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On the Extension of a Certain Class of Carleman Operators

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Dedicated to the memory of E. L. Alexandrov

Abstract. The integral operators of Carleman play an important role in the spectral theory of selfadjoint operators and made the object of several works such those of G. I. Targonski [*Proc. Amer. Math. Soc.* 18 (1967)(3)], V. B. Korotkov [*Sib. Math. J.* 11 (1970)(1)], and J. Weidmann [*Manuscripta Math.* (1970)(2)]. In the present paper, we study a certain class of these operators in the Hilbert space $L^2(X, \mu)$. Precisely, we give necessary and sufficient conditions so that they possess equal defeciency indices. Such operators find their applications in the theory of random variable approximation.

Keywords. Deficiency indices, integral operator, Carleman kernel

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1. Preliminaries

Let H be a Hilbert space endowed with the inner product $(.,.)$, and let $A : D(A) \subset H \longrightarrow H$ be a densely defined closed linear operator whose range is denoted $R(A)$. The *defect number* is the dimension of the orthogonal complement to $R(A)$

$$d_A = \dim(H \ominus R(A)) = \dim \ker(A^*),$$

where A^* is the adjoint operator of A and $\ker(A^*) = \{f \in D(A^*) : A^* f = 0\}$, $D(A^*)$ being the domain of A^*.

Let A be a symmetric operator, $\widetilde{A}$ its symmetric extension, then the following relation holds

$$A \subset \widetilde{A} \subset \widetilde{A}^* \subset A^*. \tag{1}$$

The interest of (1) resides in the following conclusion: all symmetrical extension of A comes of a restriction of the domain of A^*. So $D(\widetilde{A})$ is a subspace between $D(A)$ and $D(A^*)$. To construct the extensions $\widetilde{A}$ it is therefore well to examine the structure of the space $D(A^*)$.

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We now introduce the *defect spaces* of A

$$\mathcal{N}_\lambda = \ker(A^* - \lambda I) \text{ and } \mathcal{N}_{\overline{\lambda}} = \ker(A^* - \overline{\lambda} I), \ \ (\Im m\lambda > 0)\,. \tag{2}$$

I denotes the identity operator in H.The numbers $n_+(A) = \dim \mathcal{N}_\lambda$ and $n_-(A) = \dim \mathcal{N}_{\overline{\lambda}}$ are called *deficiency indices* of the symmetric operator A. It being, we have an important result.

Theorem 1.1 ([2]). *In the Hilbert space $D(A^*)$ we have the following Hilbertienne decomposition:*

$$D(A^*) = D(A) \oplus \mathcal{N}_\lambda \oplus \mathcal{N}_{\overline{\lambda}}.$$

Lemma 1.2 ([4]). *The operator A possesses self adjoint extensions if and only if $n_+(A) = n_-(A)$. One gets in this case all self extensions of A from all surjective isometries $\mathcal{F}$ defined from $\mathcal{N}_\lambda$ to $\mathcal{N}_{\overline{\lambda}}$.*

2. Carleman operators

One can find necessary information about Carleman operators, for example, in [3], [5]–[8]. In this section we shall present only a part of it. Let X be an arbitrary set, μ a σ-finite measure on X (μ is defined on a σ-algebra of subsets of X, we don't indicate this σ-algebra), $L_2(X, \mu)$ the Hilbert space of square integrable functions with respect to μ. Instead of writing 'μ-measurable', 'μ-almost everywhere' and '$d\mu(x)$' we write 'measurable', 'almost everywhere' and 'dx'.

