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Existence and Multiplicity of Positive Solutions for Singular $p-$Laplacian Equations

Haishen Lü and Yi Xie

Abstract. Positive solutions are obtained for the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda(u^\beta + \frac{1}{u^\alpha}) & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{$*$}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $N \geq 3$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < \alpha < 1$ and $p - 1 < \beta < p^* - 1$ $\left(p^* = \frac{Np}{N-p}\right)$ are two constants, $\lambda > 0$ is a real parameter. We obtain that Problem $(*)$ has two positive weakly solutions if λ is small enough.

Keywords. p-Laplacian, positive solution, critical point theory

Mathematics Subject Classification (2000). Primary 35J20, secondary 35J25

1. Introduction

In this paper we study the singular boundary value problem

$$\begin{cases} -\Delta_p u = \lambda(u^\beta + \frac{1}{u^\alpha}) & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $N \geq 3$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < \alpha < 1$ and $p - 1 < \beta < p^* - 1$ $\left(p^* = \frac{Np}{N-p}\right)$ are two constants, $\lambda > 0$ is a real parameter.

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Definition 1.1. A function $u \in W_0^{1,p}(\Omega)$ is called a *positive weakly solution* of Problem (1), if $u(x) > 0$ for $x \in \Omega$ and

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u, \nabla\varphi\big)dx = \lambda \int_\Omega u^\beta \varphi dx + \lambda \int_\Omega \frac{\varphi}{u^\alpha} dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega)$$

holds.

In the pioneering work [1], A. Ambrosetti, H. Brezis and G. Cerami investigated the problem

$$\begin{cases} -\Delta u = \lambda u^a + u^b & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 < a < 1 < b$. In the succeeding work [2], the above problem is extended to the p-Laplacian by A. Ambrosetti, J. G. Azorero and I. Peral. Motivated by this, this paper attempt to improve the above results to the singular p-Laplacian equation, i.e., $-1 < a < 0$. We must point out that since the functional of (1) fails to be Frechet differentiable in Ω, critical point theory where [1, 2] have used could not be applied to obtain the existence of solutions. So the method in [1, 2] could not be used. So, it is very difficult to find existence and multiplicity of positive solutions for Problem (1).

The existence of solutions to the elliptic equation

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

on a smooth domain $\Omega \subset \mathbb{R}^N$ has been extensively studied (cf. [5, 7, 8, 11, 12] and their references). For bounded Ω, in [7] it is shown that Problem (2) with $0 < \gamma < 1$ has a unique positive weakly solution in $H_0^1(\Omega)$ if $p(x)$ is a nonnegative nontrivial function in $L^2(\Omega)$. For the general problem

$$\begin{cases} -\Delta u = \frac{\sigma}{u^\gamma} + \lambda u^\beta & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

It is worth mentioning that, in [10] the existence of a unique positive solution in the cases when $\beta = 1$ and $0 < \beta < 1$ (the sub-linear problem) has been proved. On the other hand, in [4], Y. Sun, S. Wu and Y. Long have proved that Problem (3) has at least one positive weakly solution $u \in H_0^1(\Omega)$ for all $\lambda > 0$ and $\sigma \in (0, \sigma^*]$.

Our goal in this paper is to prove that Problem (1) has two positive weakly solutions for all λ small enough. In this paper, critical point theory could not be applied to obtain the existence of solutions since the associate functional fails to be Frechet differentiable in Ω. We mainly rely on the Ekeland's variational principle [6] and careful estimates inspirsed by Lair-Shaker [7] and Tarantello [3].

We work on the Sobolev space $W_0^{1,p}(\Omega)$ equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}$. For $u \in W_0^{1,p}(\Omega)$ we define $I : W_0^{1,p}(\Omega) \to R$ by

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{\beta+1}\int_\Omega |u|^{\beta+1} dx - \frac{\lambda}{1-\alpha}\int_\Omega |u|^{1-\alpha} dx.$$

On the other hand, $L^p(\Omega)$ denote Lebesgue's spaces, the norm in L^p is denoted by $\|\cdot\|_p$; C_1, C_2, $\cdots$ denote (possibly different) positive constants. Our main results is the following:

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$. Let $0 < \alpha < 1$, $p < \beta+1 < p^*$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ Problem* (1) *possesses at least two positive weakly solutions $u_1(\cdot)$, $u_2(\cdot) \in W_0^{1,p}(\Omega)$ and*

$$\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \cdot \nabla\varphi dx = \lambda\int_\Omega u_i^\beta \varphi dx + \lambda\int_\Omega \frac{\varphi}{u_i^\alpha} dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega),\ i = 1, 2.$$

Moreover, u_1 is a local minimizer of I in $W_0^{1,p}(\Omega)$ with $I(u_1) < 0$; and u_2 is a minimizer of I on Λ_- (Λ_- is defined behind) with $I(u_2) \geq 0$.

Remark 1.3. The conclusion of Theorem 1.2 can be extended to the case of the more general problem

$$\begin{cases} -\Delta_p u = \mu\Big(\frac{f(x)}{u^\rho} + g(x)u^\tau\Big) & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where f, $g : \Omega \to R$ are two given non-negative and non-trival function in $L^p(\Omega)$.

Remark 1.4. When $N = 1$, the type of equations has been studied by Agarwal and O'Regan [9] who proved that the equation

$$\begin{cases} -(|u'|^{q-2}u')' = \varsigma\Big(\frac{1}{u^{\alpha_1}} + u^{\beta_1} + 1\Big) & \text{for } 0 < t < 1,\ 1 < q < \infty \\ \qquad u(0) = u(1) = 0\,, \end{cases}$$

where $\alpha_1 > 0$, $\beta_1 > q-1$ and $0 < \varsigma < \frac{2^q}{3}\big(\frac{q}{q-1+\alpha}\big)^{q-1}$, has two solutions u_1, $u_2 \in C[0,1] \cap C^1(0,1)$ with $u_1 > 0$, $u_2 > 0$ on $(0,1)$ and $\|u_1\|_\infty < 1 < \|u_2\|_\infty$.

