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Extreme Points and Strong U-Points in Musielak-Orlicz Sequence Spaces Equipped with the Orlicz Norm

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Abstract. We give some criteria for extreme points and strong U-points in Musielak–Orlicz sequence spaces equipped with the Orlicz norm. It follows from these results that the notion of the strong U-point is essentially stronger than the notion of the extreme point in these spaces.

Keywords. Musielak–Orlicz sequence space, extreme point, strong U-point, Orlicz norm.

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1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and $S(X)$ be the unit sphere of X. By X^* denote the dual space of X. For any $x \in S(X)$, we denote by $\operatorname{Grad}(x)$ the set of all support functionals at x, that is, $\operatorname{Grad}(x) = \{f \in S(X^*) : f(x) = \|x\|\}$.

A point $x \in S(X)$ is called an extreme point if for every $y, z \in S(X)$ with $x = \frac{y+z}{2}$, we have $y = z = x$. A Banach space X is said to be rotund if every point of $S(X)$ is an extreme point.

A point $x \in S(X)$ is said to be a strong U-point (SU-point for short) if for any $y \in S(X)$ with $\|x + y\| = 2$, we have $x = y$. It is obvious that a Banach space X is rotund if and only if every $x \in S(X)$ is an SU-point.

Recall that the nature of SU-points is such that a point $x \in S(X)$ is a point of local uniform rotundity if and only if x is a point of compact local uniform rotundity and an SU-point (see [4]).

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Extreme points and strongly extreme points in Orlicz sequence spaces have been investigated in [5] and [11]. The criteria for extreme points and strong U-points in Orlicz sequence space were obtained in [2, 3, 4] and criteria for rotundity of Musielak–Orlicz spaces were presented in [7]. In this paper, we will give criteria for extreme points and SU-points in Musielak–Orlicz sequence space equipped with the Orlicz norm. As it has been noted in [4], the notions of extreme point and SU-point are different and the second notion is much stronger than the first one. As it follows from criteria presented in this paper the situation in Musielak–Orlicz sequence spaces equipped with the Orlicz norm is similar.

The sequence $M = (M_i)_{i=1}^{\infty}$ is called a Musielak–Orlicz function provided that for any $i \in \mathcal{N}$, $M_i : (-\infty, +\infty) \to [0, +\infty]$ is even, convex, left continuous on $[0, +\infty)$, $M_i(0) = 0$, and there exists $u_i > 0$ such that $M_i(u_i) < \infty$ (see [10]). By $N = (N_i)_{i=1}^{\infty}$ we denote the Musielak–Orlicz function complementary to $M = (M_i)$ in the sense of Young, i.e.,

$$N_i(v) = \sup_{u \geq 0} \{ u\,|v| - M_i(u) \}$$

for each $v \in \mathcal{R}$ and $i \in \mathcal{N}$.

Define $b(i) = \sup\{u \geq 0 : M_i(u) = 0\}$, $B(i) = \sup\{u \geq 0 : M_i(u) < \infty\}$, $\widetilde{b}(i) = \sup\{v \geq 0 : N_i(v) = 0\}$ and $\widetilde{B}(i) = \sup\{v \geq 0 : N_i(v) < \infty\}$ for each $i \in \mathcal{N}$. Let $p_i(u)$ and $p_i^-(u)$ ($q_i(v)$ and $q_i^-(v)$) stand for the right and left derivatives of M_i (of N_i) at $u \in \mathcal{R}$ with $0 \leq u < B(i)$ (at $v \in \mathcal{R}$ with $0 \leq v < \widetilde{B}(i)$), respectively. Here we define $p_i(B(i)) = \infty$, $p_i^-(u) = p_i(u) = \infty$ for $u > B(i)$, $q_i(\widetilde{B}(i)) = \infty$ and $q_i^-(v) = q_i(v) = \infty$ for $v > \widetilde{B}(i)$.

Moreover, for every $u, v \in \mathcal{R}$, we have the following Young inequality:

$$|uv| \leq M_i(u) + N_i(v).$$

Further, $|uv| = M_i(u) + N_i(v)$ if and only if $p_i^-(|u|) \leq |v| \leq p_i(|u|)$, when u is fixed, or $q_i^-(|v|) \leq |u| \leq q_i(|v|)$, when v is fixed (cf. [2], p. 5).

Let l^0 denote the space of all real sequences $x = (x(i))$. Given any Musielak–Orlicz function $M = (M_i)$, we define on l^0 the convex modular ρ_M by

$$\rho_M(x) = \sum_{i=1}^{\infty} M_i(x(i)) \quad \text{for any } x = (x(i)) \in l^0.$$

The space $\{x \in l^0 : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ \lambda > 0 : \rho_M\left(\tfrac{x}{\lambda}\right) \leq 1 \right\}$$

or the Orlicz norm

$$\|x\|^0 = \sup \left\{ \sum_i x(i)y(i) : \rho_N(y) \leq 1 \right\}$$

is a Banach space, denoted according to the norm by l_M or l_M^0 respectively, and it is called the Musielak–Orlicz sequence space (see [2, 8, 10]). The subspace

$$\Big\{x \in l_M : \text{for any } \lambda > 0, \text{ there exists } i_0 \text{ such that } \sum_{i>i_0} M_i(\lambda x(i)) < \infty\Big\}$$

equipped with the norm $\|\cdot\|$ (or $\|\cdot\|^0$) is also a Banach space, and it is denoted by h_M (resp. h_M^0). For Orlicz spaces, i.e. the spaces that are generated by the Musielak–Orlicz function $(M_i)_{i=1}^\infty$ with all M_i being the same, we refer to [9].

We say that $\phi \in (l_M^0)^*$ is a singular functional ($\phi \in F$ for short), if $\phi(x) = 0$ for any $x \in h_M^0$. The dual space of l_M^0 is represented in the form $(l_M^0)^* = l_N \oplus F$, i.e., every $f \in (l_M^0)^*$ has the unique representation $f = y + \phi$, where $\phi \in F$ and $y \in l_N$ is the regular functional defined by the formula $\langle x, y\rangle = \sum_{i=1}^\infty x(i)y(i)$ (for any $x = (x(i)) \in l_M^0$).