Definition 2.1 ([8]). A linear operator $A : D(A) \longrightarrow L_2(X, \mu)$, where the domain $D(A)$ is a dense linear manifold in $L_2(X, \mu)$, is said to be *integral* if there exists a measurable function K on $X \times X$, a kernel, such that, for every $f \in D(A)$,

$$Af(x) = \int_X K(x, y) f(y)\, dy \quad \text{almost everywhere in } X.$$

A kernel K on $X \times X$ is said to be a *Carleman kernel* if $K(x, y) \in L_2(X, \mu)$ for almost every fixed x, that is to say

$$\int_X |K(x, y)|^2\, dy \ < \infty \quad \text{almost everywhere.}$$

An integral operator A with a kernel K is called *Carleman operator* if K is a Carleman kernel. Every Carleman kernel K defines a Carleman function k from X to $L_2(X, \mu)$ by $k(x) = \overline{K(x, .)}$ for all x in X for which $K(x, .) \in L_2(X, \mu)$.

Now we shall be interested, in the Hilbert space $L^2(X,\mu)$, with a class of Carleman integral operators whose kernel is defined by

$$K(x,y)=\sum_{p=0}^{\infty} a_p\psi_p(x)\overline{\psi_p(y)}, \tag{3}$$

where $\{\psi_p(x)\}_{p=0}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ such that

$$\sum_{p=0}^{\infty} |\psi_p(x)|^2<\infty \quad \text{almost everywhere in } X, \tag{4}$$

and $\{a_p\}_{p=0}^{\infty}$ a real number sequence verifying

$$\sum_{p=0}^{\infty} a_p^2|\psi_p(x)|^2<\infty \quad \text{almost everywhere in } X. \tag{5}$$

We call $\{\psi_p(x)\}_{p=0}^{\infty}$ *Carleman sequence.*

2.1. Example of Carleman sequences. We need the following lemma.

Lemma 2.2 ([7]). *Let (X,μ) a measured space. Then it exists a sequence $\{Y_n\}_1^{\infty}$ of measurable subset's of the set X such that*

(i) $Y_m\cap Y_n=\varnothing,\ m\neq n$;

(ii) $X=\bigcup\limits_{n=1}^{\infty} Y_n$;

(iii) $\mu(Y_n)>0,\ n=1,2,\ldots$;

(iv) *for all $n\in\mathbb{N}^*$, $L_2(Y_n,\mu)$ is of infinite dimension.*

We choose a function $\varphi(x)\in L_2(X,\mu)$ verifying

1. $\varphi(x)>0$ for all $x\in X$
2. $\varphi_n(x)=\mathbf{1}_{Y_n}(x)\varphi(x),\ \in\mathbb{N}^*$, with $\|\varphi_n\|_{L_2}^2=\frac{1}{2^n}$, where $\mathbf{1}_{Y_n}(x)$ is the characteristic function of Y_n and $\{Y_n\}_1^{\infty}$ a sequence of measurable subset's of the set X verifying (i)–(iv) of Lemma 2.2. Therefore, we will have$\|\varphi\|_{L_2}^2=\sum_{n=1}^{\infty}\|\varphi_n\|_{L_2}^2=\sum_{n=1}^{\infty}\frac{1}{2^n}=1$.

This being we put

– $\psi_0(x)=\ \varphi(x),\quad x\in X$

– $\psi_1(x)=\begin{cases}-\varphi(x), & x\in Y_1\\ \ \ \varphi(x), & x\in\bigcup\limits_{k=2}^{\infty} Y_k\end{cases}$

$$- \ \psi_n(x) = \begin{cases} 0, & x \in \bigcup\limits_{k=1}^{n-1} Y_k \\ -2^{\frac{n-1}{2}} \varphi(x), & x \in Y_n \\ 2^{\frac{n-1}{2}} \varphi(x), & x \in \bigcup\limits_{k=n+1}^{\infty} Y_k \end{cases}, \quad \text{for all } n \geq 2.$$

We can show easily that the sequence $\{\psi_p(x)\}_{p=0}^{\infty}$ verifies

a) $\|\psi_n\|_{L_2}^2 = 1, \ n \in \mathbb{N}$

b) $(\psi_m, \psi_n) = 0, \ m \neq n$

c) $\sum\limits_{p=1}^{\infty} |\psi_p(x)|^2 < \infty$ almost everywhere in X

d) $\psi_0(x) + \sum\limits_{n=1}^{\infty} 2^{\frac{n-1}{2}} \psi_n(x) = 0$ almost everywhere in X.

