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Elaboration of Some Results of Srivastava and Choi

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Abstract. In this paper we shall utilize some recent results of S. Kanemitsu, H. Kumagai, H. M. Srivastava and M. Yoshimoto in *Appl. Math. Comput.* 154 (2004) on an asymptotic as well as an integral formula for the partial sum of the Hurwitz zeta-function, to elaborate on some results of Srivastava and Choi in *Series Associated with the Zeta and Related Functions* (Kluwer 2001), and in some cases to give improved generalizations thereof. More specifically, we shall give an asymptotic expansion of the sum of the values derivative of the digamma function. We shall also re-establish Bendersky–Adamchik's result and Elizalde's result.

Keywords. Hurwitz zeta-function, partial sum, digamma function, generalized Euler constant

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1. Introduction and basic results

In [7], an asymptotic as well as an integral formula for the partial sum

$$L_u(x,a) = \sum_{0\le n\le x} (n+a)^u$$

of the corresponding Hurwitz zeta-function $\zeta(-u,a)$ is obtained which we reproduce as Proposition 1 below, where u indicates a complex variable, $a>0$, $x\ge 0$ and the principal value of $\log(n+a)$ is taken. The result is far-reaching and easily applicable, entailing the corresponding result for $\zeta(-u,a)$. In a sequal paper [8] to [7], some applications were made of Proposition 1 and its corollaries (i.e., the successive derivatives), and it is shown, among other things, that the theory of the gamma function $\Gamma(a)$ can be based on that of the (special case of the derivative of) Hurwitz zeta function $\zeta'(0,a) = \frac{\partial}{\partial s}\zeta(s,a)|_{s=0}$.

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Our aim in this paper is likewise to show that by using Proposition 1 and its corollaries, we may elaborate on some of the results of Srivastava and Choi and others scattered around in their book [10] in clearer perspective, and in some cases, we may give improved generalization thereof; an example, in contrast to the case of the gamma function, is Theorem 1 which gives a complete asymptotic expansion of the sum of the values of $\psi^{(\nu)}(n)$.

In this section we shall state Proposition 1 and its corollary (Corollary 1) with its proof in the same lines as those of the proof of [5, Lemma 8]. Then in Remark 2 we shall make a more explicit statement than that of [7] to the effect that mere comparison of the Taylor coefficients of (1.11) gives the simplest proof of the fact that the k-th Laurent coefficient of $\zeta(s,a)$ is given by $\frac{(-1)^k}{k!}\gamma_k(a)$, where $\gamma_k(a)$ is defined by (1.9). In §2 we shall prove Theorem 1 and other formulas which are deducible from Proposition 1, while in §3 we shall give some results which follow from Corollary 1, i.e., from the results on $\frac{\partial}{\partial u}L_u(x,a)$.

We use the following notation: s denotes the complex variable with $\mathrm{Re}\, s=\sigma$, $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$ the *gamma function* $(\sigma>0)$ and $\psi(s)=\frac{\Gamma'}{\Gamma}(s)=(\log\Gamma(s))'$ the *digamma function*, both of which are meromorphically continued to the whole complex plane with simple poles at non-positive integers;

$\zeta(s,a)=\sum_{n=0}^\infty \frac{1}{(n+a)^s}$ denotes the *Hurwitz zeta function*, $\sigma>1$, $a>0$, the power taking the principal value, and $\zeta(s)=\zeta(s,1)$ the *Riemann zeta-function*, both of which are continued meromorphically over the complex plane with a simple pole at $s=1$;

$B_r^{(\alpha)}(x)$ denotes the generalized *Bernoulli polynomial* of degree r in x, defined through the generating function $(\frac{z}{e^z-1})^\alpha e^{zx}=\sum_{r=0}^\infty \frac{1}{r!}B_r^{(\alpha)}(x)z^n$ $(|z|<2\pi)$ ([10, p. 61]) satisfying the addition formula

$$B_r^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{r}\binom{r}{k}B_k^{(\alpha)}(x)B_{r-k}^{(\beta)}(y) \tag{1.1}$$

([10, Formula (24), p. 61]) with the properties $B_r^{(\alpha)}=B_r^{(\alpha)}(0)$, $B_r^{(1)}(x)=B_r(x)$, $B_r^{(1)}=B_r$, where $B_r(x)$ and $B_r=B_r(0)$ are the r-th Bernoulli polynomial and the r-th Bernonlli number defined by (1.1) with $\alpha=1$. For a real number y, we denote its integral part and fractional part by $[y]$ and $\{y\}$, respectively. Then $\overline{B}_r(x)=B_r(\{x\})$ denotes the r-th *periodic Bernoulli polynomial.*

We are in a position to state basic results from which we start our investigation, both results due to [7].

Proposition 1 ([7, Theorem 1], Integral Representations). *Let*

$$L_u(x,a) = \sum_{0\le n\le x} (n+a)^u.$$

Then, for any $l \in \mathbb{N}$ with $l > \operatorname{Re} u + 1$,Rea > 0 we have

$$\begin{aligned} L_u(x,a) = {} & \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r}{r!} \overline{B}_r(x)(x+a)^{u-r+1} \\ & + \frac{(-1)^l}{l!} \frac{\Gamma(u+1)}{\Gamma(u+1-l)} \int_x^\infty \overline{B}_l(t)(t+a)^{u-l} dt \\ & + \begin{cases} \dfrac{1}{u+1}(x+a)^{u+1} + \zeta(-u,a), & u \ne -1 \\ \log(x+a) - \psi(a), & u = -1. \end{cases} \end{aligned} \tag{1.2}$$