For any $i \in \mathcal{N}$, we say that a point $w \in \mathcal{R}$ is a strict convexity point of M_i, if $M_i(\frac{u+v}{2}) < \frac{1}{2}(M_i(u) + M_i(v))$ whenever $w = \frac{u+v}{2}$ and $u \neq v$. We write then $w \in SC_{M_i}$. An interval $[a, b]$ is called a structurally affine interval of M_i (or simply SAI of M_i) provided that M_i is affine on $[a, b]$ and it is not affine either on $[a-\varepsilon, b]$ or on $[a, b+\varepsilon]$ for any $\varepsilon > 0$. It is obvious that $SC_{M_i} = \mathcal{R}\backslash(\bigcup_n (a_n, b_n))$, where $[a_n, b_n] \in SAI(M_i)$, $n = 1, 2, \ldots$.

For any $i \in \mathcal{N}$, denote

$$\begin{aligned} SC^-_{M_i} &= \{u \in SC_{M_i} :\ \exists\, \varepsilon > 0 \text{ such that } M_i \text{ is affine on } [u, u+\varepsilon]\} \\ SC^+_{M_i} &= \{u \in SC_{M_i} :\ \exists\, \varepsilon > 0 \text{ such that } M_i \text{ is affine on } [u-\varepsilon, u]\}, \\ SC^0_{M_i} &= SC_{M_i} \setminus (SC^+_{M_i} \cup SC^-_{M_i}). \end{aligned}$$

For any $x \in l_M^0$, we put:

$$\begin{aligned} &\operatorname{supp} x = \{i \in \mathcal{N} : x(i) \neq 0\}, \\ &\theta(x) = \inf\Big\{\lambda > 0 : \text{there exists } i_0 \text{ such that } \sum_{i>i_0} M_i\big(\tfrac{x(i)}{\lambda}\big) < \infty\Big\}. \end{aligned}$$

Let $p \circ kx$ denote the sequence $\{p_i(kx(i))\}$ and let

$$k_x^* = \inf\Big\{k > 0 : \rho_N(p \circ kx) = \sum_{i=1}^\infty N_i\big(p_i(kx(i))\big) \geq 1\Big\}$$

$$k_x^{**} = \sup\Big\{k > 0 : \rho_N(p \circ kx) = \sum_{i=1}^\infty N_i\big(p_i(kx(i))\big) \leq 1\Big\}$$

$$k(x) = \begin{cases} [k_x^*, k_x^{**}], & \text{if } k_x^{**} < \infty \\ [k_x^*, \infty), & \text{if } k_x^* < \infty \end{cases}$$

and $k_x^{**} = \infty$, and $k(x) = \emptyset$, if $k_x^* = \infty$.

For the convenience of reading, we first list some known results.

Lemma 1.1 (see [12]). *If* $x \in l_M^0 \setminus \{0\}$, *then* $k(x) \neq \emptyset$ *if and only if* $\sum_{i\in \operatorname{supp} x} N_i(\widetilde{B}(i)) > 1$ *or* $\sum_{i\in \operatorname{supp} x} N_i(\widetilde{B}(i)) = 1$ *and* $\sup_{i\in \operatorname{supp} x} \frac{q_i^-(\widetilde{B}(i))}{|x(i)|} < \infty$.

Lemma 1.2 (see [1]). *Let* $x \in l_M^0 \setminus \{0\}$. *If* $\sum_{i\in \operatorname{supp} x} N_i(\widetilde{B}(i)) > 1$, *then* $\|x\|^0 = \frac{1}{k}(1+\rho_M(kx))$ *if and only if* $k \in k(x)$, *and if* $\sum_{i\in \operatorname{supp} x} N_i(\widetilde{B}(i)) \leq 1$, *then* $\|x\|^0 = \sum_{i\in \operatorname{supp} x} |x(i)|\,(\widetilde{B}(i))$.

Lemma 1.3. *If* $1 = \|x\|_M^0 = \frac{1}{k}(1+\rho_M(kx))$, *then* $f = y + \phi$ *is a support functional of* x *if and only if*

1. $\rho_N(y) + \|\phi\| = 1$,
2. $\|\phi\| = \phi(kx)$,
3. $x(i)y(i) \geq 0$ *and* $p_i^-(k\,|x(i)|) \leq |y(i)| \leq p_i(k\,|x(i)|)$ *for any* $i \in \mathcal{N}$.

Proof. The proof of this lemma is similar to that of Theorem 1.77 in [12] and [6], so we omit it here. □

Lemma 1.4 (see [6]). *For any* $\phi \in F$, *we have*

$$\|\phi\| = \sup\{\phi(x) : \rho_M(x) < \infty\} = \sup_{\theta(x)\neq 0} \frac{\phi(x)}{\theta(x)}.$$

Lemma 1.5. *Let* $x \in S(l_M^0)$. *If* $\theta(kx) < 1$ *for some* $k \in k(x)$, *then all support functionals of* x *are in* l_N.

Proof. If $\theta(kx) = 0$, then the implication is obvious. Let us suppose that $0 < \theta(kx) < 1$. Take any support functional $f = y + \phi$ of x. By Lemma 1.4, we have $\|\phi\| = \sup_{\theta(y)\neq 0} \frac{\phi(y)}{\theta(y)} \geq \frac{\phi(kx)}{\theta(kx)} > \phi(kx)$. From Lemma 1.3, it follows that $\phi = 0$, which completes the proof of the lemma. □

2. Main results

We start with a criterion for extreme points of $S(l_M^0)$.

Theorem 2.1. *A point* $x = (x(i)) \in S(l_M^0)$ *is an extreme point of* $S(l_M^0)$ *if and only if:*

(i) $k(x) = \emptyset$ *and* $\operatorname{card}(\operatorname{supp} x) = 1$, *or*

(ii) $k(x) \neq \emptyset$ *and*

(ii-a) $\operatorname{card}(\operatorname{supp} x) = 1$ *and* $b(i) = 0$ *for any* $i \notin \operatorname{supp} x$, *or*

(ii-b) $\operatorname{card}(\operatorname{supp} x) > 1$ *and* $kx(i) \in SC_{M_i}$ *for any* $k \in k(x)$ *and any* $i \in \mathcal{N}$.

Proof. `Necessity`. If (i) does not hold, without loss of generality we may assume that $x(1) > 0, x(2) > 0$ and $k(x) = \emptyset$. By Lemma 1.1 and Lemma 1.2, we have $1 = \|x\|^0 = \sum_{i=1}^{\infty} |x(i)| \widetilde{B}(i)$.