2. Preliminary lemmas

Let us define

$$\Lambda = \left\{ u \in W_0^{1,p}(\Omega) : \|u\|^p - \lambda \|u\|_{\beta+1}^{\beta+1} - \lambda \int_\Omega |u|^{1-\alpha} = 0 \right\}.$$

It is easy to see that $\Lambda \backslash \{0\}$ is a Nehari manifold, see [14]. Notice that if u is a weak of (1), then $u \in \Lambda$. For the sake of the convenience, we record

$$\begin{aligned} A &= \frac{p-1+\alpha}{\beta+\alpha}, \quad B = \frac{\beta-p+1}{\beta+\alpha}, \quad D = \frac{p+\alpha-1}{\beta+1-p} \\ E &= \frac{p^*-\beta-1}{p^*(\beta+1)}, \quad F = \frac{\beta+\alpha}{\beta+1}. \end{aligned} \tag{4}$$

Further, we define $G : W_0^{1,p}(\Omega) \to R$ by

$$G(u) = A\|u\|^p - \lambda \|u\|_{\beta+1}^{\beta+1}.$$

In succession, let

$$\begin{aligned} \Lambda_+ &= \{u \in \Lambda : G(u) > 0\} \\ \Lambda_0 &= \{u \in \Lambda : G(u) = 0\} \\ \Lambda_- &= \{u \in \Lambda : G(u) < 0\}. \end{aligned}$$

For the sake of the convenience, we list some inequalities which we will use in the next section. By Sobolev's embedding Theorem, we have

$$\|u\|_p \le C_0 \|u\| \quad \forall u \in W_0^{1,p}(\Omega)$$

$$\|u\|_{p^*} \le \left(\frac{1}{S}\right)^{\frac{1}{p}} \|u\| \quad \forall u \in W_0^{1,p}(\Omega), \tag{5}$$

where $C_0 > 0$ is a constant and $S > 0$ is the best Sobolev constant. By Hölder inequalities we have

$$\|u\|_{\beta+1} \le |\Omega|^E \|u\|_{p^*} \qquad \forall u \in W_0^{1,p}(\Omega) \tag{6}$$

$$\int_\Omega |u|^{1-\alpha} dx \le |\Omega|^F \|u\|_{\beta+1}^{1-\alpha} \quad \forall u \in W_0^{1,p}(\Omega). \tag{7}$$

By (5) and (6), we have

$$\|u\|_{\beta+1} \le C_1 \|u\| \qquad \forall u \in W_0^{1,p}(\Omega) \tag{8}$$

where $C_1 = |\Omega|^E \left(\frac{1}{S}\right)^{\frac{1}{p}}$. By (7) and (8), we have

$$\int_\Omega |u|^{1-\alpha} dx \le C_2 \|u\|^{1-\alpha} \qquad \forall u \in W_0^{1,p}(\Omega) \tag{9}$$

where $C_2 = |\Omega|^{F+E(1-\alpha)} S^{\frac{\alpha-1}{p}}$.

Lemma 2.1. *Let*

$$\lambda_1 = \left(\frac{A^\delta B S^{\frac{1}{p}}}{|\Omega|^{E+F}}\right)^{\frac{1}{D}}, \tag{10}$$

where A, B, D, E, F, S are defined in (4) and (5). Then, for all $\lambda \in (0, \lambda_1)$, we have the following conclusions:

1. *For every $u \in \Lambda$, $u \not\equiv 0$, then $G(u) \neq 0$, (i.e., $\Lambda_0 = \{0\}$);*
2. *Λ_- is closed in $W_0^{1,p}(\Omega)$.*

Proof. 1. Suppose, by contradiction that there exists some $u \in \Lambda$, $u \not\equiv 0$ such that $G(u) = 0$. Then

$$\|u\|_{\beta+1}^{\beta+1} = \frac{A}{\lambda}\|u\|^p. \tag{11}$$

So

$$0 = \|u\|^p - \lambda\|u\|_{\beta+1}^{\beta+1} - \lambda\int_\Omega |u|^{1-\alpha}dx = \|u\|^p - A\|u\|^p - \lambda\int_\Omega |u|^{1-\alpha}dx.$$

Thus

$$\int_\Omega |u|^{1-\alpha}dx = \frac{1-A}{\lambda}\|u\|^p = \frac{B}{\lambda}\|u\|^p. \tag{12}$$

By (11) and (12) we have

$$\frac{B}{\lambda}\|u\|^p\left(\frac{A}{\lambda}\right)^D \frac{\|u\|^{pD}}{\|u\|_{\beta+1}^{(\beta+1)D}} - \int_\Omega |u|^{1-\alpha}dx = 0. \tag{13}$$

On the other hand, by (7) and (8) we have

$$\begin{aligned}\frac{B}{\lambda}\|u\|^p\left(\frac{A}{\lambda}\right)^D \frac{\|u\|^{pD}}{\|u\|_{\beta+1}^{(\beta+1)D}} - \int_\Omega |u|^{1-\alpha}dx &\geq \frac{B}{\lambda}\left(\frac{A}{\lambda}\right)^D \frac{S^{\frac{1}{p}}}{|\Omega|^E}\frac{\|u\|_{\beta+1}^{pD+p}}{\|u\|_{\beta+1}^{(\beta+1)D}} - |\Omega|^F\|u\|_{\beta+1}^{1-\alpha} \\ &= \left(\frac{A^D B}{\lambda^{1+D}}\frac{S^{\frac{1}{p}}}{|\Omega|^E} - |\Omega|^F\right)\|u\|_{\beta+1}^{1-\alpha}.\end{aligned}$$

If $0 < \lambda < \lambda_1$, then $\frac{A^D B}{\lambda^{1+D}}\frac{S^{\frac{1}{p}}}{|\Omega|^E} - |\Omega|^F > 0$. Thus

$$\frac{B}{\lambda}\|u\|^p \cdot \left(\frac{A}{\lambda}\right)^D \frac{\|u\|^{p\sigma}}{\|u\|_{\beta+1}^{(\beta+1)D}} - \int_\Omega |u|^{1-\alpha}dx > 0\,,$$

which yields a contraction by (13). So $\Lambda_0 = \{0\}$.

2. Let $\{u_n\} \subset \Lambda_-$ be a sequence such that $u_n \to u_0$ in $W_0^{1,p}(\Omega)$. Then $u_n \to u_0$ in $L^{\beta+1}(\Omega)$ and $u_0 \in \Lambda_- \cup \Lambda_0$. Now we prove $u_0 \in \Lambda_-$. Suppose

$u_0 \in \Lambda_0$. Since $\Lambda_0 = \{0\}$, it follows that $u_0 = 0$. On the other hand, for all $u \in \Lambda_-$,

$$\frac{A}{\lambda} \le \frac{\|u\|_{\beta+1}^{\beta+1}}{\|u\|^p}.$$

By (8), we have

$$\frac{AS}{\lambda|\Omega|^{Ep}} \le \|u\|_{\beta+1}^{\beta+1-p}. \tag{14}$$

Thus

$$\frac{AS}{\lambda|\Omega|^{Ep}} \le \|u_n\|_{\beta+1}^{\beta+1-p} \quad \text{for } n \in N.$$

Let $n \to \infty$, we have

$$\frac{AS}{\lambda|\Omega|^{Ep}} \le \|u_0\|_{\beta+1}^{\beta+1-p}.$$

So $u_0 \not\equiv 0$. Hence $u_0 \in \Lambda_-$. □

Lemma 2.2. *Let*

$$\lambda_2 = A^{\frac{D}{1-D}} \cdot B^{\frac{1}{1-D}} \cdot \frac{S}{|\Omega|^{pE+F}} \tag{15}$$

where A, B, D, E, F, S *are defined in* (4) *and* (5). *If* $0 < \lambda < \lambda_2$, *then for all* $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, *there exists a unique* $t^+ = t^+(u) > 0$ *such that* $t^+ u \in \Lambda_-$.