Also the asymptotic formula

$$\begin{aligned} L_u(x,a) = {} & \sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{u}{r-1} \overline{B}_r(x)(x+a)^{u-r+1} + O\left(x^{\operatorname{Re} u - l}\right) \\ & + \begin{cases} \dfrac{1}{u+1}(x+a)^{u+1} + \zeta(-u,a), & u \ne -1 \\ \log(x+a) - \psi(a), & u = -1 \end{cases} \end{aligned} \tag{1.3}$$

holds true as $x \to \infty$. On the other hand, formula (1.2) with $x = 0$ yields the integral representation

$$\begin{aligned} \zeta(-u,a) = {} & a^u - \frac{1}{u+1} a^{u+1} - \sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{u}{r-1} B_r a^{u-r+1} \\ & + (-1)^{l+1} \binom{u}{l} \int_0^\infty \overline{B}_l(t)(t+a)^{u-l} dt, \end{aligned} \tag{1.4}$$

which is true for all $u \ne -1$, and l can be any natural number satisfying $l >$ $\operatorname{Re} u + 1$; the integral being absolutely convergent in the region $\operatorname{Re} u < l - 1$, where it is analytic except at $u = -1$. The counterpart of (1.4) for $u = -1$ reads

$$\psi(a) = \log a - \frac{1}{2} a^{-1} - \sum_{r=2}^{l} \frac{B_r}{r} a^{-r} + \int_0^\infty \overline{B_l}(t)(t+a)^{-l-1} dt. \tag{1.5}$$

Corollary 1. *For any complex u and any natural number $l \in \mathbb{N}$ with $l > \operatorname{Re} u+1$ we have*

$$\begin{aligned}\frac{d}{du}L_u(x,a) &= \sum_{0\le n\le x}(n+a)^u\log(n+a)\\ &= \sum_{r=1}^{l}\frac{(-1)^r}{r!}\overline{B}_r(x)(x+a)^{u-r+1}\frac{\Gamma(u+1)}{\Gamma(u+2-r)}\\ &\quad\times\big\{(\psi(u+1)-\psi(u+2-r))+\log(x+a)\big\}\\ &\quad+\frac{(-1)^l}{l!}\int_x^\infty\overline{B}_l(t)(t+a)^{u-l}\frac{\Gamma(u+1)}{\Gamma(u+1-l)}\\ &\quad\times\big\{(\psi(u+1)-\psi(u+1-l))+\log(t+a)\big\}dt\\ &\quad+\begin{cases}\frac{(x+a)^{u+1}}{u+1}\log(x+a)-\frac{(x+a)^{u+1}}{(u+1)^2}-\zeta'(-u,a), & u\ne -1\\ \frac{1}{2}\log^2(x+a)+\gamma_1(a), & u=-1,\end{cases}\end{aligned}$$

where in general $\log^k(x+a)$ is an abbreviation for $(\log(x+a))^k$.

Remark 1. The notation $\frac{\Gamma(u+1)}{\Gamma(u+1-r)}$ is preferred to $r!\binom{u}{r}$, where $\binom{u}{r}$ means the binomial coefficient $\frac{u(u-1)\cdots(u-r+1)}{r!}$ just because it is feasible for differentiation, and in applications, the binomial coefficients are to be used. Similarly, for negative integer values of u, the seemingly singular function $\psi(u+1)-\psi(u+1-l)$ is to mean $\psi(-u)-\psi(l-u)$ (cf. Lemma 2 below).

The proof of Proposition 1 is given in [7]. Since the integral is an analytic function in the region $\operatorname{Re} u < l-1$, we may deduce Corollary 1 for $u \ne -1$, by differentiation with respect to the complex variable u. But the case $u=-1$ is exceptional and needs a special treatment. In view of this, we shall give a direct proof of Corollary 1, modelled on the proof of [5, Lemma 8]. We need the following two lemmas.

Lemma 1 ([11, p. 26]). *Suppose f is piecewise of class C^1 on $[a,b]$. Then for any $l \in \mathbb{N}$,*

$$\sum_{a<n\le b}f(n)=\int_a^b f(t)dt+\sum_{r=1}^{l}\frac{(-1)^r}{r!}\Big[\overline{B}_r(t)f^{(r-1)}(t)\Big]_a^b+\frac{(-1)^{l+1}}{l!}\int_a^b\overline{B}_l(t)f^{(l)}(t)dt.$$

Lemma 2 ([5, (2.6)]). *For any $u \in \mathbb{C}$ and $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n-1}(-1)^k\binom{u}{k}\frac{1}{n-k}=(-1)^{n-1}\binom{u}{n}\left(\psi(u+1)-\psi(u+1-n)\right),$$

which is the most convenient for most cases, but if $0 \ge u \in \mathbb{Z}$, the right-hand side is to be interpreted as $(-1)^n\binom{u}{n}\left(\psi(n-u)-\psi(-u)\right)$.

Proof of Corollary 1. We need an expression for the n-th derivative of the function $f(t) = (t+a)^u \log(t+a)$. By Leibniz's rule we obtain

$$f^{(n)}(t) =(-1)^{n-1}(t+a)^{u-n}\sum_{k=0}^{n-1}(-1)^k\binom{u}{k}\frac{1}{n-k}+n!\binom{u}{n}(t+a)^{u-n}\log(t+a)$$

to which we apply Lemma 2 to rewrite it in the form

$$f^{(n)}(t) = n!\binom{u}{n}\big\{\psi(u+1)-\psi(u+1-n)+\log(t+a)\big\}(t+a)^{u-n} \tag{1.6}$$

or in the alternative form stated in Lemma 2. Applying Lemma 1 to the sum $\sum_{0<n\le x}(n+a)^u\log(n+a)$, thereby substituting (1.6), we may deduce the formula for it, corresponding to [7, (6)], whence we may argue as in the proof of [7, Theorem 1] to deduce Corollary 1 in the case $u \neq -1$.