Take $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfying $\varepsilon_1 \widetilde{B}(1) = \varepsilon_2 \widetilde{B}(2)$ and $x(1) - \varepsilon_1 > 0$, $x(2) - \varepsilon_2 > 0$. Define

$$y = (x(1) - \varepsilon_1, x(2) + \varepsilon_2, x(3), x(4), \dots)$$
$$z = (x(1) + \varepsilon_1, x(2) - \varepsilon_2, x(3), x(4), \dots).$$

It is obvious that $y + z = 2x$ and $y \neq z$. Moreover, by the definitions of the Orlicz norm and of $\widetilde{B}(i)$, we can easily obtain that $\|y\|^0 \le \sum_{i=1}^{\infty} |y(i)| \widetilde{B}(i) = \sum_{i=1}^{\infty} |x(i)| \widetilde{B}(i) = 1$. In fact, for any $v \in l_N$ with $\rho_N(v) \le 1$, by the definition of $\widetilde{B}(i)$, we have $|v(i)| \le \widetilde{B}(i)$ for all $i \in \mathcal{N}$ and hence $\sum_{i=1}^{\infty} y(i)v(i) \le \sum_{i=1}^{\infty} |y(i)| \widetilde{B}(i)$. From the definition of the Orlicz norm, it follows that $\|y\|^0 = \sup\left\{\sum_{i=1}^{\infty} y(i)v(i) : \rho_N(y) \le 1\right\} \le \sum_{i=1}^{\infty} |y(i)| \widetilde{B}(i)$. Similarly, we have $\|z\|^0 \le 1$. Using $\|x+y\|^0 = 2$, we get $\|y\|^0 = \|z\|^0 = 1$, which contradicts the fact that x is an extreme point.

Suppose (ii-a) fails. Then we may assume without loss of generality that $x = (x(1), 0, 0, \dots)$ and $b(i_0) > 0$ for some $i_0 > 1$. Take $k \in k(x)$ and put

$$y(i) = \begin{cases} x(i), & i \neq i_0 \\ \frac{b(i_0)}{k}, & i = i_0 \end{cases} \quad \text{and} \quad z(i) = \begin{cases} x(i), & i \neq i_0 \\ -\frac{b(i_0)}{k}, & i = i_0 \,. \end{cases}$$

Then $y + z = 2x$ and $y \neq z$. We can get a contradiction with the assumption that x is an extreme point by showing that $\|y\|^0 \le 1$ and $\|z\|^0 \le 1$. Note that from the definition of $b(i_0)$ we get

$$\|y\|^0 \le \frac{1}{k}(1 + \rho_M(ky)) = \frac{1}{k}(1 + \rho_M(kx)) = \|x\|^0 = 1.$$

Similarly, we have $\|z\|^0 \le 1$.

Now we verify the necessity of (ii-b). Otherwise, without loss of generality, we may assume that $x(1) > 0$, $x(2) > 0$, and there exists $k \in k(x)$ such that $kx(1) \in (a_1, b_1)$, where $[a_1, b_1] \in SAI(M_1)$ and $M_1(u) = Au + B$ for $u \in [a_1, b_1]$. Take $u_0 > 0$ such that $kx(1) \pm u_0 \in (a_1, b_1)$. By $1 = \|x\|^0 = \frac{1}{k}(1 + \rho_M(kx))$, we have $k = 1 + Akx(1) + B + \sum_{i \neq 1} M_i(kx(i))$. Put

$$h = 1 + A(kx(1) + u_0) + B + \sum_{i \neq 1} M_i(kx(i))$$
$$l = 1 + A(kx(1) - u_0) + B + \sum_{i \neq 1} M_i(kx(i)),$$

and $y = \frac{k}{h}(x(1) + \frac{u_0}{k}, x(2), x(3), \ldots)$, $z = \frac{k}{l}(x(1) - \frac{u_0}{k}, x(2), x(3), \ldots)$. Then $h + l = 2k$ and $hy + lz = 2kx$, i.e., $x = \frac{h}{2k}y + \frac{l}{2k}z$. Moreover, by the left continuity of p_i^-, right continuity of p_i and the fact that $k(x) = [k_x^*, k_x^{**}]$, we have

$$\begin{aligned}\rho_N(p^- \circ hy) &= N_1(p_1^-(kx(1) + u_0)) + \sum_{i \neq 1} N_i(p_i^-(kx(i)))\\ &= N_1(p_1^-(kx(1))) + \sum_{i \neq 1} N_i(p_i^-(kx(i)))\\ &= \rho_N(p^- \circ kx) \leq 1\end{aligned}$$

and for any $\eta > 0$

$$\begin{aligned}\rho_N(p \circ (1+\eta)hy) &= N_1\big(p_1((1+\eta)(kx(1) + u_0))\big) + \sum_{i \neq 1} N_i\big(p_i((1+\eta)kx(i))\big)\\ &= N_1\big(p_1((1+\eta)kx(1))\big) + \sum_{i \neq 1} N_i\big(p_i((1+\eta)kx(i))\big)\\ &= \rho_N(p \circ (1+\eta)kx) \geq 1.\end{aligned}$$

Hence $h \in k(y)$, and so

$$\|y\|^0 = \frac{1}{h}\big(1 + \rho_M(hy)\big) = \frac{1}{h}\left(1 + A(kx(1) + u_0) + B + \sum_{i \neq 1} M_i(kx(i))\right) = 1.$$

Similarly we can prove that $\|z\|^0 = 1$. Noticing that $y \neq z$, we conclude that x is not an extreme point. This contradiction shows that condition (ii-b) is necessary.

`Sufficiency`. Let $y + z = 2x$, $y, z \in S(l_M^0)$. We should show that $y = z = x$. First, we assume that $k(x) = \emptyset$. By (i), without loss of generality, we assume that $x = (x(1), 0, 0, \cdots)$ and $x(1) > 0$. Then by Lemma 1.1 and Lemma 1.2, we have $1 = \|x\|^0 = x(1)\widetilde{B}(1)$. Hence, by $\|x\|^0 = x(1)\|e_1\|^0$, we get $\|e_1\|^0 = \widetilde{B}(1)$, where $e_1 = (1, 0, 0, \ldots)$.