Proof. For all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, define $H : [0,\infty) \to (-\infty,\infty)$ by

$$H(t) = t^{p-1+\alpha}\|u\|^p - \lambda t^{\beta+\alpha}\|u\|_{\beta+1}^{\beta+1}.$$

Easy computations show that H achieves its maximum at

$$t_0 = \left(\frac{A}{\lambda}\,\frac{\|u\|^p}{\|u\|_{\beta+1}^{\beta+1}}\right)^{\frac{1}{\beta+1-p}}.$$

So

$$H(t_0) = \left(\frac{A}{\lambda}\right)^D B \cdot \left[\frac{\|u\|^{p(\beta+\alpha)}}{\|u\|_{\beta+1}^{(\beta+1)(p+\alpha-1)}}\right]^{\frac{1}{\beta+1-p}}.$$

If $\lambda \in (0, \lambda_2)$, then $\lambda|\Omega|^F \|u\|_{\beta+1}^{1-\alpha} < H(t_0)$. By (7), $\lambda \int |u|^{1-\alpha} dx \le \lambda|\Omega|^F \|u\|_{\beta+1}^{1-\alpha}$. So $\lambda \int |u|^{1-\alpha} dx < H(t_0)$.

On the other hand, $H'(t) < 0$ for $t \in (t_0, \infty)$ and $\lim_{t\to+\infty} H(t) = -\infty$. So, there exists a unique $t^+ \in (t_0,\infty)$ such that $H(t^+) = \lambda \int |u|^{1-\alpha} dx$, i.e., $\|t^+u\|^p - \lambda\|t^+u\|_{\beta+1}^{\beta+1} = \lambda \int_\Omega |tu|^{1-\alpha} dx$. So $t^+u \in \Lambda$. By

$$H'(t^+) = (p-1+\alpha)(t^+)^{p-2+\alpha}\|u\|^p - \lambda(\beta+\alpha)(t^+)^{\beta+\alpha-1}\|u\|_{\beta+1}^{\beta+1} < 0,$$

we have $G(t^+u) = (A\|t^+u\|^p - \lambda\|t^+u\|_{\beta+1}^{\beta+1}) \le 0$. So $t^+u \in \Lambda_-$. □

Remark 2.3. From Lemma 2.2 it follows that the set Λ_- is nonempty.

Lemma 2.4. *Given $u \in \Lambda_-$, then there exist $\varepsilon > 0$ and a continuous function $f = f(w) > 0$, $w \in W_0^{1,p}(\Omega)$, $\|w\| < \varepsilon$, satisfying*

$$f(0) = 1, \quad f(w)(u + w) \in \Lambda_- \text{ for all } w \in W_0^{1,p}(\Omega),\ \|w\| < \varepsilon.$$

Proof. Define $F : R \times W_0^{1,p}(\Omega) \to R$ as follows:

$$F(t, w) = t^{p-1+\alpha}\|u + w\|^p - \lambda t^{\beta+\alpha}\|u + w\|_{\beta+1}^{\beta+1} - \lambda \int_\Omega |u + w|^{1-\alpha} dx.$$

Since $u \in \Lambda_-(\subset \Lambda)$, it follows that $F(1, 0) = 0$ and

$$F_t(1, 0) = (p - 1 + \alpha)\|u\|^p - \lambda(\beta + \alpha)\|u\|_{\beta+1}^{\beta+1} < 0,$$

then we can apply the implicit function theorem at the point $(1, 0)$ and obtain $\overline{\varepsilon} > 0$ and a continuous function $f = f(w) > 0$, $w \in W_0^{1,p}(\Omega)$, $\|w\| < \overline{\varepsilon}$, satisfying $f(0) = 1$, $F(f(w), w) = 0$ for all $w \in W_0^{1,p}(\Omega)$, $\|w\| < \overline{\varepsilon}$. Hence $f(w)(u + w) \in \Lambda$. Let $\varepsilon \in (0, \overline{\varepsilon})$ small enough, we have $f(w)(u + w) \in \Lambda_-$ for all $w \in W_0^{1,p}(\Omega)$, $\|w\| < \varepsilon$. □

Lemma 2.5. *Let*

$$\lambda_3 = \left(\frac{\beta + 1}{1 - \alpha}\right)^B D^B |\Omega|^{FB} \frac{AS}{|\Omega|^{Ep}}. \tag{16}$$

Then, for all $\lambda \in (0, \lambda_3]$, the whole set Λ_- lies at the nonnegative level, that is $I(u) \geq 0$, for all $u \in \Lambda_-$.

Proof. We argue by contradiction. Suppose that exists $u_0 \in \Lambda_- \subset \Lambda$ such that $I(u_0) < 0$, i.e.,

$$\frac{1}{p}\|u_0\|^p - \frac{\lambda}{\beta + 1}\|u_0\|_{\beta+1}^{\beta+1} - \frac{\lambda}{1 - \alpha} \int_\Omega |u_0|^{1-\alpha} dx < 0. \tag{17}$$

By $u_0 \in \Lambda$, we have $\|u_0\|^p = \lambda\|u_0\|_{\beta+1}^{\beta+1} + \lambda \int_\Omega |u_0|^{1-\alpha} dx$. By (17), we have

$$\lambda\left(\frac{1}{p} - \frac{1}{\beta + 1}\right)\|u_0\|_{\beta+1}^{\beta+1} + \lambda\left(\frac{1}{p} - \frac{1}{1 - \alpha}\right)\int_\Omega |u_0|^{1-\alpha} dx < 0,$$

and by (7), we have

$$\|u_0\|_{\beta+1}^{\beta+\alpha} \leq \frac{D(1 + \beta)}{1 - \alpha}|\Omega|^F.$$

By (14) (noticing $u_0 \in \Lambda_-$), we have

$$\left(\frac{AS}{\lambda|\Omega|^{Ep}}\right)^{\frac{\beta+\alpha}{\beta+1-p}} \leq \|u_0\|_{\beta+1}^{\beta+\alpha}.$$

If $0 < \lambda < \lambda_3$, we have

$$\|u_0\|_{\beta+1}^{\beta+\alpha} \leq \frac{D(1 + \beta)}{1 - \alpha}|\Omega|^F < (\frac{AS}{\lambda|\Omega|^{Ep}})^{\frac{\beta+\alpha}{\beta+1-p}} \leq \|u_0\|_{\beta+1}^{\beta+\alpha}.$$

This is a contradiction. So $I(u_0) \geq 0$. □

3. Proof of Theorem 1.2

In this section, we prove that there exist $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, there exist at least two positive functions $u_1(\cdot)$, $u_2(\cdot) \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \cdot \nabla \varphi dx = \lambda \int_\Omega u_i^\beta \varphi dx + \lambda \int_\Omega \frac{\varphi}{u_i^\alpha} dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega),\ i = 1, 2.$$

Moreover u_1 is a local minimizer of I in $W_0^{1,p}(\Omega)$ with $I(u_1) < 0$; and u_2 is a minimizer of I on Λ_-.