Now we restrict to the case $u = -1$. Then noting that $\int_0^x \frac{1}{t+a}\log(t+a)dt = \frac{1}{2}\log^2(x+a) - \frac{1}{2}\log^2 a$, we obtain

$$\begin{aligned}
&\sum_{0\le n\le x}\frac{1}{n+a}\log(n+a)\\
&= \frac{1}{a}\log a + \frac{1}{2}\log^2(x+a) - \frac{1}{2}\log^2 a + \sum_{r=1}^{l}\frac{B_r}{r}(\psi(1)-\psi(r)+\log a)a^{-r} \qquad (1.7)\\
&\quad -\int_0^\infty \overline{B}_l(t)(\psi(1)-\psi(l+1)+\log(t+a))(t+a)^{-1-l}dt + O\left(x^{-1}\log x\right),
\end{aligned}$$

by estimating the integral by the second mean value theorem (or by integration by parts). From (1.7) it follows that as $x\to\infty$,

$$\begin{aligned}
\gamma_1(a) &= \lim_{x\to\infty}\left(\sum_{0\le n\le x}\frac{1}{n+a}\log(n+a) - \frac{1}{2}\log^2(x+a)\right)\\
&= \frac{1}{a}\log a - \frac{1}{2}\log^2 a + \sum_{r=1}^{l}\frac{B_r}{r}(\psi(1)-\psi(r)+\log a)a^{-r} \qquad (1.8)\\
&\quad -\int_0^\infty \overline{B}_l(t)(\psi(1)-\psi(l+1)+\log(t+a))(t+a)^{-l-1}dt.
\end{aligned}$$

This completes the proof. □

Remark 2. In [7, Corollary 1] it is proved that the k-th Laurent coefficient of the Hurwitz zeta-function is given by $\frac{(-1)^k}{k!}\gamma_k(a)$, where

$$\gamma_k(a) = \lim_{x\to\infty}\left(\sum_{0\le n\le x}\frac{\log^k(n+a)}{n+a} - \frac{\log^{k+1}(x+a)}{k+1}\right) \tag{1.9}$$

by the simplest possible method and that $\gamma_k(a)$ admits the integral representation

$$\gamma_k(a) = \frac{1}{2a}\log^k a - \frac{1}{k+1}\log^{k+1} a - \int_0^\infty \frac{\overline{B}_1(t)}{(t+a)^2}\left(\log^k(t+a) - k\log^{k1}(t+a)\right) \tag{1.10}$$

However, the statement of [7, Corollary 1] does not seem to imply these, Formula [7, (9)] reducing to the same result as [7, (8)]. Thus it would be worth recovering both (1.9) and (1.10) here.

The starting point is [7, (11)], i.e., Proposition 1 with $l = 1$ and $-s$ ($s \neq 1$, $\sigma > 0$) for u:

$$L_{-s}(x,a) = \frac{(x+a)^{1-s}}{1-s} + \zeta(s,a) - \frac{\overline{B}_1(x)}{(x+a)^s} + s\int_x^\infty \frac{\overline{B}_1(t)}{(t+a)^{s+1}}dt\,. \tag{1.11}$$

Since both sides of (1.11) are analytic in $\sigma > 0$, we may compute the k-th Taylor coefficient around $s = 1$. The k-th Taylor coefficient of the left-hand side (as [7, (12)] should read) is

$$\frac{1}{k!}\frac{\partial^k}{\partial s^k}L_{-s}(x,a)|_{s=1} = \frac{(-1)^k}{k!}\sum_{0\le n\le x}(n+a)^{-1}\log^k(n+a) \tag{1.12}$$

and that of the right-hand side is

$$\frac{(-1)^k}{k!}\left(\frac{\log^{k+1}(x+a)}{k+1} + \gamma_k(a) - \frac{\overline{B}_1(x)}{x+a}\log^k(x+a) + \int_x^\infty \frac{\overline{B}_1(t)}{(t+a)^2}\left(\log^k(t+a) - k\log^{k-1}(t+a)\right)dt\right); \tag{1.13}$$

equating (1.12) and (1.13), we conclude that

$$\begin{aligned}\gamma_k(a) = &\sum_{0\le n\le x}(n+a)^{-1}\log^k(n+a)\\ &- \frac{\log^{k+1}(x+a)}{k+1} + \frac{\overline{B}_1(x)}{x+a}\log^k(x+a)\\ &- \int_x^\infty \frac{\overline{B}_1(t)}{(t+a)^2}\left(\log^k(t+a) - k\log^{k-1}(t+a)\right)dt,\end{aligned} \tag{1.14}$$

which should substitute [7, (9)].

We now note that (1.14), being valid for any $x \ge 0$, implies both (1.9) and (1.10) by letting $x \to \infty$ and $x = 0$ respectively, a point more advanced than in Berndt [3].

Needless to say, (1.8) with $l = 1$ and (1.10) with $k = 1$ coincide with each other, and (1.10) with $k = 0$ is a special case of (1.5) (Cf. (2.13) below).