Now, we are going to prove that $y(1) = x(1)$. In fact, if $y(1) > x(1)$, then there exist $a > 0$ such that $y(1) > x(1) + a$. Therefore

$$1 = \|y\|^0 \geq \|y(1)e_1\|^0 = y(1)\widetilde{B}(1) > (x(1) + a)\widetilde{B}(1) > 1.$$

This contradiction shows that $y(1) \leq x(1)$. If we suppose that $y(1) < x(1) - b$ for some $b > 0$, then $z(1) > x(1) + b$. Using similar arguments as above we get a contradiction. So $y(1) = x(1)$.

Next, we shall show that $k(y) = \emptyset$. Otherwise, there exists $k_0 > 0$ such that $\|y\|^0 = \frac{1}{k_0}(1 + \rho_M(k_0 y))$. Since $k(x) = \emptyset$, we have

$$1 = \|y\|^0 = \frac{1}{k_0}\left(1 + \sum_{i=1}^{\infty} M_i(k_0 y(i))\right) \geq \frac{1}{k_0}\big(1 + M_1(k_0 y(1))\big) > \|x\|^0 = 1,$$

a contradiction. Therefore

$$1 = \|y\|^0 = \sum_{i=1}^{\infty} |y(i)| \widetilde{B}(i) = x(1)\widetilde{B}(1) + \sum_{i=2}^{\infty} |y(i)| \widetilde{B}(i) = 1 + \sum_{i=2}^{\infty} |y(i)| \widetilde{B}(i),$$

which yields that $\sum_{i=2}^{\infty} |y(i)| \widetilde{B}(i) = 0$. This means that $y(i) = 0 = x(i)$ for any $i > 1$. Using the equality $y + z = 2x$, we get that $y = z = x$.

Assume now that $k(x) \neq \emptyset$ and $k \in k(x)$. We will consider the following three cases.

`Case I`. $k(y) \neq \emptyset$, $k(z) \neq \emptyset$, $k_1 \in k(y)$ and $k_2 \in k(z)$. In this case, by the same method as in the proof of Theorem 2.8 in [2], we can prove that x is an extreme point.

`Case II`. $k(y) = \emptyset$ and $k(z) \neq \emptyset$. Since $\left\|\frac{y+z}{2}\right\|^0 = 1$ and $\|\cdot\|_M$ is a convex function, we have $\left\|\frac{x+y}{2}\right\|^0 = \left\|\frac{3}{4}y + \frac{1}{4}z\right\|^0 = 1$ and $\left\|\frac{x+z}{2}\right\|^0 = 1$.

Take a sequence $\{k_n\}_{n=1}^{\infty}$ of positive numbers such that $\frac{1}{k_n}(1 + \rho_M(k_n y)) < \|y\|^0 + \frac{1}{n}$ and put $h_n = \frac{2kk_n}{k+k_n}$. Then we have

$$\begin{aligned}
1 = \left\|\frac{x+y}{2}\right\|^0 &\leq \frac{1}{h_n}\left(1 + \rho_M\left(h_n \frac{x+y}{2}\right)\right) \\
&= \frac{k+k_n}{2kk_n}\left(1 + \rho_M\left(\frac{2kk_n}{k+k_n} \cdot \frac{x+y}{2}\right)\right) \\
&\leq \frac{k+k_n}{2kk_n}\left(1 + \frac{k_n}{k+k_n}\rho_M(kx) + \frac{k}{k+k_n}\rho_M(k_n y)\right) \\
&\leq \frac{1}{2}\left(\frac{1}{k}(1 + \rho_M(kx)) + \frac{1}{k_n}(1 + \rho_M(k_n y))\right) \\
&< \frac{1}{2}\left(\|x\|^0 + \|y\|^0 + \frac{1}{n}\right) \to 1 \quad (\text{as } n \to \infty).
\end{aligned}$$

Hence $\lim_{n\to\infty} \frac{1}{h_n}(1 + \rho_M(h_n \frac{x+y}{2})) = 1$. Since the sequence $\{h_n\}$ is bounded, we may assume (passing to a subsequence if necessary) that $\lim_{n\to\infty} h_n = h$. If we assume that $k(\frac{x+y}{2}) = \emptyset$, then $1 = \frac{\|x+y\|^0}{2} < \frac{1}{h}(1 + \rho_M(h\frac{x+y}{2}))$. Next, we take $i_0 \in \mathcal{N}$ such that $1 < \frac{1}{h}(1 + \sum_{i=1}^{i_0} M_i(h\frac{x(i)+y(i)}{2}))$, whence

$$\begin{aligned}
1 &< \frac{1}{h}\left(1 + \sum_{i=1}^{i_0} M_i\left(h\frac{x(i)+y(i)}{2}\right)\right) \\
&= \lim_{n\to\infty} \frac{1}{h_n}\left(1 + \sum_{i=1}^{i_0} M_i\left(h_n\frac{x(i)+y(i)}{2}\right)\right) \\
&\leq \lim_{n\to\infty} \frac{1}{h_n}\left(1 + \rho_M\left(h_n\frac{x+y}{2}\right)\right) = 1.
\end{aligned}$$

This is a contradiction, which shows that $k(\frac{x+y}{2}) \neq \emptyset$.

Taking $h \in k(z)$, the condition

$$\begin{aligned} 0 &= \frac{\|x\|^0 + \|z\|^0}{2} - \left\|\frac{x+z}{2}\right\|^0 \\ &\geq \frac{1}{2k}\big(1+\rho_M(kx)\big) + \frac{1}{2h}\big(1+\rho_M(hz)\big) - \frac{k+h}{2kh}\left(1+\rho_M\left(\frac{2kh}{k+h}\Big(\frac{x+z}{2}\Big)\right)\right) \geq 0, \end{aligned}$$

implies that $\left\|\frac{x+z}{2}\right\|^0 = \frac{k+h}{2kh}(1+\rho_M(\frac{2kh}{k+h}(\frac{x+z}{2})))$, i.e., $k(\frac{x+z}{2}) \neq \emptyset$.

Put $y' = \frac{x+y}{2}$ and $z' = \frac{x+z}{2}$. Then, by Case I, we have $y' = z'$, i.e., $y = z$. So, Case II can not occur if $y \neq z$.