Proof of Theorem 1.2. Using the inequalities (8) and (9), we have

$$I(u) \geq \frac{1}{p}\|u\|^p - \lambda C_3 \|u\|^{\beta+1} - \lambda C_4 \|u\|^{1-\alpha}, \quad \forall u \in W_0^{1,p}(\Omega),$$

where $C_3, C_4 > 0$ are positive constants. From this we readily find that there exists $\lambda_4 > 0$ such that for all $\lambda \in (0, \lambda_4]$ there are $r, a > 0$ such that

(i) $I(u) \geq a$ for all $\|u\| = r$;

(ii) I is bounded on $B_r = \left\{u \in W_0^{1,p}(\Omega) : \|u\| \leq r\right\}$;

Let $\lambda_0 = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where $\lambda_i (i = 1, 2, 3)$ are the values found in (10), (15), (16), and λ_4 is defined as above. Next, we fix $\lambda \in (0, \lambda_0)$.

Existence of u_1. In view of of [6, Theorem 1.2] the infinimum of I on B_r can be achieved at a point $u_1 \in B_r$. Note that, since $1 - \alpha < 1$, it follows that for every $v > 0$, $I(tv) < 0$ as $t > 0$ small. So there exists $v_1 \in B_r$ such that $I(v_1) < 0$. Hence $I(u_1) = \inf_{u \in B_r} I(u) \leq I(v_1) < 0$. This, together with (i), implies that $u_1 \notin \partial B_1$. Hence u_1 is a local minimizer of I in the $W_0^{1,p}$ topology. Clearly, $u_1 \not\equiv 0$. Moreover, since $I(|u|) = I(u)$, we may assume that $u_1 \geq 0$ in Ω. Then, for any $\varphi \in W_0^{1,p}$, $\varphi \geq 0$,

$$\begin{aligned}
0 &\leq I(u_1 + t\varphi_1) - I(u_1) \\
&= \frac{1}{p}\left(\|u_1 + t\varphi\|^p - \|u_1\|^p\right) + \frac{\lambda}{\beta+1}\left(\|u_1\|_{\beta+1}^{\beta+1} - \|u_1 + t\varphi\|_{\beta+1}^{\beta+1}\right) \\
&\quad + \frac{\lambda}{1-\alpha}\left(\int_\Omega |u_1|^{1-\alpha} dx - \int_\Omega |u_1 + t\varphi|^{1-\alpha} dx\right) \\
&\leq \frac{1}{p}\left(\|u_1 + t\varphi\|^p - \|u_1\|^p\right),
\end{aligned}$$

i.e.,

$$0 \leq \frac{1}{p}\int_\Omega \left(|\nabla(u_1 + t\varphi)|^p - |\nabla u_1|^p\right) dx \tag{18}$$

provided $t > 0$ small enough. Dividing (18) by $t > 0$ and passing to the limit as $t \to 0$, we derive

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla \varphi dx \geq 0 \quad \text{for } \varphi \in W_0^{1,p}(\Omega),\ \varphi \geq 0,$$

which means $u_1 \in W_0^{1,p}(\Omega)$ satisfies in a weak sense that $-\Delta_p u_1 \geq 0$ in Ω. Since $u_1 \geq 0$, $u_1 \not\equiv 0$, then the strong maximum principle yields

$$u_1 > 0 \quad \text{in } \Omega.$$

On the other hand, from (18), we have

$$\begin{aligned} &\frac{\lambda}{1-\alpha}\left(\int_\Omega |u_1 + t\varphi|^{1-\alpha}dx - \int_\Omega |u_1|^{1-\alpha}dx\right) \\ &\quad \leq \frac{1}{p}\left(\|u_1 + t\varphi\|^p - \|u_1\|^p\right) - \frac{\lambda}{\beta+1}\left(\|u_1 + t\varphi\|_{\beta+1}^{\beta+1} - \|u_1\|_{\beta+1}^{\beta+1}\right). \end{aligned} \tag{19}$$

Dividing (19) by $t > 0$ and passing to the limit, it follows that

$$\begin{aligned} &\frac{\lambda}{1-\alpha}\liminf_{t\to 0^+}\frac{\int_\Omega |u_1 + t\varphi|^{1-\alpha}dx - \int_\Omega |u_1|^{1-\alpha}dx}{t} \\ &\qquad \leq \int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\varphi dx - \lambda\int_\Omega u_1^\beta \varphi dx. \end{aligned} \tag{20}$$

Observing

$$\frac{1}{1-\alpha}\int_\Omega \frac{(u_1 + t\varphi)^{1-\alpha} - u_1^{1-\alpha}}{t}dx = \int_\Omega (u_1 + \theta t\varphi)^{-\alpha}\varphi dx,$$

where $\theta \to 0^+$ as $t \to 0^+$ and $(u_1 + \theta t\varphi)^{-\alpha}\varphi \to u_1^{-\alpha}\varphi$ a.e. in Ω as $t \to 0^+$. Since $0 \leq (u_1 + \theta t\varphi)^{-\alpha}\varphi$, for all $x \in \Omega$. By Fatou's Lemma, we have

$$\frac{1}{1-\alpha}\liminf_{t\to 0^+}\int_\Omega \frac{(u_1 + t\varphi)^{1-\alpha} - u_1^{1-\alpha}}{t}dx \geq \int_\Omega u_1^{-\alpha}\varphi dx. \tag{21}$$

Combining (20) and (21), we have, for all $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$,

$$0 \leq \int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\varphi dx - \lambda\int_\Omega u_1^\beta\varphi dx - \lambda\int_\Omega u_1^{-\alpha}\varphi dx\,. \tag{22}$$

On the other hand, there exists $\eta_1 \in (0,1)$ such that $u_1 + tu_1 \in B_r$ for $|t| \leq \eta_1$. We define $h_1 : [-\eta_1, \eta_1] \to R$ by $h_1(t) \equiv I((1+t)u_1)$. We have that $h_1(t)$ achieves its minimum at $t = 0$. Therefore,

$$\left.\frac{dh_1}{dt}\right|_{t=0} = \int_\Omega \left[|\nabla u_1|^p - \lambda u_1^{\beta+1} - \lambda u_1^{1-\alpha}\right]dx = 0. \tag{23}$$

Therefore, $u_1 \in \Lambda$.