2. Applications of Proposition 1

We shall improve and generalize results of Srivastava and Choi on the asymptotic formula for the sum of the ν-th derivative of ψ: $\sum_{k\le x}\psi^{(\nu)}(k)$, which may be expressed as

$$(-1)^\nu \nu! \left([x]L_{-\nu-1}(x) - L_{-\nu}(x) - \begin{cases}\zeta(\nu+1)[x], & \nu\neq 0\\ \gamma[x], & \nu=0\end{cases}\right), \tag{2.1}$$

where $L_u(x) = L_u(x-1,1)$ throughout this section. We shall prove a general result for the sum

$$S_u(x) := [x]L_{u-1}(x) - L_u(x) - \begin{cases}\zeta(-u+1)[x], & u\neq 0\\ \gamma[x], & u=0\end{cases} \tag{2.2}$$

for any $u\in\mathbb{C}$. Our result reads

Theorem 1. *The above sum $S_u(x)$ has the asymptotic formula*

$$S_u(x) = \begin{cases}\frac{1}{u(u+1)}x^{u+1} - \zeta(-u) - \frac{1}{u}\left(\overline{B}_1(x)+\frac12\right)x^u, & u\neq 0,-1\\ x\log x - x - \zeta(0) - \left(\overline{B}_1(x)+\frac12\right)\log x, & u=0\\ -\log x - \gamma - 1 + \left(\overline{B}_1(x)+\frac12\right)x^{-1}, & u=-1\end{cases}$$
$$+\sum_{r=2}^{l}\frac{(-1)^r}{r(r-1)}\binom{u-1}{r-2}B_r^{(2)}(\{x\}+1)x^{u-r+1} + O(x^{\operatorname{Re}u-l}),$$

where $B_r^{(2)}(x+1) = \sum_{k=0}^{r}\binom{r}{k}B_{r-k}(1)\overline{B}_k(x)$.

Corollary 2. *For $n\in\mathbb{N}$ we have*

$$\sum_{k=1}^{n}\psi(k) = S_0(n) = n\log n - n + \frac12 + \sum_{r=2}^{l}\frac{1}{r(r-1)}B_r^{(2)}(1)n^{1-r} + O\left(n^{-l}\right) \tag{2.3}$$

$$\sum_{k=1}^{n}\psi'(k) = -S_{-1}(n) = \log n + \gamma + 1 - \sum_{r=2}^{l}\frac{1}{r}B_r^{(2)}(1)n^{-r} + O(n^{-1-l}), \tag{2.4}$$

and for $\nu\in\mathbb{N}$, $\nu\ge 2$,

$$\begin{aligned}\sum_{k=1}^{n}\psi^{(\nu)}(k) &= (-1)^\nu\nu! S_{-\nu}(n)\\ &= (-1)^\nu\nu!\left(\frac{1}{\nu(\nu-1)}n^{1-\nu} - \zeta(\nu)\right.\\ &\quad\left.+\sum_{r=2}^{l}\frac{1}{r(r-1)}\binom{\nu+r-2}{r-2}B_r^{(2)}(1)n^{1-r-\nu}\right) + O(n^{-\nu-l}),\end{aligned} \tag{2.5}$$

where in (2.4) *and* (2.5)

$$B_r^{(2)}(1) = \sum_{k=0}^{r} \binom{r}{k} B_k B_{r-k}(n).$$

Before turning to the proof of Theorem 1 we shall show that the sum $\sum_{k\le x} \psi^{(\nu)}(k)$ may be expressed as (2.1): From the fundamental difference equation satisfied by $\psi(z)$:

$$\psi(z+1) - \psi(z) = \frac{1}{z} \quad (z \neq 0),$$

we may easily deduce that

$$\psi^{(\nu)}(z+m) - \psi^{(\nu)}(z) = (-1)^{\nu}\nu! \sum_{k=1}^{m} \frac{1}{(z+k-1)^{\nu+1}},$$

whence that

$$\psi^{(\nu)}(k) = (-1)^{\nu}\nu! \sum_{n=1}^{k-1} \frac{1}{n^{\nu+1}} + \psi^{(\nu)}(1). \tag{2.6}$$

Summing (2.6) over $k \le x$, we obtain

$$\begin{aligned} \sum_{k\le x} \psi^{(\nu)}(k) &= (-1)^{\nu}\nu! \sum_{k\le x} \left(\sum_{n=1}^{k} \frac{1}{n^{\nu+1}} - \frac{1}{k^{\nu+1}} \right) + \psi^{(\nu)}(1)[x] \\ &= (-1)^{\nu}\nu! \left(\sum_{n\le x} \frac{1}{n^{\nu+1}} \sum_{n\le k\le x} 1 - L_{-\nu-1}(x) \right) + \psi^{(\nu)}(1)[x] \end{aligned}$$

after changing the order of summation. Since $\sum_{n\le k\le x} 1 = [x]-n+1$, we see that the first term reduces to $(-1)^{\nu}\nu!\,([x]L_{-\nu-1}(x) - L_{-\nu}(x))$, Taking into account that ([10, p. 22])

$$\psi^{(\nu)}(1) = \begin{cases} (-1)^{\nu+1}\nu!\zeta(\nu+1), & \nu \neq 0 \\ -\gamma, & \nu = 0, \end{cases}$$

we conclude the assertion.

We now go on to the proof of Theorem 1. First we state a lemma.

