty \, ,$$

and consequently $\|f\|_{\mathcal{B}^1} \leq |f(0)| + 4n\|f\|_\infty \leq (4n + 1)\|f\|_\infty$, from which the result follows. □

Lemma 4. *Let $a \geq 1$, and $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a holomorphic self-map of U^n. If $C_\varphi : H^\infty(U^n) \to \mathcal{B}^a(U^n)$, then there exists a constant C such that*

$$\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^a}{(1 - |\varphi_l(z)|^2)} \leq C, \tag{4}$$

for all $z \in U^n$.

Proof. As we know the operator $C_\varphi : H^\infty(U^n) \to \mathcal{B}^a(U^n)$ is bounded. Hence, there is a positive constant C such that

$$\|C_\varphi f\|_{\mathcal{B}^a} \leq C\|f\|_\infty \tag{5}$$

for every $f \in H^\infty(U^n)$. For $f_l(z) = z_l$, $l \in \{1, \ldots, n\}$, we obtain

$$\sum_{k=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| (1 - |z_k|^2)^a < \infty, \tag{6}$$

that is, $\varphi_l \in \mathcal{B}^a(U^n)$, for each $l \in \{1, \ldots, n\}$. For fixed $l \in \{1, \ldots, n\}$, we use a family of test functions $\{f_w \mid w \in \mathbb{C}, |w| < 1\}$ in $H^\infty(U^n)$ defined as follows:

$$f_w(z) = \frac{1 - |w|^2}{(1 - \bar{w}z_l)^a} \, .$$

We have

$$\|f\|_\infty = \sup_{z \in U^n} \frac{1 - |w|^2}{|1 - \bar{w}z_l|^a} \leq \frac{2}{(1 - |w|)^{a-1}} \, .$$

From (5) it follows that

$$\sum_{k=1}^{n} \left| \sum_{j=1}^{n} \frac{\partial f_w(\varphi(z))}{\partial \zeta_j} \frac{\partial \varphi_j}{\partial z_k}(z) \right| (1 - |z_k|^2)^a \leq \frac{C}{(1 - |w|)^{a-1}}, \tag{7}$$

for every $z \in U^n$. Let $w = \varphi_l(z)$, then (7) becomes

$$a|\varphi_l(z)| \sum_{k=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^a}{(1 - |\varphi_l(z)|^2)} \leq C.$$

Let $\delta \in (0,1)$ be fixed and $E_{\delta,l} = \{z \in U^n \,|\, |\varphi_l(z)| > \delta\,\}$, then

$$\sup_{z \in E_{\delta,l}} \sum_{k=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^a}{(1-|\varphi_l(z)|^2)} \leq \frac{C}{a\delta} < \infty. \tag{8}$$

For $z \in U^n \setminus E_{\delta,l}$, using (6) we have

$$\sum_{k=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^a}{(1-|\varphi_l(z)|^2)} \leq \sum_{k=1}^{n} \frac{1}{1-\delta^2} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \left(1-|z_k|^2\right)^a < \infty. \tag{9}$$

Since (8) and (9) hold for each $l \in \{1,\dots,n\}$, we have that for any $z \in U^n$,

$$\sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^a}{(1-|\varphi_l(z)|^2)} \leq C,$$

finishing the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Proof of the main result

In this section we prove the main result in this paper.

Proof of Theorem 1. Assume that $\|f_j\|_\infty \leq C, j \in \mathbb{N}$ and that $(f_j)_{j \in \mathbb{N}}$ converges to zero uniformly on compact subsets of U^n as $j \to \infty$. By Lemma 2 it is enough to prove that $\|C_\varphi f_j\|_{\mathcal{B}^a} \to 0$ as $j \to \infty$.