We next prove that u_1 is a positive weakly solution. Suppose $\phi \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$. Let $\Psi \equiv (u_1 + \varepsilon\phi)^+$, where $(u_1 + \varepsilon\phi)^+ = \max\{u_1 + \varepsilon\phi, 0\}$. Then

$\Psi \in W_0^{1,p}(\Omega)$ and $\Psi \geq 0$. Inserting Ψ into (22) and using (23) again, we infer that

$$\begin{aligned}
0 &\leq \int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\Psi dx - \lambda \int_\Omega u_1^\beta \Psi dx - \lambda \int_\Omega u_1^{-\alpha}\Psi dx \\
&= \int_{\Omega\setminus\Omega_\varepsilon} \left[|\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla(u_1+\varepsilon\phi) - \lambda u_1^\beta(u_1+\varepsilon\phi) - \lambda u_1^{-\alpha}(u_1+\varepsilon\phi)\right] dx \\
&= \int_\Omega \left[|\nabla u_1|^p - \lambda u_1^{\beta+1} - \lambda u_1^{1-\alpha}\right] dx \\
&\quad + \varepsilon \int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\phi dx - \lambda\varepsilon \int_\Omega u_1^\beta \phi dx - \lambda\varepsilon \int_\Omega u_1^{-\alpha}\phi dx \\
&\quad - \int_{\Omega_\varepsilon} \left[|\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla(u_1+\varepsilon\phi) - \lambda u_1^\beta(u_1+\varepsilon\phi) - \lambda u_1^{-\alpha}(u_1+\varepsilon\phi)\right] dx \\
&\leq \varepsilon \int_\Omega \left[|\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\phi - \lambda u_1^\beta\phi - \lambda u_1^{-\alpha}\phi\right] dx - \varepsilon \int_{\Omega_\varepsilon} |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\phi dx,
\end{aligned}$$

where $\Omega_\varepsilon = \{x \in \Omega : u_1(x) + \varepsilon\phi(x) < 0\}$. Since the measure of Ω_ε tends to zero as $\varepsilon \to 0$, it follows that $\int_{\Omega_\varepsilon} |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\phi dx \to 0$ as $\varepsilon \to 0$. Dividing by ε and letting $\varepsilon \to 0$ therefore shows

$$\int_\Omega \left[|\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\phi - \lambda u_1^\beta\phi - \lambda u_1^{-\alpha}\phi\right] dx \geq 0.$$

Noting that ϕ is arbitrary, this holds equally for $-\phi$. So

$$\int_\Omega \left[|\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\phi - \lambda u_1^\beta\phi - \lambda u_1^{-\alpha}\phi\right] dx = 0, \quad \text{for all } \phi \in W_0^{1,p}(\Omega).$$

Hence, u_1 is a positive weak solution of (1) and $I(u_1) < 0$

Next, we prove that (1) has another positive weakly solution u_2 such that $I(u_2) > 0$. We first show that I is coercive on Λ. Indeed, for $u \in \Lambda$, we have

$$\|u\|^p - \lambda\|u\|_{\beta+1}^{\beta+1} - \lambda \int_\Omega |u|^{1-\alpha} dx = 0. \tag{24}$$

By (24) and (9), we have

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \frac{\lambda}{\beta+1}\|u\|_{\beta+1}^{\beta+1} - \frac{\lambda}{1-\alpha}\int_\Omega |u|^{1-\alpha} dx \\
&\geq \left(\frac{1}{p} - \frac{1}{\beta+1}\right)\|u\|^p - \lambda C_2 \left(\frac{1}{1-\alpha} - \frac{1}{\beta+1}\right)\|u\|^{1-\alpha}.
\end{aligned}$$

So, I is coercive on Λ. Since Λ_- is a closed set in $W_0^{1,p}(\Omega)$, we apply Ekeland's variational Principle to the minimization problem $\inf_{\Lambda_-} I$. It gives a minimizing sequence $\{w_n\} \subset \Lambda_-$ with the following properties:

(i) $I(w_n) < \inf_{\Lambda_-} I + \frac{1}{n}$;
(ii) $I(w) \geq I(w_n) - \frac{1}{n}\|w - w_n\|$, $\forall w \in \Lambda_-$.

Since $I(|u|) = I(u)$, we may assume that $w_n \geq 0$ in Ω. By coerciveness, $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$, i.e.,

$$\|w_n\| \leq C_5, \quad n = 1, 2, \ldots, \tag{25}$$

where $C_5 > 0$ is some constant independent on n. So there exists a subsequence (without loss of generality, suppose it is itself) and a function $u_2 \geq 0$ such that

$$\begin{array}{lcll} w_n & \rightarrow & u_2 & \text{a.e. } x \in \Omega \\ w_n & \overset{\text{strongly}}{\rightarrow} & u_2 & \text{in } L^{\beta+1} \\ w_n & \overset{\text{weakly}}{\rightarrow} & u_2 & \text{in } W_0^{1,p}. \end{array}$$

On the other hand, by (14)

$$\frac{AS}{\lambda|\Omega|^{Ep}} \leq \|w_n\|_{\beta+1}^{\beta+1-p}, \tag{26}$$

so $u_2 \not\equiv 0$. In addition, for the minimizing sequence $\{w_n\}$ there exists a suitable constant $C_6 > 0$ such that

$$A\|w_n\|^p - \lambda\|w_n\|_{\beta+1}^{\beta+1} \leq -C_6 \quad n = 1, 2, \ldots . \tag{27}$$

Suppose, by contradiction, that for a subsequence, which is still denoted by $\{w_n\}$, we have

$$A\|w_n\|^p - \lambda\|w_n\|_{\beta+1}^{\beta+1} = o(1).$$

Using $\{w_n\} \subset \Lambda_-$ and (26), we have

$$\begin{aligned} I(w_n) &= \frac{1}{p}\|w_n\|^p - \frac{\lambda}{\beta+1}\|w_n\|_{\beta+1}^{\beta+1} - \frac{1}{1-\alpha}\|w_n\|^p + \frac{\lambda}{1-\alpha}\|w_n\|_{\beta+1}^{\beta+1} \\ &= -\frac{\beta+\alpha}{p(1-\alpha)}G(w_n) - \frac{\lambda(\beta+1-p)}{\beta+1}\|w_n\|_{\beta+1}^{\beta+1} \\ &\leq -\frac{\beta+\alpha}{p(1-\alpha)}G(w_n) - C_7 \quad \text{for } n = 1, 2, \ldots , \end{aligned}$$

where $C_7 > 0$ is some constant independent of n. Passing to the limit as $n \to \infty$, we get $\lim_{n\to\infty} I(w_n) \leq -C_7$. This, together with $I(w_n) \geq \inf_{\Lambda_-} I(u)$ implies $\inf_{u\in\Lambda_-} I(u) \leq -C_7 < 0$, which is clearly impossible because from Lemma 2.5. It follows that $\inf_{u\in\Lambda_-} I(u) \geq 0$.