Lemma 3 ([6, Lemma 6], cf. also [9, p. 75]). *We have*

$$\left(\overline{B}_1(x) + \frac{1}{2}\right)\overline{B}_r(x) = \frac{1}{r+1} \sum_{k=0}^{r} (-1)^{r-k+1} \binom{r+1}{k} B_{r-k+1}\overline{B}_k(x) + \overline{B}_{r+1}(x).$$

Proof. Recall the formula

$$\left(\overline{B}_1(x)-\frac{1}{2}\right)B_r(x)=\frac{1}{r+1}\sum_{k=0}^{r}\binom{r+1}{k}B_{r-k+1}B_k(x)+B_{r+1}(x). \tag{2.7}$$

Restricting the range of x to $0\le x<1$ and adding $\overline{B}_r(x)$ to both sides of (2.7), we have

$$\begin{aligned}\left(\overline{B}_1(x)+\frac{1}{2}\right)&\overline{B}_r(x)\\ &=\frac{1}{r+1}\sum_{k=0}^{r-1}\binom{r+1}{k}B_{r-k+1}\overline{B}_k(x)+\frac{1}{2}\overline{B}_r(x)+\overline{B}_{r+1}(x).\end{aligned} \tag{2.8}$$

Recall the relation $(-1)^{r-k+1}B_{r-k+1}=B_{r-k+1}$ except for $k=r$, in which case it is $-B_1=\frac{1}{2}$. Then the result follows on replacing $\binom{r+1}{k}$ by $(-1)^{r-k+1}\binom{r+1}{k}$ and putting the penultimate term into the sum. $\square$

We note that the sum on the right of (2.8) is $B_r^{(2)}(x)$.

Proof of Theorem 1. Using $[x]=x-\left(\overline{B}_1(x)+\frac{1}{2}\right)$, we rewrite (2.2) as

$$\begin{aligned}S_u(x)=xL_{u-1}(x)-L_u(x)-\left(\overline{B}_1(x)+\frac{1}{2}\right)L_{u-1}(x)\\ -\left(x-\left(\overline{B}_1(x)+\frac{1}{2}\right)\right)\begin{cases}\zeta(1-u), & u\neq 0,\\ \gamma, & u=0,\end{cases}\end{aligned}$$

to which we apply (1.3) to obtain

$$\begin{aligned}S_u(x)=&\sum_{r=1}^{l}\frac{(-1)^r}{r}\binom{u-1}{r-1}\overline{B}_r(x)x^{u-r+1}\\ &-\sum_{r=1}^{l}\frac{(-1)^r}{r}\binom{u}{r-1}\overline{B}_r(x)x^{u-r+1}\\ &-\sum_{r=1}^{l-1}\frac{(-1)^r}{r}\binom{u-1}{r-1}\left(\overline{B}_1(x)+\frac{1}{2}\right)\overline{B}_r(x)x^{u-r}\\ &+\begin{cases}\frac{1}{u}x^{u+1}+\zeta(1-u)x-\frac{1}{u+1}x^{u+1}-\zeta(-u), & u\neq 0,-1\\ x(\log x+\gamma)-x-\zeta(0), & u=0\\ x\left(-x^{-1}+\zeta(2)\right)-\log x-\gamma, & u=-1\end{cases}\end{aligned} \tag{2.9}$$

$$-\begin{cases}\left(\overline{B}_1(x)+\dfrac{1}{2}\right)\left(\dfrac{1}{u}x^u+\zeta(1-u)\right), & u\neq 0\\ \left(\overline{B}_1(x)+\dfrac{1}{2}\right)(\log x+\gamma), & u=0\end{cases}$$
$$-\begin{cases}\zeta(1-u)x+\left(\overline{B}_1(x)+\dfrac{1}{2}\right)\zeta(1-u), & u\neq 0\\ \gamma x+\left(\overline{B}_1(x)+\dfrac{1}{2}\right)\gamma, & u=0\end{cases}$$
$$+O\left(x^{\operatorname{Re}u-l}\right),$$

where the first and the second terms combine to yield

$$-\sum_{r=2}^{l}\frac{(-1)^r}{r}\binom{u-1}{r-2}\overline{B}_r(x)x^{u-r+1},$$

while the third term may be written as

$$\sum_{r=2}^{l}\frac{(-1)^r}{r-1}\binom{u-1}{r-2}\left(\overline{B}_1(x)+\frac{1}{2}\right)\overline{B}_{r-1}(x)x^{u-r+1}.$$

Hence we transform (2.9) into

$$\begin{aligned}&S_u(x)\\ &=\begin{cases}\frac{1}{u(u+1)}x^{u+1}-\zeta(-u)-\frac{1}{u}\left(\overline{B}_1(x)+\frac{1}{2}\right)x^u, & u\neq 0,-1\\ x\log x-\gamma-\zeta(0)-\left(\overline{B}_1(x)+\frac{1}{2}\right)\log x, & u=0\\ -\log x-\gamma-1+\left(\overline{B}_1(x)+\frac{1}{2}\right)x^{-1}, & u=-1\end{cases}\\ &\quad-\sum_{r=2}^{l}(-1)^r\binom{u-1}{r-2}\left(\frac{1}{r}\overline{B}_r(x)-\frac{1}{r-1}\left(\overline{B}_1(x)+\frac{1}{2}\right)\overline{B}_{r-1}(x)\right)x^{u-r+1}\\ &\quad+O\left(x^{\operatorname{Re}u-l}\right)\end{aligned}\tag{2.10}$$

Applying Lemma 3, we may transform the penultimate term further. Since the third factor of x^{u-r+1} is

$$\begin{aligned}&-\frac{1}{r(r-1)}\left(\overline{B}_r(x)+\sum_{k=0}^{r-1}(-1)^{r-k}\binom{r}{k}B_{r-k}\overline{B}_k(x)\right)\\ &\qquad\qquad=-\frac{1}{r(r-1)}\sum_{k=0}^{r}\binom{r}{k}B_{r-k}(1)\overline{B}_k(x),\end{aligned}$$

which is, by (1.1), equal to $-\frac{1}{r(r-1)}B_r^{(2)}(\{x\}+1)$, we conclude that the penultimate term in (2.10) is

$$\sum_{r=2}^{l-1}\frac{(-1)^r}{r(r-1)}\binom{u-1}{r-2}B_r^{(2)}(\{x\}+1)x^{u-r+1}.$$

Substituting this in (2.10) completes the proof. $\square$

Remark 3. (2.3) and (2.4) give improved generalizations of [10, (57), p. 22] and [10, (58), p. 23], respectively, while (2.5) gives the value $\zeta(\nu)$ for the limit [10, l.10, p. 27] $\lim_{n\to\infty}\sum_{k=0}^{n}\sum_{m=1}^{\infty}\frac{1}{(m+k)^{\nu+1}}$.