`Case III`. $k(y) = \emptyset$ and $k(z) = \emptyset$. Put $y' = \frac{x+y}{2}$ and $z' = \frac{x+z}{2}$. Clearly $y' + z' = 2x$. Similarly as in Case II, we can prove that $\|y'\|^0 = \|z'\|^0 = 1$, $k(y') \neq \emptyset$ and $k(z') \neq \emptyset$. By Case I, we conclude that $y' = z'$. Consequently $y = z$, and the result follows. □

Theorem 2.2. *A point $x \in S(l_M^0)$ with $k(x) = \emptyset$ is an SU-point of $S(l_M^0)$ if and only if:*

(1) $\operatorname{card}(\operatorname{supp} x) = 1$, *say* $\operatorname{supp} x = \{j\}$,

(2) *for any $i \neq j$, we have* $N_j(\widetilde{B}(j)) + N_i(\widetilde{B}(i)) > 1$,

(3) $q_j^-(\widetilde{B}(j)) = \infty$ *if* $N_j(\widetilde{B}(j)) < 1$.

If $x \in S(\ell_M^0)$ and $k(x) \neq \emptyset$, then x is an SU-point of $S(l_M^0)$ if and only if:

(I) $\operatorname{card}(\operatorname{supp} x) = 1$ *and $b(i) = 0$ for any $i \notin \operatorname{supp} x$, or*

(II) $\operatorname{card}(\operatorname{supp} x) > 1$, *and for any $k \in k(x)$ we have*

(i) $kx(i) \in SC_{M_i}$ *for all* $i \in \mathcal{N}$,

(ii) $\{i \in \mathcal{N} : k|x(i)| \in SC_{M_i}^+\} = \emptyset$ *if* $\theta_M(kx) = 1$,

(iii) $\sum_{i\neq j} N_i(p_i(k\,|x(i)|)) + N_j(p_j^-(k\,|x(j)|)) < 1$ *if* $k\,|x(j)| \in SC_{M_j}^+$ *for some* $j \in \mathcal{N}$,

(iv) $\sum_{i\neq j} N_i(p_i^-(k\,|x(i)|)) + N_j(p_j(k\,|x(j)|)) > 1$ *if* $k\,|x(j)| \in SC_{M_j}^-$ *for some* $j \in \mathcal{N}$.

Proof. Without loss of generality, we may assume that $x(i) \geq 0$ for all $i \in \mathcal{N}$.

At first, we suppose that $k(x) = \emptyset$.

`Necessity`. Since any SU-point is an extreme point, by Theorem 2.1, condition (1) holds and we assume, without loss of generality, that $j = 1$.

Let us suppose that (2) fails. Then there exists an $i_0 > 1$ such that $N_1(\widetilde{B}(1)) + N_{i_0}(\widetilde{B}(i_0)) \leq 1$. Put

$$y(i) = \begin{cases} \frac{1}{\widetilde{B}(i_0)}, & i = i_0 \\ 0, & i \neq i_0\,. \end{cases}$$

Then, by Lemma 1.2, we have $\|y\|^0 = \frac{1}{\widetilde{B}(i_0)}\widetilde{B}(i_0) = 1$ and $\|x+y\|^0 = x(1)\widetilde{B}(1) + \frac{1}{\widetilde{B}(i_0)}\widetilde{B}(i_0) = 2$. But it is obvious that $x \neq y$, which means that x is not an SU-point.

If (3) does not hold, then $N_1(\widetilde{B}(1)) < 1$ and $q_1^-(\widetilde{B}(1)) < \infty$. Since $N_1(\widetilde{B}(1)) + N_2(\widetilde{B}(2)) > 1$, there exists $\beta_2 \in (\widetilde{b}(2), \widetilde{B}(2))$ such that

$$N_1(\widetilde{B}(1)) + N_2(\beta_2) = 1. \tag{$*$}$$

We have $0 < q_1^-(\widetilde{B}(1)) < \infty$ and $0 < q_2^-(\beta_2) < \infty$. Consider the following system of equations:

$$\begin{cases} w_1\widetilde{B}(1) + w_2\beta_2 = 1 \\ w_1 q_2^-(\beta_2) - w_2 q_1^-(\widetilde{B}(1)) = 0\,, \end{cases}$$

where we are looking for w_1 and w_2. Denoting the solution of this system of equations by (x_1, x_2), we have $x_1 > 0$ and $x_2 > 0$. Let $y = (x_1,\ x_2,\ 0,\ 0, \ldots)$. It was already proved in Theorem 9 in [7] that $\|y\|^0 = x_1\widetilde{B}(1) + x_2\beta_2 = 1$. Therefore, by $(*)$, we have

$$1 = \frac{x_1\widetilde{B}(1) + x_2\beta_2 + x(1)\widetilde{B}(1)}{2} = \frac{x_1 + x(1)}{2}\widetilde{B}(1) + \frac{x_2}{2}\beta_2 \le \left\|\frac{x+y}{2}\right\|^0 \le 1,$$

i.e., $\|x+y\|^0 = 2$. But it is obvious that $x \neq y$, which means that x is not an SU-point.

`Sufficiency`. For convenience, let $x = (x(1), 0, 0.\cdots)$, $y \in S(l_M^0)$ and $\|x+y\|^0 = 2$. Choose $f \in (l_M^0)$ such that $\|f\| = 1$ and $f(x+y) = \|x+y\|^0 = 2$. Hence we obtain $f(x) = f(y) = 1$. Notice that $x \in h_M^0$, so we have by $k(x) = \emptyset$ that $f \in S(l_N)$ and $f(1) = \frac{1}{x(1)} = \widetilde{B}(1)$.

Now, we shall prove that $|f(i)| < \widetilde{B}(i)$ for any $i > 1$. Otherwise, there exists $i_0 > 1$ such that $|f(i_0)| = \widetilde{B}(i_0)$. Hence, by (2), we have

$$1 \ge \rho_N(f) \ge N_1(f(1)) + N_{i_0}(f(i_0)) = N_1(\widetilde{B}(1)) + N_{i_0}(\widetilde{B}(i_0)) > 1,$$

which is a contradiction, proving the claim.