From (1) we have that for every $\varepsilon > 0$, there exists an r, $0 < r < 1$, such that

$$\sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^a}{(1-|\varphi_l(z)|^2)} < \varepsilon, \tag{10}$$

whenever $dist(\varphi(z), \partial U^n) < r$. Let

$$I = \sum_{k=1}^{n} \left|\frac{\partial\left(C_\varphi f(z)\right)}{\partial z_k}\right| (1-|z_k|^2)^a,$$

then

$$\begin{aligned}
I &= \sum_{k=1}^{n} \left|\sum_{l=1}^{n} \frac{\partial f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z)\right| (1-|z_k|^2)^a \\
&\leq \sum_{k,l=1}^{n} \left|\frac{\partial f}{\partial w_l}(\varphi(z)) \frac{\partial \varphi_l}{\partial z_k}(z)\right| (1-|z_k|^2)^a \\
&\leq \sum_{l=1}^{n} \left|\frac{\partial f}{\partial w_l}(\varphi(z))\right| \left(1-|\varphi_l(z)|^2\right) \sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^a}{(1-|\varphi_l(z)|^2)}. \tag{11}
\end{aligned}$$

From (10), (11), and by Lemma 3, it follows that for sufficiently large j

$$\sum_{k=1}^{n}\left|\frac{\partial\left(C_\varphi f_j\right)}{\partial z_k}(z)\right|(1-|z_k|^2)^a \leq C\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^a}{(1-|\varphi_l(z)|^2)} \leq C\varepsilon, \qquad (12)$$

whenever $dist\left(\varphi(z),\partial U^n\right) < r$. Since $E_r = \{w \in U^n \,|\, dist(w,\partial U^n) \geq r\}$ is a compact subset of U^n, we have that $f_j(w)$ and $\frac{\partial f_j}{\partial z_k}(w)$ tend to zero uniformly on E_r, as $j \to \infty$.

On the other hand, since $C_\varphi : H^\infty(U^n) \to \mathcal{B}^a(U^n)$ is bounded then $\varphi_l \in \mathcal{B}^a(U^n)$ for every $l \in \{1,\dots,n\}$, which implies that there is a positive constant M such that

$$\max_{z\in\varphi^{-1}(E_r)}(1-|z_k|^2)^a\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right| \leq M$$

for every $k,l \in \{1,\dots,n\}$. Using this fact and (11), we have that

$$\sum_{k=1}^{n}\left|\frac{\partial\left(C_\varphi f_j\right)}{\partial z_k}(z)\right|(1-|z_k|^2)^a \leq C\sum_{l=1}^{n}\left|\frac{\partial f_j}{\partial w_l}(w)\right| \leq C\varepsilon, \qquad (13)$$

for every z such that $dist(\varphi(z),\partial U^n) \geq r$. From (12) and (13), and using the fact $\lim_{j\to\infty} f_j(\varphi(0)) = 0$, we obtain $\|C_\varphi f_j\|_{\mathcal{B}^a} \to 0$, as $j \to \infty$, as desired.

Now assume $C_\varphi : H^\infty(U^n) \to \mathcal{B}^a(U^n)$ is compact. We need to prove (1). Assume that (1) fails, then there is a sequence $(z^j)_{j\in\mathbb{N}}$ in U^n such that $w^j = \varphi(z^j) \to \partial U^n$, as $j \to \infty$, and $\varepsilon_0 > 0$, such that

$$\sum_{k,l=1}^{n}\left|\frac{\partial\varphi_l}{\partial z_k}(z^j)\right|\frac{(1-|z_k^j|^2)^a}{(1-|\varphi_l(z^j)|^2)} \geq \varepsilon_0, \qquad (14)$$

for all $j \in \mathbb{N}$. On the other hand, by Lemma 4 condition (4) holds. Hence for any $k,l \in \{1,\dots,n\}$, there is a subsequence of (z^j) (we keep the same notation (z^j)), such that

$$\left|\frac{\partial\varphi_l}{\partial z_k}(z^j)\right|\frac{(1-|z_k^j|^2)^a}{(1-|\varphi_l(z^j)|^2)}$$

converges to a finite number as $j \to \infty$. Also we may assume that for every $l \in \{1,\dots,n\}$ there is finite limit $\lim_{j\to\infty}|w_l^j|$, where w_l^j denotes $\varphi_l(z^j)$. By (14), without loss of generality we may assume that

$$\left|\frac{\partial\varphi_1}{\partial z_{k_0}}(z^j)\right|\frac{(1-|z_{k_0}^j|^2)^a}{(1-|\varphi_1(z^j)|^2)} \to \varepsilon_1 > 0 \quad (j \to \infty), \qquad (15)$$

for some $k_0 \in \{1,\dots,n\}$.