For all $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$, applying Lemma 2.4, with $u = w_n$, $w = t\varphi$, $t > 0$ small, we find $f_n(t) = f_n(t\varphi)$ such that $f_n(0) = 1$ and $f_n(t)(w_n + t\varphi) \in \Lambda_-$. Note that, since

$$0 = f_n^p(t)\|w_n + t\varphi\|^p - \lambda f_n^{\beta+1}\|w_n + t\varphi\|_{\beta+1}^{\beta+1} - f_n^{1-\alpha}(t)\int_\Omega (w_n + t\varphi)^{1-\alpha} dx$$

and $0 = \|w_n\|^p - \lambda\|w_n\|_{\beta+1}^{\beta+1} - \lambda\int_\Omega w_n^{1-\alpha} dx$, so

$$\begin{aligned}
0 &= f_n^p(t)\|w_n + t\varphi\|^p - \lambda f_n^{\beta+1}\|w_n + t\varphi\|_{\beta+1}^{\beta+1} - \lambda f_n^{1-\alpha}(t)\int_\Omega (w_n + t\varphi)^{1-\alpha} dx \\
&\quad - \|w_n\|^p + \lambda\|w_n\|_{\beta+1}^{\beta+1} + \lambda\int_\Omega w_n^{1-\alpha} dx \\
&= \big(f_n^p(t) - 1\big)\|w_n + t\varphi\|^p + \big(\|w_n + t\varphi\|^p - \|w_n\|^p\big) \\
&\quad - \lambda\big(f_n^{\beta+1} - 1\big)\|w_n + t\varphi\|_{\beta+1}^{\beta+1} - \lambda\Big(\|w_n + t\varphi\|_{\beta+1}^{\beta+1} - \|w_n\|_{\beta+1}^{\beta+1}\Big) \\
&\quad - \lambda\big(f_n^{1-\alpha} - 1\big)\int_\Omega (w_n + t\varphi)^{1-\alpha} dx - \lambda\int_\Omega \big[(w_n + t\varphi)^{1-\alpha} - w_n^{1-\alpha}\big] dx \\
&\leq \big(f_n^p(t) - 1\big)\|w_n + t\varphi\|^p + \big(\|w_n + t\varphi\|^p - \|w_n\|^p\big) \\
&\quad - \lambda\big(f_n^{\beta+1} - 1\big)\|w_n + t\varphi\|_{\beta+1}^{\beta+1} - \lambda\Big(\|w_n + t\varphi\|_{\beta+1}^{\beta+1} - \|w_n\|_{\beta+1}^{\beta+1}\Big) \\
&\quad - \lambda\big(f_n^{1-\alpha} - 1\big)\int_\Omega (w_n + t\varphi)^{1-\alpha} dx.
\end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0$, we infer that

$$\begin{aligned}
0 &\leq p f_{n+}'(0)\|w_n\|^p + p\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla\varphi dx \\
&\quad - \lambda f_{n+}'(0)(\beta+1)\|w_n\|_{\beta+1}^{\beta+1} - \lambda(\beta+1)\int_\Omega w_n^\beta \varphi dx - \lambda(1-\alpha) f_{n+}'(0)\int_\Omega w_n^{1-\alpha} dx \\
&= f_{n+}'(0)\Big[p\|w_n\|^p - \lambda(\beta+1)\|w_n\|_{\beta+1}^{\beta+1} - \lambda(1-\alpha)\|w_n\|_{1-\alpha}^{1-\alpha}\Big] \\
&\quad + p\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla\varphi dx - \lambda(\beta+1)\int_\Omega w_n^\beta \varphi dx \\
&= f_{n+}'(0)\Big[(p+\alpha-1)\|w_n\|^p - \lambda(\beta+\alpha)\|w_n\|_{\beta+1}^{\beta+1}\Big] \\
&\quad + p\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla\varphi dx - \lambda(\beta+1)\int_\Omega w_n^\beta \varphi dx\,,
\end{aligned}$$

i.e.,

$$\begin{aligned}
0 &\leq f_{n+}'(0)\Big[(p+\alpha-1)\|w_n\|^p - \lambda(\beta+\alpha)\|w_n\|_{\beta+1}^{\beta+1}\Big] \\
&\quad + p\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla\varphi dx - \lambda(\beta+1)\int_\Omega w_n^\beta \varphi dx\,,
\end{aligned} \tag{28}$$

where $f_{n+}^{'}(0) = \lim_{t\to 0^+} \frac{f_n(t)-f_n(0)}{t}$. For the sake of simplicity, we assume henceforth that the right derivate of f_n at $t = 0$ exists. Indeed, if it doesn't exist, we let $t_k \to 0$ (instead of $t \to 0$), $t_k > 0$ is chosen in such a way that f_n satisfies $q_n := \lim_{k\to\infty} \frac{f_n(t_k)-f_n(0)}{t_k}$, then replace $f_{n+}^{'}(0)$ by q_n. We next prove that $f_{n+}^{'}(0) \neq \pm\infty$.

By (8) and (25)

$$\begin{aligned}\left|p\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx - \lambda(\beta+1)\int_\Omega w_n^\beta \varphi dx\right| \\ \leq p\|w_n\|^{p-1}\|\varphi\|^p + \lambda(\beta+1)\|w_n\|_{\beta+1}^\beta \|\varphi\|_{\beta+1} \leq C_8,\end{aligned} \tag{29}$$

where $C_8 > 0$ is a positive constant. For (27), (28) and (29), we know immediately that $f_{n+}^{'}(0) \neq +\infty$. Now we prove that $f_{n+}^{'}(0) \neq -\infty$. By contradiction, we assume that $f_{n+}^{'}(0) = -\infty$, and so for $t > 0$ small there holds $f_n(t) < 1$. Then

$$\begin{aligned}\|f_n(t)(w_n+t\varphi)-w_n\| &= \left(\int_\Omega \big|f_n(t)(\nabla w_n + t\nabla\varphi) - \nabla w_n\big|^p dx\right)^{\frac{1}{p}} \\ &= \left(\int_\Omega \big|(f_n(t)-1)\nabla w_n + tf_n(t)\nabla\varphi\big|^p dx\right)^{\frac{1}{p}} \\ &\leq \big[1-f_n(t)\big]\|w_n\| + tf_n(t)\|\varphi\|\end{aligned}$$