Next, we are going to show that $y(i) = 0$ for any $i > 1$. Indeed, if we suppose that $y(i_0) \neq 0$ for some $i_0 > 1$, then

$$\begin{aligned}\sum_{i=1}^{\infty} |y(i)|\,\widetilde{B}(i) &= |y(i_0)|\,\widetilde{B}(i_0) + \sum_{i\neq i_0} |y(i)|\,\widetilde{B}(i) \\ &> |y(i_0)|\,|f(i_0)| + \sum_{i\neq i_0} |y(i)|\,|f(i)| \\ &\ge \sum_{i=1}^{\infty} y(i)f(i) = f(y) = 1.\end{aligned}$$

By Lemma 1.1 and Lemma 1.2, we conclude that $k(y) \neq \emptyset$. Take $k > 0$ satisfying $\frac{1}{k}(1 + \rho_M(ky)) = \|y\|^0 = 1$. Then, by the Young inequality, we get

$$k = k\sum_{i=1}^{\infty} f(i)y(i) \leq \rho_M(ky) + \rho_N(f) \leq \rho_M(ky) + 1 = k.$$

Therefore, the above inequalities are equalities in fact, whence $q_i^-(|f(i)|) \leq k\,|y(i)| \leq q_i(|f(i)|)$, $i = 1, 2, \ldots$. In particular, we have the inequality

$$q_1^-(\widetilde{B}(1)) \leq k\,|y(1)| \leq q_1(\widetilde{B}(1)). \qquad (**)$$

By $k(x) = \emptyset$, if $N_1(\widetilde{B}(1)) = 1$, then by Lemma 1.1, we have $q_1^-(\widetilde{B}(1)) = \infty$; if $N_1(\widetilde{B}(1)) < 1$, then by condition (3), we also have $q_1^-(\widetilde{B}(1)) = \infty$. So we always have $q_1^-(\widetilde{B}(1)) = \infty$, which contradicts the inequality $(**)$. This contradiction shows that $y(i) = 0$ for any $i > 1$.

Therefore, from $\|y\|^0 = \|x\|^0 = \left\|\frac{x+y}{2}\right\|^0 = 1$, it follows that $x(1) = y(1)$. Consequently, we have $x = y$, which means that x is an SU-point.

Now, we shall consider the case when $k(x) \neq \emptyset$.

`Necessity`. Clearly x is an extreme point. So, from Theorem 2.1, we get that conditions (I) and (II)-(i) are necessary.

We are going to prove that (ii) in condition (II) holds. If not we may assume, without loss of generality, that $kx(1) = b_1 \in SC_{M_1}^+$ and $\theta(kx) = 1$ for some $k \in k(x)$. Take $c_1 < b_1$ satisfying $p_1^-(c_1) = p_1(c_1) = p_1^-(b_1)$ and let $y = (\frac{c_1}{k}$, $x(2)$, $x(3), \ldots)$. Then we have $\rho_N(p^-(ky)) = \rho_N(p^-(kx)) \leq 1$. Moreover, for any $\eta > 0$, by $\theta_M(kx) = 1$, we get $\sum_{i>1} M_i((1+\eta)kx(i)) = \infty$. From the Young inequality, it follows that $\sum_{i>1} N_i(p_i((1+\eta)kx(i))) = \infty$. Thus, for any $\eta > 0$, we have

$$\rho_N(p \circ (1+\eta)ky) \geq \sum_{i>1} N_i\big(p_i((1+\eta)kx(i))\big) = \infty.$$

So $k \in k(y)$. Take $h = k\,\|y\|^0 \in k\big(\frac{y}{\|y\|^0}\big)$. Then

$$\begin{aligned}\rho_N\left(p^- \circ \frac{kh}{k+h}\Big(x + \frac{y}{\|y\|^0}\Big)\right) &= \sum_{i>1} N_i\big(p_i^-(kx(i))\big) + N_1\left(p_1^-\Big(\frac{h}{k+h}b_1 + \frac{k}{k+h}c_1\Big)\right)\\ &= \sum_{i>1} N_i\big(p_i^-(kx(i))\big) + N_1(p_1^-(b_1))\\ &= \rho_N(p^- \circ kx) \leq 1\end{aligned}$$

and for any $\eta > 0$

$$\rho_N\left(p \circ (1+\eta)\frac{kh}{k+h}\Big(x + \frac{y}{\|y\|^0}\Big)\right) \geq \sum_{i>1} N_i\big(p_i((1+\eta)kx(i))\big) = \infty,$$

i.e., $\frac{kh}{k+h} \in k\big(x + \frac{y}{\|y\|^0}\big)$. Therefore

$$\begin{aligned}\left\|x + \frac{y}{\|y\|^0}\right\|^0 &= \frac{k+h}{kh}\left(1 + \rho_M\Big(\frac{kh}{k+h}\Big(x + \frac{y}{\|y\|^0}\Big)\Big)\right) \\ &= \frac{k+h}{kh}\left(1 + \sum_{i\neq 1} M_i(kx(i)) + M_1\Big(\frac{h}{k+h}b_1 + \frac{k}{k+h}c_1\Big)\right) \\ &= \frac{k+h}{kh}\left(1 + \sum_{i\neq 1} M_i(kx(i)) + \frac{h}{k+h}M_1(b_1) + \frac{k}{k+h}M_1(c_1)\right) \\ &= \frac{1}{k}(1 + \rho_M(kx)) + \frac{1}{h}(1 + \rho_M(ky)) \\ &= 2.\end{aligned}$$

But it obvious that $x \neq \frac{y}{\|y\|^0}$. This shows that x is not an SU-point if (ii) does not hold.

If (iii) does not hold, we may assume, without loss of generality, that $kx(1) = b_1 \in SC^+_{M_i}$ and $\sum_{i\neq 1} N_i(p_i(kx(i))) + N_1(p_1^-(b_1)) \geq 1$ for some $k \in k(x)$. In view of $\sum_{i\neq 1} N_i(p_i^-(kx(i)) + N_1(p_1^-(b_1)) \leq 1$, there exists $v \in l_N$ such that $\rho_N(v) = 1$ and $v(1) = p_1^-(b_1)$, $p_i^-(kx(i)) \leq v(i) \leq p_i(kx(i))$ for any $i > 1$. From Lemma 1.3, it follows that $v \in \mathrm{Grad}(x)$. Pick $c_1 < b_1$ such that $p_1^-(c_1) = p_1(c_1) = p_1^-(b_1)$ and put $y = (\frac{c_1}{k}, x(2), x(3), \ldots)$. Then $\rho_N(p_-(ky)) = \rho_N(p_-(kx)) \leq 1$ and $\rho_N(p(ky)) = \sum_{i\neq 1} N_i(p_i kx(i)) + N_1(p_1^-(b_1)) \geq 1$. So $k \in k(y)$. Thus, by Lemma 1.3, we get that $v \in \mathrm{Grad}(y)$. Therefore

$$2 \geq \left\|x + \frac{y}{\|y\|^0}\right\|^0 \geq \left\langle x + \frac{y}{\|y\|^0}, v\right\rangle = \langle v, x\rangle + \frac{1}{\|y\|^0}\langle v, y\rangle = 2,$$

i.e., $\big\|x + \frac{y}{\|y\|^0}\big\|^0 = 2$. This leads to the conclusion that x is not an SU-point.