2.2. Deficiency indices. Here, we give the necessary and sufficient conditions so that the Carleman operator A possesses equal deficiency indices $n_+(A) = n_-(A)$.

The domain of the adjoint operator A^* of A is defined by

$$D(A^*) = \left\{ f \in L^2(X, \mu) : \int_X K(x, y) f(y)\, dy \in L_2(X, \mu) \right\}.$$

The kernel (3) verifying condition (4) will be a Hilbert–Schmidt kernel if and only if $\sum_{p=0}^{\infty} a_p^2 < \infty$. In this work we consider the case $\sum_{p=0}^{\infty} a_p^2 = \infty$. Let $L(\psi)$ be the closed set of linear combinations of elements of the orthogonal sequence $\{\psi_p(x)\}_{p=0}^{\infty}$. It is lucid that the orthogonal complement $L^{\perp}(\psi) = L_2(X, \mu) \ominus L(\psi)$ is contained in $D(A)$ and annul the operator A.

Theorem 2.3. *The operator A possesses equal deficiency indices $n_+(A) = n_-(A) = m$, $(m < \infty)$, if and only if there exist sequences $\left\{\gamma_p^{(k)}\right\}_{p=0}^{\infty}$ $(k = 1, 2, \ldots, m)$ verifying*

1. *for all k,*

$$\theta_k(x) = \sum_{p=0}^{\infty} \gamma_p^{(k)} \psi_p(x) \in L^{\perp}(\psi) \quad (k = 1, 2, \ldots, m); \tag{6}$$

2. *for all λ $(\Im m\lambda \neq 0)$,*

$$\sum_{p=0}^{\infty} \left| \frac{\gamma_p^{(k)}}{a_p - \lambda} \right|^2 < \infty \quad (k = 1, 2, \ldots, m); \tag{7}$$

3. *the linear space of the sequences $\left\{\gamma_p^{(k)}\right\}_{p=0}^{\infty}$ $(k = 1, 2, \ldots, m)$ verifying (6) and (7) is m-dimensional.*

Proof. First we have for all $f \in D(A)$, $f(x) = \sum_{p=0}^{\infty} c_p\ \psi_p(x) + q(x)$, $q \in L^{\perp}(\psi)$, then

$$A^* f(x) = \sum_{p=0}^{\infty} a_p c_p \psi_p(x), \quad c_p = (f, \psi_p).$$

Indeed, let $f_n(x) = \sum_{p=0}^{\infty} c_p \psi_p(x) + q(x)$, as $\sum_{p=0}^{n} c_p \psi_p(x) \in L(\psi)$, then $\lim_{n\to\infty} f_n(x) = f(x)$ and

$$A^* f(x) = \lim_{n\to\infty} \big(f_n(x), K(x,y)\big) = \sum_{p=0}^{\infty} a_p (f, \psi_p)\, \psi_p(x).$$

Necessary condition. We suppose that A possesses equal deficiency indices $n_+(A) = n_-(A) = m$ $(m < \infty)$. Let φ_λ^1, $\varphi_\lambda^2, \ldots,$ φ_λ^m be a base of the defect space (2)

$$\mathcal{N}_{\overline{\lambda}} = L_2(X, \mu) \ominus \big(A - \overline{\lambda} I\big)\, D(A).$$

Let us write φ_λ^k $(k = 1, 2, \ldots, m)$ as

$$\varphi_\lambda^k(x) = \sum_{p=0}^{\infty} \beta_p^{(k)}(\lambda)\ \ \psi_p(x) + q_k(x) \quad (k = 1, 2, \ldots, m)$$

with

$$\sum_{p=0}^{\infty} \left|\beta_p^{(k)}(\lambda)\right|^2 < \infty \quad \text{and} \quad q_k \in L^{\perp}(\psi) \quad (k = 1, 2, \ldots, m).$$