We will construct a sequence of functions $(f_j)_{j\in\mathbb{N}}$ satisfying the following conditions:

(a) $(f_j)_{j \in \mathbb{N}}$ is a bounded sequence in $H^\infty(U^n)$;
(b) $(f_j)_{j \in \mathbb{N}}$ tends to zero uniformly on compacts of U^n;
(c) $\|C_\varphi f_j\|_{\mathcal{B}^a} \not\to 0$, as $j \to \infty$.

By Lemma 2 we will arrive at a contradiction, hence $C_\varphi : H^\infty(U^n) \to \mathcal{B}^a(U^n)$ will be compact.

Case 1. Let $|w_1^j| \to 1$ as $j \to \infty$ and

$$f_j(z) = \frac{1 - |w_1^j|^2}{1 - z_1 \overline{w_1^j}}, \quad j = 1, 2, \dots . \tag{16}$$

Then clearly $\|f_j\|_\infty \le 2$, and (f_j) tends to zero uniformly on compact subsets of U^n. We show that $\|C_\varphi f_j\|_{\mathcal{B}^a} \not\to 0$ as $j \to \infty$. Let

$$I_j = \sum_{k=1}^{n} (1 - |z_k^j|^2)^a \left| \frac{\partial f_j}{\partial z_k}(\varphi(z^j)) \right|.$$

Then we have

$$
\begin{aligned}
I_j &= \sum_{k=1}^{n} (1 - |z_k^j|^2)^a \left| \frac{\partial f_j}{\partial \zeta_1}(\varphi(z^j)) \frac{\partial \varphi_1}{\partial z_k}(z^j) \right| \\
&= |w_1^j| \sum_{k=1}^{n} \frac{(1 - |z_k^j|^2)^a}{(1 - |w_1^j|^2)} \left| \frac{\partial \varphi_1}{\partial z_k}(z^j) \right| \\
&\ge |w_1^j| \left| \frac{\partial \varphi_1}{\partial z_{k_0}}(z^j) \right| \frac{(1 - |z_{k_0}^j|^2)^a}{(1 - |\varphi_1(z^j)|^2)} \to \varepsilon_1 > 0 \quad \text{as } j \to \infty, \tag{17}
\end{aligned}
$$

from which the result follows in this case.

Case 2. Let now $|w_1^j| \to \rho < 1$ as $j \to \infty$. Since $w^j \to \partial U^n$ there is an $l \in \{2, \dots, n\}$, such that $|w_l^j| \to 1$ as $j \to \infty$. If there is an $k_1 \in \{1, \dots, n\}$ and $\varepsilon_2 > 0$ such that

$$\left| \frac{\partial \varphi_l}{\partial z_{k_1}}(z^j) \right| \frac{(1 - |z_{k_1}^j|^2)^a}{(1 - |\varphi_l(z^j)|^2)} \to \varepsilon_2 > 0 \quad \text{as } j \to \infty,$$

then we obtain a contradiction using the following test functions

$$g_j(z) = \frac{1 - |w_l^j|^2}{1 - z_l \overline{w_l^j}}, \quad j = 1, 2, \dots ,$$

similarly as in Case 1. Hence, we may assume that

$$\left| \frac{\partial \varphi_l}{\partial z_k}(z^j) \right| \frac{(1 - |z_k^j|^2)^a}{(1 - |\varphi_l(z^j)|^2)} \to 0 \quad \text{as } j \to \infty, \tag{18}$$

for each $k \in \{1, \ldots, n\}$. The functions

$$f_j(z) = (z_1 + 2)\frac{1 - |w_l^j|^2}{1 - z_l\overline{w_l^j}}, \quad j = 1, 2, \ldots, \tag{19}$$

satisfy the condition $\|f_j\|_\infty \le 6$ and since for $|z| \le r < 1$

$$|f_j(z)| \le (2 + r)\frac{1 - |w_l^j|^2}{1 - r}, \quad j = 1, 2, \ldots,$$

we have that the sequence $(f_j)_{j\in\mathbb{N}}$ tends to zero uniformly on compact subsets of U^n.