Suppose (iv) fails. Then we may assume that $kx(1) = a_1 \in SC^-_{M_i}$ and $\sum_{i\neq 1} N_i(p_i^-(kx(i))) + N_1(p_1(a_1)) \leq 1$ for some $k \in k(x)$. Take $c_1 > a_1$ satisfying $p_1^-(c_1) = p_1(c_1) = p_1(a_1)$ and put $y = (\frac{c_1}{k}, x(2), x(3), \ldots)$. Then

$$\rho_N(p^- \circ ky) = \sum_{i\neq 1} N_i(p_i^- kx(i)) + N_1(p_1^-(c_1)) = \sum_{i\neq 1} N_i(p_i^-(kx(i))) + N_1(p_1(a_1)) \leq 1$$

and, for any $\eta > 0$, we have $\rho_N(p \circ (1+\eta)ky) = \rho_N(p \circ (1+\eta)kx) \geq 1$, i.e., $k \in k(y)$. Put $h = k\,\|y\|^0 \in k\big(\frac{y}{\|y\|^0}\big)$. By the argumentation as above, we can finish the proof of (iv), so we omit the remaining procedure of the proof.

`Sufficiency`. Let $y \in S(l_M^0)$, $\|x + y\|^0 = 2$ and $k \in k(x)$. In the following we will investigate two cases.

`Case 1`: $k(y) \neq \emptyset, h \in k(y)$. In order to show that $x = y$, we only need to prove that $kx = ky$. From the inequalities

$$\begin{aligned}
0 &= \|x\|^0 + \|x\|^0 - \|x+y\|^0 \\
&\geq \frac{1}{k}(1+\rho_M(kx)) + \frac{1}{h}(1+\rho_M(hy)) - \frac{k+h}{kh}\Big(1+\rho_M\Big(\frac{kh}{k+h}(x+y)\Big)\Big) \\
&= \frac{k+h}{kh}\Big(\frac{h}{k+h}\rho_M(kx) + \frac{k}{k+h}\rho_M(hy) - \rho_M\Big(\frac{kh}{k+h}(x+y)\Big)\Big) \\
&= \frac{k+h}{kh}\sum_{i=1}^{\infty}\Big(\frac{h}{k+h}M_i(kx(i)) + \frac{k}{k+h}M_i(hy(i)) \\
&\quad - M_i\Big(\frac{h}{k+h}kx(i) + \frac{k}{k+h}hy(i)\Big)\Big) \geq 0,
\end{aligned}$$

we obtain that $kx(i) = hy(i)$ or $kx(i)$ and $hy(i)$ belong to the same affine interval of M_i for all $i \in \mathcal{N}$ and $\frac{kh}{k+h} \in k(x+y)$.

If $\mathrm{card}(\mathrm{supp}\, x) = 1$, without loss of generality, we may assume that $x(1) \neq 0$. Then by (I), we have $b(i) = 0$ for all $i > 1$, i.e., $0 \in SC^0_{M_i}$ for all $i > 1$. Therefore, $y(i) = 0$ for any $i > 1$. Using $\|x\|^0 = \|y\|^0 = \big\|\frac{x+y}{2}\big\|^0 = 1$, we get that $x(1) = y(1)$.

If card $(\mathrm{supp}\, x) > 1$, we will consider again two cases.

`(A)`. $\theta(kx) < 1$. Since $\theta(kx) < 1$, there exist $\tau > 0$ and $i_0 \in \mathcal{N}$ such that $\sum_{i>i_0} M_i((1+\tau)kx(i)) < \infty$. Take $\varepsilon > 0$ small enough so that $\frac{1+\varepsilon}{1-\frac{\varepsilon k}{h}} < 1+\tau$. Then

$$\begin{aligned}
&\sum_{i>i_0} M_i\Big((1+\varepsilon)\frac{kh}{k+h}(x(i)+y(i))\Big) \\
&\qquad = \sum_{i>i_0} M_i\Big(\frac{(1+\varepsilon)k}{k+h}hy(i) + \frac{(1-\frac{\varepsilon k}{h})h}{k+h}\,\frac{1+\varepsilon}{1-\frac{\varepsilon k}{h}}kx(i)\Big) \\
&\qquad \leq \frac{(1+\varepsilon)k}{k+h}\sum_{i>i_0} M_i(hy(i)) + \frac{(1-\frac{\varepsilon k}{h})h}{k+h}\sum_{i>i_0} M_i\Big(\frac{1+\varepsilon}{1-\frac{\varepsilon k}{h}}kx(i)\Big) \\
&\qquad \leq \frac{(1+\varepsilon)k}{k+h}\rho_M(hy) + \frac{h-\varepsilon k}{k+h}\sum_{i>i_0} M_i((1+\tau)kx(i)) < \infty.
\end{aligned}$$

This means that $\theta\big(\frac{kh}{k+h}(x+y)\big) \leq \frac{1}{1+\varepsilon} < 1$. Then, by Lemma 1.5 and Lemma 1.3, we have $\rho_N\big(p \circ \frac{kh}{k+h}(x+y)\big) \geq 1$.