provided $t > 0$ small. Thus, from (ii) we have $\frac{1}{n}\|w - w_n\| \geq I(w_n) - I(w)$. So

$$\begin{aligned}&\big[1-f_n(t)\big]\frac{\|w_n\|}{n} + tf_n(t)\frac{\|\varphi\|}{n} \\ &\quad\geq \frac{1}{n}\|f_n(t)(w_n+t\varphi)-w_n\| \\ &\quad\geq I(w_n) - I\big(f_n(t)(w_n+t\varphi)\big) \\ &\quad= \frac{1}{p}\|w_n\|^p - \frac{\lambda}{\beta+1}\|w_n\|_{\beta+1}^{\beta+1} - \frac{\lambda}{1-\alpha}\int_\Omega |w_n|^{1-\alpha}dx - \frac{1}{p}\|f_n(t)(w_n+t\varphi)\|^p \\ &\qquad+ \frac{\lambda}{\beta+1}\|f_n(t)(w_n+t\varphi)\|_{\beta+1}^{\beta+1} + \frac{\lambda}{1-\alpha}\int_\Omega \big|f_n(t)(w_n+t\varphi)\big|^{1-\alpha}dx\end{aligned}$$

Using

$$-\frac{\lambda}{1-\alpha}\int_\Omega |w_n|^{1-\alpha}dx = -\frac{1}{1-\alpha}\|w_n\|^p + \frac{\lambda}{1-\alpha}\|w_n\|_{\beta+1}^{\beta+1}$$

and

$$\begin{aligned}\frac{\lambda}{1-\alpha}\int_\Omega \big|f_n(t)(w_n+t\varphi)\big|^{1-\alpha}dx &= \frac{1}{1-\alpha}f_n^p(t)\|w_n+t\varphi\|^p \\ &\quad- \frac{\lambda}{1-\alpha}f_n^{\beta+1}(t)\|w_n+t\varphi\|_{\beta+1}^{\beta+1},\end{aligned}$$

we have

$$
\begin{aligned}
&\big[1-f_n(t)\big]\frac{\|w_n\|}{n}+tf_n(t)\frac{\|\varphi\|}{n}\\
&\quad\geq\left(\frac{1}{p}-\frac{1}{1-\alpha}\right)\|w_n\|^p-\left(\frac{\lambda}{\beta+1}-\frac{\lambda}{1-\alpha}\right)\|w_n\|_{\beta+1}^{\beta+1}\\
&\qquad-\left(\frac{1}{p}-\frac{1}{1-\alpha}\right)f_n^p(t)\|w_n+t\varphi\|^p\\
&\qquad+\lambda\left(\frac{1}{\beta+1}-\frac{1}{1-\alpha}\right)f_n^{\beta+1}(t)\|w_n+t\varphi\|_{\beta+1}^{\beta+1}\\
&\quad=\frac{p+\alpha-1}{p(1-\alpha)}\big(\|w_n+t\varphi\|^p-\|w_n\|^p\big)+\frac{p+\alpha-1}{p(1-\alpha)}\big(f_n^p(t)-1\big)\|w_n+t\varphi\|^p\\
&\qquad-\lambda\frac{\beta+\alpha}{(\beta+1)(1-\alpha)}f_n^{\beta+1}(t)\Big(\|w_n+t\varphi\|_{\beta+1}^{\beta+1}-\|w_n\|_{\beta+1}^{\beta+1}\Big)\\
&\qquad-\lambda\frac{\beta+\alpha}{(\beta+1)(1-\alpha)}\big[f_n^{\beta+1}(t)-1\big]\|w_n\|_{\beta+1}^{\beta+1}\,.
\end{aligned}
$$

Dividing by $t>0$ and passing to the limit as $t\to 0$, we have

$$
\begin{aligned}
&-f_{n+}'(0)\frac{\|w_n\|}{n}+\frac{\|\varphi\|}{n}\\
&\quad\geq\frac{p+\alpha-1}{p(1-\alpha)}\int_\Omega|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx+\frac{p+\alpha-1}{1-\alpha}f_{n+}'(0)\|w_n\|^p\\
&\qquad-\frac{\beta+\alpha}{1-\alpha}\int_\Omega|w_n|^\beta\varphi dx-\lambda\frac{\beta+\alpha}{1-\alpha}f_{n+}'(0)\|w_n\|_{\beta+1}^{\beta+1}\\
&\quad=\frac{1}{1-\alpha}\Big[(p+\alpha-1)\|w_n\|^p-\lambda(\beta+\alpha)\|w_n\|_{\beta+1}^{\beta+1}\Big]f_{n+}'(0)\\
&\qquad+\frac{1}{1-\alpha}\Big[(p+\alpha-1)\int_\Omega|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx-\lambda(\beta+\alpha)\int_\Omega|w_n|^\beta\varphi dx\Big],
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\frac{\|\varphi\|}{n}\geq&\frac{1}{1-\alpha}\Big[(p+\alpha-1)\|w_n\|^p-\lambda(\beta+\alpha)\|w_n\|_{\beta+1}^{\beta+1}+\frac{1-\alpha}{n}\|w_n\|\Big]f_{n+}'(0)\\
&+\frac{1}{1-\alpha}\Big[(p+\alpha-1)\int_\Omega|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx\\
&-\lambda(\beta+\alpha)\int_\Omega|w_n|^\beta\varphi dx\Big].
\end{aligned}
\tag{30}
$$

By (25) and (27), there exist $N_0>0$ and $C_9>0$ (independent of n) such that, for $n\geq N_0$,

$$
\frac{1}{1-\alpha}\left[(p+\alpha-1)\|w_n\|^p-\lambda(\beta+\alpha)\|w_n\|_{\beta+1}^{\beta+1}+\frac{1-\alpha}{n}\|w_n\|\right]\leq -C_9\,.
$$

On the other hand, by (8) and (25), we have, for $n \geq N_0$,

$$\left|\frac{1}{1-\alpha}\left[(p+\alpha-1)\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx - \lambda(\beta+\alpha)\int_\Omega |w_n|^\beta\varphi dx\right]\right| \leq C_{10}\,,$$

where $C_{10} > 0$ (independent of n) is a suitable constant. By (30), it is impossible that $f'_{n+}(0) = -\infty$. Furthermore, (28) and (30) imply that $|f'_{n+}(0)| \leq C_{11}$ for $n = 1, 2, \ldots$, where $C_{11} > 0$ is a suitable constant.