We show that $\|C_\varphi f_j\|_{\mathcal{B}^a} \not\to 0$ as $j \to \infty$. We have

$$I_j = \sum_{k=1}^n \left(1 - |z_k^j|^2\right)^a \left| \frac{\partial f_j}{\partial \zeta_1}(\varphi(z^j))\frac{\partial \varphi_1}{\partial z_k}(z^j) + \frac{\partial f_j}{\partial \zeta_l}(\varphi(z^j))\frac{\partial \varphi_l}{\partial z_k}(z^j) \right|.$$

Since

$$\frac{\partial f_j}{\partial \zeta_1}(\varphi(z^j)) = \frac{1 - |w_l^j|^2}{(1 - w_l^j\overline{w_l^j})} = 1, \qquad \frac{\partial f_j}{\partial \zeta_l}(\varphi(z^j)) = \frac{(2 + w_1^j)\overline{w_l^j}}{(1 - |w_l^j|^2)},$$

and since $|w_l^j| \to 1$ as $j \to \infty$, it follows that

$$\begin{aligned}
I_j &= \sum_{k=1}^n (1 - |z_k^j|^2)^a \left| \frac{(2 + w_1^j)\overline{w_l^j}}{(1 - |w_l^j|^2)}\frac{\partial \varphi_l}{\partial z_k}(z^j) + \frac{\partial \varphi_1}{\partial z_k}(z^j) \right| \\
&\sim \sum_{k=1}^n (1 - |z_k^j|^2)^a \left| \frac{\partial \varphi_1}{\partial z_k}(z^j) \right| \\
&\ge (1 - |z_{k_0}^j|^2)^a \left| \frac{\partial \varphi_1}{\partial z_{k_0}}(z^j) \right| \\
&\succeq \left| \frac{\partial \varphi_1}{\partial z_{k_0}}(z^j) \right| \frac{(1 - |z_{k_0}^j|^2)^a}{(1 - |\varphi_1(z^j)|^2)} \to \varepsilon_1 > 0 \quad \text{as } j \to \infty. \tag{20}
\end{aligned}$$

From (20) we obtain a contradiction, finishing the proof. $\qquad\square$

Proof of the Corollary. Using (2), we see that the condition

$$H_z(u, \bar{u}) \le CH_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u}),$$

is equivalent to

$$\sum_{k=1}^n \frac{|u_k|^2}{(1 - |z_k|^2)^2} \le C\sum_{l=1}^n \frac{\left| \sum_{k=1}^n \frac{\partial \varphi_l}{\partial z_k}u_k \right|^2}{(1 - |\varphi_l(z)|^2)^2}.$$

By an obvious inequality and choosing $u_k = 1 - |z_k|^2$, $k = 1, \ldots, n$, we have

$$n = \sum_{k=1}^{n} \frac{|u_k|^2}{(1-|z_k|^2)^2} \leq C \sum_{l=1}^{n} \frac{\left(\sum_{k=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}\right| |u_k|\right)^2}{(1-|\varphi_l(z)|^2)^2} \leq C \left(\sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}\right| \frac{1-|z_k|^2}{1-|\varphi_l(z)|^2}\right)^2,$$

and consequently

$$\sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}\right| \frac{1-|z_k|^2}{1-|\varphi_l(z)|^2} \geq \frac{\sqrt{n}}{C},$$

for each $z \in U^n$. Form this and Theorem 1, we obtain the result. $\square$

Remark. It is known that if $\varphi \in Aut(U^n)$, then

$$H_z(u, \bar{u}) = H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u}),$$

for each $z \in U^n$ and $u \in \mathbb{C}^n$. Applying the corollary we obtain that, if $\varphi \in Aut(U^n)$, then $C_\varphi : H^\infty(U^n) \to \mathcal{B}^1(U^n)$ is not compact.

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