We have on the one hand $\varphi_\lambda^k \in \mathcal{N}_{\overline{\lambda}}$, then

$$A^* \varphi_\lambda^k(x) = \lambda \varphi_\lambda^k(x) \quad (k = 1, 2, \ldots, m).$$

On the other hand

$$A^* \varphi_\lambda^k(x) = \sum_{p=0}^{\infty} a_p \beta_p^{(k)}(\lambda)\, \psi_p(x) \quad (k = 1, 2, \ldots, m).$$

Thus

$$\sum_{p=0}^{\infty} (a_p - \lambda)\, \beta_p^{(k)}(\lambda)\, \psi_p(x) = \lambda q_k(x) \quad (k = 1, 2, \ldots, m).$$

Therefore if we put

$$\gamma_p^{(k)}(\lambda) = (a_p - \lambda)\, \beta_p^{(k)}(\lambda) \quad (k = 1, 2, \ldots, m)$$

it ensures that

$$\theta_k(x) = \sum_{p=0}^{\infty} \gamma_p^{(k)} \psi_p(x) \in L^{\perp}(\psi) \quad (k = 1, 2, \ldots, m)$$

and

$$\sum_{p=0}^{\infty}\left|\frac{\gamma_p^{(k)}}{a_p-\lambda}\right|^2=\sum_{p=0}^{\infty}\left|\beta_p^{(k)}(\lambda)\right|^2<\infty.$$

For condition 3), let us suppose that there exist numbers $x_k\,(k=1,2,\dots,m)$ no all hopeless as $\sum_{k=1}^{m}x_k\gamma_p^{(k)}=0$. For $\lambda\neq 0$,

$$\begin{aligned}\sum_{k=1}^{m}x_k\varphi_\lambda^k(x)&=\sum_{k=1}^{m}x_k\left(\sum_{p=0}^{\infty}\frac{\gamma_p^{(k)}}{a_p-\lambda}\psi_p(x)+\frac{1}{\lambda}\theta_k(x)\right)\\&=\sum_{p=0}^{\infty}\frac{\gamma_p^{(k)}}{a_p-\lambda}\psi_p(x)\left(\sum_{k=1}^{m}x_k\varphi_\lambda^k(x)\right)+\frac{1}{\lambda}\sum_{k=1}^{m}x_k\theta_k(x)\\&=\frac{1}{\lambda}\sum_{k=1}^{m}x_k\theta_k(x)\in L^{\perp}(\psi).\end{aligned}$$

Or if $f\in L^{\perp}(\psi)$, then $A^*f=0$, from where

$$A^*\left(\sum_{k=1}^{m}x_k\varphi_\lambda^k(x)\right)=\lambda\sum_{k=1}^{m}x_k\varphi_\lambda^k(x)=0,$$

so as the $\varphi_\lambda^k\ (k=1,2,\dots,m)$ are linearly independent it follows that $x_1=x_2=\dots=x_m=0$.

Sufficient condition. We suppose that there exist sequences $\left\{\gamma_p^{(k)}\right\}_{p=0}^{\infty}$ $(k=1,2,\dots,m)$ verifying conditions 1–3. It is easy to see that the functions

$$\varphi_\lambda^{(k)}(x)=\sum_{p=0}^{\infty}\frac{\gamma_p^{(k)}}{a_p-\lambda}\psi_p(x)\quad(k=1,2,\dots,m)$$

are solutions of the equation $A^*y=\lambda y$. Indeed, we have

$$\begin{aligned}A^*\varphi_\lambda^{(k)}(x)&=\sum_{p=0}^{\infty}a_p\frac{\gamma_p^{(k)}}{a_p-\lambda}\psi_p(x)\\&=\sum_{p=0}^{\infty}(a_p-\lambda+\lambda)\frac{\gamma_p^{(k)}}{a_p-\lambda}\psi_p(x)\\&=\lambda\sum_{p=0}^{\infty}\frac{\gamma_p^{(k)}}{a_p-\lambda}\psi_p(x),\end{aligned}$$

from where $\varphi_\lambda^{(k)}\in N_{\overline{\lambda}}\ (k=1,2,\dots,m)$. Therefore the operator A possesses equal deficiency indices $n_+(A)=n_-(A)=m\ (m<\infty)$. □