Now we prove that $u_2 \in \Lambda_-$ is a positive weakly solution of (1). From condition (ii) we infer $\frac{1}{n}\|w - w_n\| \geq I(w_n) - I(w)$, i.e.,

$$\begin{aligned}
&\frac{1}{n}[|f_n(t)-1|\|w_n\| + tf_n(t)\|\varphi\|]\\
&\quad\geq \frac{1}{n}\|f_n(t)(w_n+t\varphi) - w_n\|\\
&\quad\geq I(w_n) - I\big(f_n(t)(w_n+t\varphi)\big)\\
&\quad= \frac{1}{p}\|w_n\|^p - \frac{\lambda}{\beta+1}\|w_n\|_{\beta+1}^{\beta+1} - \frac{\lambda}{1-\alpha}\int_\Omega |w_n|^{1-\alpha}dx\\
&\qquad - \frac{1}{p}\|f_n(t)(w_n+t\varphi)\|^p + \frac{\lambda}{\beta+1}\|f_n(t)(w_n+t\varphi)\|_{\beta+1}^{\beta+1}\\
&\qquad + \frac{\lambda}{1-\alpha}\int_\Omega \big|f_n(t)(w_n+t\varphi)\big|^{1-\alpha}dx\\
&\quad= -\frac{f_n^p(t)-1}{p}\|w_n\|^p + \lambda\frac{f_n^{\beta+1}(t)-1}{\beta+1}\|w_n\|_{\beta+1}^{\beta+1} + \lambda\frac{f_n^{1-\alpha}(t)-1}{1-\alpha}\int_\Omega |w_n|^{1-\alpha}dx\\
&\qquad - \frac{f_n^p(t)}{p}\big(\|w_n+t\varphi\|^p - \|w_n\|^p\big) + \frac{\lambda}{\beta+1}f_n^{\beta+1}(t)\left(\|w_n+t\varphi\|_{\beta+1}^{\beta+1} - \|w_n\|_{\beta+1}^{\beta+1}\right)\\
&\qquad + \frac{\lambda}{1-\alpha}f_n^{1-\alpha}(t)\int_\Omega [(w_n+t\varphi)^{1-\alpha} - w_n^{1-\alpha}]dx.
\end{aligned}$$

Dividing by $t > 0$ and passing to the limit as $t \to 0$, this yields

$$\begin{aligned}
&\frac{1}{n}\left[|f'_{n+}(0)|\|w_n\| + \|\varphi\|\right]\\
&\qquad\geq -f'_{n+}(0)\|w_n\|^p + \lambda f'_{n+}(0)\|w_n\|_{\beta+1}^{\beta+1} + \lambda f'_{n+}(0)\int_\Omega |w_n|^{1-\alpha}dx\\
&\qquad\quad - \int_\Omega |\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx + \lambda\int_\Omega w_n^\beta\varphi dx\\
&\qquad\quad + \liminf_{t\to 0^+}\frac{\lambda}{1-\alpha}\int_\Omega \frac{(w_n+t\varphi)^{1-\alpha} - w_n^{1-\alpha}}{t}dx\\
&\qquad= -\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx + \lambda\int_\Omega w_n^\beta\varphi dx\\
&\qquad\quad + \liminf_{t\to 0^+}\frac{\lambda}{1-\alpha}\int_\Omega \frac{(w_n+t\varphi)^{1-\alpha} - w_n^{1-\alpha}}{t}dx.
\end{aligned}$$

Since $(w_n(x)+t\varphi(x))^{1-\alpha}-w_n^{1-\alpha}(x)\geq 0$, for all $x\in\Omega$, $t>0$, then by Fatou's Lemma, we have

$$\lambda\int_\Omega w_n^{1-\alpha}\varphi dx\leq\liminf_{t\to 0^+}\frac{\lambda}{1-\alpha}\int_\Omega\frac{(w_n+t\varphi)^{1-\alpha}-w_n^{1-\alpha}}{t}dx.$$

So

$$\begin{aligned}\lambda\int_\Omega w_n^{-\alpha}\varphi dx&\leq\frac{1}{n}\left[|f_{n+}'(0)|\|w_n\|+\|\varphi\|\right]\\&\quad+\int_\Omega|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx-\lambda\int_\Omega w_n^\beta\varphi dx\\&\leq\frac{C_{11}C_5+\|\varphi\|}{n}+\int_\Omega|\nabla w_n|^{p-2}\nabla w_n\cdot\nabla\varphi dx-\lambda\int_\Omega w_n^\beta\varphi dx.\end{aligned}$$

Let $n\to\infty$, we have

$$\liminf_{n\to\infty}\lambda\int_\Omega w_n^{-\alpha}\varphi dx\leq\int_\Omega|\nabla u_2|^{p-2}\nabla u_2\cdot\nabla\varphi dx-\lambda\int_\Omega u_2^\beta\varphi dx;$$

then using once more Fatou's Lemma, we infer that, for all $\varphi\in W_0^{1,p}(\Omega)$, $\varphi\geq 0$,

$$\int_\Omega|\nabla u_2|^{p-2}\nabla u_2\cdot\nabla\varphi dx-\lambda\int_\Omega u_2^\beta\varphi dx-\lambda\int_\Omega w_2^{-\alpha}\varphi dx\geq 0\,, \tag{31}$$

which means that u_2 satisfies $-\Delta_p u_2\geq 0$ in Ω. Since $u_2\geq 0$ and $u_2\not\equiv 0$ in Ω, then the strong maximum principle yields $u_2>0$ in Ω. In particular, using (31) with $\varphi=u_2$, we infer that

$$\|u_2\|^p-\lambda\|u_2\|_{\beta+1}^{\beta+1}-\lambda\int_\Omega u_2^{1-\alpha}dx\geq 0.$$

On the other hand, by weakly lower semi-continuity of the norm

$$\|u_2\|^p\leq\lambda\|u_2\|_{\beta+1}^{\beta+1}+\lambda\int_\Omega u_2^{1-\alpha}dx.$$

So

$$\|u_2\|^p=\lim_{n\to\infty}\|w_n\|^p=\lambda\|u_2\|_{\beta+1}^{\beta+1}+\lambda\int_\Omega u_2^{1-\alpha}dx. \tag{32}$$

Consequently

$$w_n\overset{\text{strongly}}{\to}u_2\quad\text{in } W_0^{1,p}(\Omega)$$

and $I(u_2)=\inf_{\Lambda_-}I$. Also from Lemma 2.1, it follows that necessarily $u_2\in\Lambda_-$. Then, following the same arguments as in proving the existence of u_1 and using (31)–(32), we obtain $u_2\in\Lambda_-$ is a positive weakly solution of (1). This completes the proof of Theorem 1.2. □

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