We collect here some consequences of Proposition 1. Formula (1.2) in the case $u=-1$ reads

$$\begin{aligned}\sum_{n=0}^{N}\frac{1}{n+a}&=\log(N+a)-\psi(a)+\frac{1}{2(N+a)}\\&\quad-\sum_{r=2}^{l}\frac{B_r}{r}\frac{1}{(N+a)^r}+\int_N^\infty\overline{B}_l(t)(t+a)^{-1-l}dt,\end{aligned}\tag{2.11}$$

which gives the generic definition ([10, (2), p. 14]) as implied by (1.3), too,

$$-\psi(a)=\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{1}{n+a}-\log(N+a)\right);$$

cf. (1.9) with $k=0$, the Laurent constant.

With $a=1$, (2.11) reads

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n}&=\log N+\gamma+\frac{1}{2N}-\frac{1}{12N^2}+\frac{1}{120N^4}\\&\quad-\sum_{r=6}^{l}\frac{B_r}{r}\frac{1}{N^r}+\int_N^\infty\overline{B}_l(t)t^{-1-l}dt,\end{aligned}$$

which corrects [10, (59), p. 23].

We restate Formula (1.5) with first a few terms in explicit form:

$$\psi(a)=\log a-\frac{1}{2a}-\frac{1}{12a^2}+\frac{1}{120a^4}-\sum_{r=6}^{l}\frac{B_r}{r}a^{-r}+\int_0^\infty\overline{B}_l(t)(t+a)^{-1-l}dt\tag{2.12}$$

in conformity with [10, Problem 16, p. 69]. Formula (2.12) corrects [10, (56), p. 22]. It should be remarked that the statement in [10, p. 22] that [10, (56), p. 22] (=(2.12) above) follows from [10, (54), p. 8] (=the asymptotic formula for $\log\Gamma(z+a)$, [7, Corollary 2]), is not precise enough and a sound proof is given in [7]. Cf. also the remark at the end of 3 of [7]. Formula (2.12) reduces to [10, (37), (38), p. 6] in the case $a=1$, and in the case $l=1$, it reduces to

$$\int_0^\infty \overline{B}_1(t)(t+a)^{-2}dt = \log a + \frac{1}{2a} + \psi(a), \tag{2.13}$$

which includes [10, (4), p. 345] as a special case.

We turn to the case $u=-s\neq -1$. Formula (1.3) with $x=N-1\in\mathbb{N}$, $a=1$ reads

$$\sum_{n=1}^{N} n^{-s} = -\sum_{r=1}^{l} \frac{B_r}{r!} s\cdots(s+r-2)N^{-s-r+1} + \frac{1}{1-s}N^{1-s} + \zeta(s) + O\left(N^{-\sigma-l}\right),$$

which gives a generic definition for $\sigma > -l$:

$$\zeta(s) = \lim_{N\to\infty}\left(\sum_{n=1}^{N} n^{-s} + \sum_{r=1}^{l}\frac{B_r}{r!}s\cdots(s+r-2)N^{-s-r+1} - \frac{1}{1-s}N^{1-s}\right)$$

generalizing [10, (23),(24),(25), p. 99], and for $s=2$:

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n^2} &= -\sum_{r=1}^{l} B_r N^{-r-1} - N^{-1} + \zeta(2) + O(N^{-2-l})\\ &= \frac{\pi^2}{6} - \frac{1}{N} + \frac{1}{2N^2} - \frac{1}{6N^3} + \frac{1}{30N^5} - \cdots\end{aligned}$$

coinciding with [10, (60), p. 23].

Finally, we remark that (1.4) with $l=1$ and $u=1-n$ ($2<n\in\mathbb{N}$) gives

$$\int_0^\infty \overline{B}_1(t)(t+a)^{-n}dt = \frac{1}{n-1}\left(\frac{1}{2}a^{1-n} + \frac{1}{n-2}a^{2-n} - \zeta(n-1,a)\right), \tag{2.14}$$

which is more general than [10, (7), p. 346].

3. Applications of Corollary 1

In this section we shall collect miscellaneous consequences of Corollary 1. The highlights are Proposition 2 and Theorem 2, which re-establishes Bendersky–Adamchik's result and recovers Elizalde's result, respectively.