Remark 2.4. The sequences $\left\{\gamma_p^{(k)}\right\}_{p=0}^{\infty}$ $(k = 1, 2, \ldots, m)$ are as $\sum_{p=0}^{\infty} \left|\gamma_p^{(k)}\right|^2 = \infty$. Indeed, let us suppose that it exists $k_\circ$ such that

$$\sum_{p=0}^{\infty} \left|\gamma_p^{(k_\circ)}\right|^2 < \infty \quad (k = 1, 2, \ldots, m),$$

then from $\sum_{p=0}^{\infty} |\psi_p(x)|^2 < \infty$, we have $\sum_{p=0}^{\infty} \left|\gamma_p^{(k_\circ)} \psi_p(x)\right|^2 < \infty$, from where $\theta_{k_\circ}(x) = \sum_{p=0}^{\infty} \gamma_p^{(k_\circ)} \psi_p(x) \in L(\psi)$, therefore the condition (7) of Theorem 2.3 is not verified anymore.

Corollary 2.5. *If for all* p $(p = 1, 2, \ldots)$ $|a_p| \le c$ *(constant), then the operator* A *is self adjoint.*

Proof. Indeed, we will have

$$\sum_{p=0}^{\infty} \frac{\left|\gamma_p^{(k)}\right|^2}{a_p^2 + 1} \ge \frac{1}{c^2 + 1} \sum_{p=0}^{\infty} \left|\gamma_p^{(k)}\right|^2 = \infty. \qquad \square$$

3. Stochastic processes with Carleman operators

The class of Carleman integral operators studied here find its application in the stochastic processes survey. Let $(\Omega;\ \mathcal{F};\ \mathcal{P})$ a probability space, ξ a random variable. We have

$$\forall t \in \mathbb{R},\ \{\omega : \xi(\omega) < t\} = \{\xi < t\} \in \mathcal{F}.$$

It means that $\{\xi < t\}$ is an event. We designate by $L_2(\Omega) = L_2(\Omega;\ \mathcal{F};\ \mathcal{P})$ the Hilbert space of random variables ξ such that $E|\xi|^2 < \infty$ (E : expectation) provided of the scalar product $\langle \xi, \eta \rangle = E\xi\overline{\eta}$. A function $\nu : \Omega \to H$ is said to be a random element if for all $f \in H$, (f, ν) is a random variable. The operator

$$A : H \to L_2(\Omega)$$

defined by

$$Af = (f, \nu), \quad f \in D(A) = \{f \in H : (f, \nu) \in L_2(\Omega)\}$$

is an operator of Carleman. The *correlation* operator $K = A^* A$ is also an operator of Carleman.

The considered questions are the following:

(i) decomposition of the process by the eigenfunctions of K;

(ii) linear approximation of the random variable η by the random variables $(f, \nu) = Af$, i.e., to find $f_0 \in H_-$ such that

$$\|\xi - Af_0\|_\Omega = \min_{f \in H_-} \|\xi - Af\|_\Omega ,$$

where $H_- = \overline{H}$ $\left(\text{for a certain norm } \|.\|_-\right)$.

We will treat the case which corresponds to our class of Carleman operators:

$$L_2(\Omega) = H = L_2(X;\ \mu)$$

$$\nu(x) = \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)}$$

$$Af = (f, \nu) = \sum_{p=0}^{\infty} a_p (f, \psi_p) \psi_p$$

$$Kf = A^* Af = \sum_{p=0}^{\infty} |a_p|^2 (f, \psi_p) \psi_p$$

$$H_- = \left\{ f = \sum_{p=0}^{\infty} \alpha_p \psi_p : \sum_{p=0}^{\infty} |\alpha_p \psi_p|^2 < \infty \right\}.$$

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