Proposition 2. *For $m \in \mathbb{N} \cup \{0\}$, we have*

$$
\begin{aligned}
-\zeta'(-m,a) = \lim_{N\to\infty} \Bigg(& \sum_{n=0}^{N} (n+a)^m \log(n+a) \\
& - \frac{1}{m+1}(N+a)^{m+1}\log(N+a) + \frac{1}{(m+1)^2}(N+a)^{m+1} \\
& - \frac{1}{2}(N+a)^m \log(N+a) - \sum_{r=2}^{m+1} \binom{m}{r-1} \frac{B_r}{r} \\
& \cdot \left(\frac{1}{m} + \cdots + \frac{1}{m-r+2} + \log(N+a) \right) (N+a)^{m-r+1} \Bigg)
\end{aligned}
$$

Theorem 2. *For $m \in \mathbb{N} \cup \{0\}$, $\operatorname{Re} a > 0$ and $m+2 \le l \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\zeta'(-m,a) \\
&= \frac{1}{m+1} a^{m+1} \log a - \frac{1}{(m+1)^2} a^{m+1} - \frac{1}{2} a^m \log a + \frac{1}{12} a^{m-1} \log a \\
&\quad + \sum_{r=4}^{m+1} \frac{B_r}{r} \left(\sum_{j=0}^{r-2} (-1)^j \binom{m}{j} \frac{1}{r-1-j} + \binom{m}{r-1} \log a \right) a^{m-r+1} \\
&\quad + \frac{1}{m+1} \sum_{r=m+2}^{l} B_r \left(\sum_{j=0}^{r-1} (-1)^j \binom{r-m-2}{j} \frac{1}{r-j} \right) a^{m-r+1} \\
&\quad + (-1)^{l+1} \int_0^\infty \left(\sum_{j=0}^{l-1} (-1)^j \binom{l-m-1}{j} \frac{1}{l-j} \overline{B}_l(t)(t+a)^{m-l} \right) dt
\end{aligned} \tag{3.1}
$$

where the last integral is $O\big(|a|^{m-l}\big)$, so that (3.1) *gives an asymptotic expansion for $\zeta'(-m,a)$ for $0 < \operatorname{Re} a$, $|a| \le 1$.*

Corollary 3. *For $\operatorname{Re} a > 0$ or more generally for $|\arg a| < \pi$, we have*

$$
\log \frac{\Gamma(a)}{\sqrt{2\pi}} = \left(a - \frac{1}{2} \right) \log a - a - \int_0^\infty \overline{B}_1(t)(t+a)^{-1} dt.
$$

Proposition 2 and Theorem 2 are restatement of Formulas (1.7) and (1.8) of [8], respectively. Since the proofs are not given in [8], we shall prove Theorem 2.

Proof of Theorem 2. Since those terms of the sum with $r \ge m+2$ for $\frac{\partial}{\partial u} L_u(x,a)$ have singularities at non-positive integers, we have to take the limit as $u \to m$ of $\frac{\Gamma(u+1)}{\Gamma(u+2-r)}(-\psi(u+2-r))$ or $\frac{\Gamma(u+1)}{\Gamma(u+1-l)}(-\psi(u+1-l))$ as the case may be, on noting that other terms vanish because of simple zeros of $\Gamma(z)^{-1}$ at non-positive integers.

By the fundamental difference equation satisfied by the gamma function we deduce for $\nu \in \mathbb{N} \cup \{0\}$ that

$$\Gamma(z) = \frac{\Gamma(z+\nu+1)}{z \cdots (z+\nu)}. \tag{3.2}$$

We contend that

$$\lim_{z \to -\nu} \frac{\psi(z)}{\Gamma(z)} = -(-1)^\nu \nu!.$$

To prove this it suffices to note from (3.2) that in taking the limit

$$\lim_{z \to -\nu} \frac{\psi(z)}{\Gamma(z)} = \lim_{z \to -\nu} \frac{\Gamma'(z)}{\Gamma(z)^2},$$

only one term counts, i.e., it is equal to

$$-\lim_{z \to -\nu} \frac{\Gamma(z+\nu+1)}{z \cdots (z+\nu-1) \left((z+\nu)\Gamma(z)\right)^2},$$

which is $-(-1)^\nu \nu!$ on recalling the fact $Res_{z=-\nu}\, \Gamma(z) = \frac{(-1)^\nu}{\nu!}$, which in fact follows from (3.2). Hence

$$\lim_{u \to m} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \left(-\psi(u+2-r)\right) = (-1)^{r-m} \frac{r!}{(m+1)(m+2)\binom{r}{m+2}}$$

and

$$\lim_{u \to m} \frac{\Gamma(u+1)}{\Gamma(u+1-l)} \left(-\psi(u+1-l)\right) = \frac{(-1)^{l-m}}{(m+1)\binom{l}{m+1}}.$$

Hence we conclude that

$$\begin{aligned}
\zeta'(-m,a) = {} & \frac{1}{m+1} a^{m+1} \log a - \frac{1}{(m+1)^2} a^{m+1} \\
& - \frac{1}{2} a^m \log a + \frac{1}{12} a^{m-1} (1 + m \log a) \\
& + \sum_{r=4}^{m+1} \frac{B_r}{r} \binom{m}{r-1} \left(\frac{1}{m} + \cdots + \frac{1}{m-(r-2)} + \log a \right) a^{m-r+1} \\
& + \frac{(-1)^m}{m+1} \sum_{r=m+2}^{l} B_r \frac{1}{(m+2)\binom{r}{m+2}} a^{m-r+1} \\
& + \frac{(-1)^m}{(m+1)\binom{l}{m+1}} \int_0^\infty \overline{B}_l(t) (t+a)^{m-l} dt,
\end{aligned} \tag{3.3}$$

which is Formula (1.7) of [8].

In order to deduce (3.1) from (3.3) we have recourse to Lemma 2 and its counterpart

$$\sum_{j=0}^{n-1}(-1)^j\binom{u}{j}\frac{1}{n-j}=(-1)^u\frac{1}{(n-u)\binom{n}{u}}$$

for $n>u\in\mathbb{N}$ ([5, Formula (2.6)]). Substituting the formulas

$$\begin{aligned}(-1)^{r-2}\binom{m}{r-1}\big(\psi(m+1)-\psi(m-r+2)\big)\\ &=\sum_{j=0}^{r-2}(-1)^j\binom{m}{j}\frac{1}{r-1-j} && (r\le m+1)\\ \frac{1}{(m+2)\binom{r}{m+2}}&=(-1)^{r-m}\sum_{j=0}^{r-1}(-1)^j\binom{r-m-2}{j}\frac{1}{r-j} && (r\ge m+2)\end{aligned}$$

and

$$\frac{1}{(m+1)\binom{l}{m+1}}=(-1)^{l-m-1}\sum_{j=0}^{l-1}(-1)^j\binom{l-m-1}{j}\frac{1}{l-j}$$

in (3.3), we now complete the proof of Theorem 2. □

Remark 4. (i) In the notation

$$\begin{aligned}p(N,m)={}&\frac{1}{2}N^m\log N+\frac{1}{m+1}N^{m+1}\left(\log N-\frac{1}{m+1}\right)\\ &+m!\sum_{j=1}^{m}\frac{B_{j+1}}{(j+1)!\,(m-j)!}\left(\log N+(1-\delta_{mj})\sum_{k=1}^{j}\frac{1}{m-k+1}\right)N^{m-j}\end{aligned}$$

of Bendersky [2, Proposition 2], in the case $a=1$ reads

$$-\zeta'(-m)=\lim_{N\to\infty}\left(\sum_{n=1}^{N}n^m\log n-p(N,m)\right)-\frac{B_{m+1}}{m+1}H_m, \tag{3.4}$$

where H_m signifies the m-th harmonic number equal to $\psi(m+1)-\psi(1)$. Formula (3.4) gives Adamchik's result [1, (24)] which in turn is posed as Problems 2 and 3 [10, p. 128]. The cases $m=1,2,3$ of (3.4) give (2) [10, p. 25], (69) [10, p. 36], and (26) [10, p. 99], and (70) [10, p. 37], and (27) [10, p. 100], respectively, and the limit on the right of (3.4), in the case $m=1$ is denoted by $\log A$, A being called the Glaisher-Kinkelin constant.

(ii) Although Problem 39 [10, p. 139] looks as if asking for a proof of an equality, what is asked for is a proof of an asymptotic formula as stated by Elizalde [4, (17)], and this is given in Theorem 2. We note that our method is much easier than Elizalde's who uses

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty \left(a^2+t^2\right)^{-\frac{s}{2}} \sin\left(s\tan^{-1}\frac{t}{a}\right)\frac{dt}{e^{2\pi t}-1},$$

and automatically gives corresponding asymptotic formulas for higher derivatives, to the study of which we will return at another occasion.

(iii) Corollary 3 is the special case $m = 0$ of (3.1) (cf. (3.6) below), with Lerch's formula $\zeta'(0,a) = \log\frac{\Gamma(a)}{\sqrt{2\pi}}$ incorporated. Solving for $\int_0^\infty \overline{B}_1(t+a)^{-1}dt$, it gives Problem 1 in [10, p. 350] whose special case with $a = 1$ is [10, (3), p. 345] and whose another special case is [10, (5), p. 346]. Corollary 3 already provides us with [10, (56), p. 9], which would follow from [10, (56), p. 22] (see (1.5) above), but not conversely, in general.

Correspondingly to (2.14), we now prove

Proposition 3. *With $\Gamma_2(a)$ $\left(= G(a)^{-1}\right)$ denoting the double gamma (or the G-) function of Barnes, we have for $a, b > 0$*

$$\begin{aligned}
&\frac{1}{2}(a^2\log a - b^2\log b) - \frac{1}{4}(a^2-b^2)\\
&-\frac{1}{2}(a\log a - b\log b) - \int_0^\infty \overline{B_1}(t)\log\frac{t+a}{t+b}dt\\
&\qquad = \zeta'(-1,a) - \zeta'(-1,b)\\
&\qquad = \log\frac{\Gamma_2(a)}{\Gamma_2(b)} + (a-1)\log\Gamma(a) - (b-1)\log\Gamma(b).
\end{aligned} \tag{3.5}$$

Example. We have

$$\int_0^\infty \overline{B}_1(t)\log\frac{t+2}{t+1}dt = \log 2 - \frac{3}{4}$$

((6) in [10, p. 346]) and

$$\int_0^\infty \overline{B}_1(t)\log\frac{t+\frac{1}{2}}{t-\frac{1}{2}}dt = \frac{5}{4}\log 2 + \frac{3}{8}\log 3 - \frac{1}{2} + \frac{1}{2}\log 2\pi$$

(Problem 3 in [10, p. 350]).

Proof of Proposition 3. In the first instance, for Re $u < 0$, $u \neq -1$, we have

$$\begin{aligned}
\zeta'(-u,a) = {}& \frac{1}{u+1}a^{u+1}\log a - \frac{1}{(u+1)^2}a^{u+1} - \frac{1}{2}a^u\log a\\
&- \int_0^\infty (1+u\log(t+a))\overline{B}_1(t)(t+a)^{u-1}dt.
\end{aligned} \tag{3.6}$$

We may, however, by Abel's continuity theorem for infinite integrals, take the limit as $u \to 1$ of the difference $\zeta'(-u, a) - \zeta'(-u, b)$ of (3.6), which gives the first equality of (3.5). The second equality follows from Formula (33) [10, p. 94] $\zeta'(-1, a) - \frac{1}{12} = \log \Gamma_2(a) - \log A + (a-1) \log \Gamma(a)$, where A indicates the Glaisher–Kinkelin constant ((3.4) with $m = 1$). $\square$

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