For any $i \in \mathcal{N}$, if $kx(i) \in SC^0_{M_i}$, then it is obvious that $hy(i) = kx(i)$. Now we want to prove that if $kx(i) = b_i \in SC^+_{M_i} \setminus SC^-_{M_i}$ (that is $b_i \in SC^+_{M_i}$ and $b_i \notin SC^-_{M_i}$), then $hy(i) = b_i$. Otherwise, there exists $i_0 \in \mathcal{N}$ such that

$kx(i_0) = b_{i_0} \in SC^{+}_{M_{i_0}} \setminus SC^{-}_{M_{i_0}}$ and $hy(i_0) < b_{i_0}$. Then $\frac{kh}{k+h}(x(i_0) + y(i_0)) < b_{i_0}$. Therefore, by (iii), we have

$$\begin{aligned} 1 &\le \rho_N\Big(p \circ \frac{kh}{k+h}(x+y)\Big) \\ &= \sum_{i\neq i_0} N_i\Big(p_i\Big(\frac{kh}{k+h}(x(i)+y(i))\Big)\Big) + N_{i_0}\Big(p_{i_0}\Big(\frac{kh}{k+h}x(i_0)+y(i_0)\Big)\Big) \\ &\le \sum_{i\neq i_0} N_i\big(p_i(kx(i))\big) + N_{i_0}\big(p^{-}_{i_0}(b_{i_0})\big) < 1. \end{aligned}$$

This is a contradiction, proving the claim.

By a similar argumentation, we can deduce that for any $i \in \mathcal{N}$, if $kx(i) = a_i \in SC^{-}_{M_i} \setminus SC^{+}_{M_i}$, then $hy(i) = a_i$.

For each $i \in \mathcal{N}$, if $kx(i) = a_i \in SC^{-}_{M_i} \setminus SC^{+}_{M_i}$, then by the same way as above, we can obtain that $hy(i) = a_i$.

(B). $\theta(kx) = 1$. ¿From (ii), it follows that $\{i \in \mathcal{N} : kx(i) \in SC^{+}_{M_i}\} = \emptyset$. So, it is enough to prove that if $kx(i) = a_i \in SC^{-}_{M_i} \setminus SC^{+}_{M_i}$, then $hy(i) = a_i$. In fact, if there exists $i_0 \in \mathcal{N}$ satisfying $kx(i_0) = a_{i_0} < hy(i_0)$, then $\frac{kh}{k+h}(x(i_0)+y(i_0)) > a_{i_0}$. Hence

$$\begin{aligned} 1 &\ge \rho_N\Big(p^{-} \circ \frac{kh}{k+h}(x+y)\Big) \\ &= \sum_{i\neq i_0} N_i\Big(p^{-}_i\Big(\frac{kh}{k+h}(x(i)+y(i))\Big)\Big) + N_{i_0}\Big(p^{-}_{i_0}\Big(\frac{kh}{k+h}(x(i_0)+y(i_0))\Big)\Big) \\ &\ge \sum_{i\neq i_0} N_i\big(p^{-}_i(kx(i))\big) + N_{i_0}\big(p_{i_0}(a_{i_0})\big) > 1, \end{aligned}$$

a contradiction.

Case 2: $k(y) = \emptyset$. Using the same argumentation as in the proof of Case II in Theorem 2.1, we can deduce that $k(\frac{x+y}{2}) \neq \emptyset$ and $\big\|\frac{x+\frac{x+y}{2}}{2}\big\|^0 = 1$. Thus, by case 1 above, we obtain $\frac{x+y}{2} = x$. Consequently $x = y$. But $x \neq y$, so Case 2 can not take place. Thus, we finished the proof of Theorem 2.2. □

Remark 2.3. By comparing the criterion for extreme points with the criterion for SU-points in Musielak–Orlicz sequence spaces equipped with the Orlicz norm, we conclude that strong U-points are essentially stronger than extreme points in this class of spaces what is illustrated by the following example.

Let $M_i(u) = 0$ if $|u| \le 1$ and $M_i(u) = \infty$ if $|u| > 1$ for any $i \in \mathcal{N}$ and $M = (M_i)_{i=1}^{\infty}$. Then it is easy to see that $l^0_M = l_\infty$. Notice that $\|x\|^0 = \sup_{i\in\mathcal{N}} |x(i)| = \|x\|_\infty$ for any $x \in l^0_M$. This follows by the fact that for $x \neq 0$ and $k_0 = \|x\|_\infty^{-1}$ we have $I_M(k_0x) = 0$, whence $\frac{1}{k_0}\big(1 + I_M(k_0x)\big) = \|x\|_\infty$. Moreover, for any $k < \|x\|_\infty^{-1}$, we have $\frac{1}{k}\big(1 + I_M(kx)\big) \geq \frac{1}{k} > \|x\|_\infty$. Finally, for any $k > \|x\|_\infty^{-1}$, there exists $i \in \mathcal{N}$ such that $k|x(i)| > 1$, whence $I_M(kx) = \infty$ and so $k^{-1}\big(1+I_M(kx)\big) = \infty$. Since, for the function M, we can apply to the Orlicz norm $\|x\|^0$ the Amemiya formula (cf. [9]), we get

$$\|x\|^0 = \inf_{k>0} \frac{1}{k}\big(1 + I_M(kx)\big) = \|x\|_\infty .$$

Define $x = (1, 1, \ldots)$. Then $x \in l_M^0$ and $\|x\|^0 = \|x\|_\infty = 1$. Notice that $k(x) = \{1\}$. This follows by the fact that $I_M(x) = 0$, whence $1 + I_M(x) = 1$ and for all $k > 0$ with $k \neq 1$ we have $k^{-1}(1+I_M(kx)) > 1$. Evidently $\operatorname{card}(\operatorname{supp} x) = \infty$, so applying Theorem 2.1 (ii-b) we see that x is an extreme point of the unit ball of l_M^0.

Notice that x is not an SU-point of the unit ball of l_M^0 because taking $y = (1, 0, 0, \ldots)$ we get $\|x + y\|^0 = \|x\|_\infty = 2$ and $\|y\|^0 = \|y\|_\infty = 1$ and $x \neq y$. This fact follows also from our Theorem 2.2. Since $k \in k(x)$ only if $k = 1$ and $\operatorname{card}(\operatorname{supp} x) = \infty$, we should apply Case II of Theorem 2.2. Since the functions M_j are affine to the left of $kx_j = k|x_j| = 1$, we have $kx(j) \in SC_{M_j} \cap SC_{M_j}^+$ for any $j \in \mathcal{N}$. However condition (iii) of Case (II) does not hold, since $p_i(k\,|x(i)|) = p_i(1) = \infty$ for any $i \in \mathcal{N}$ and, for any $j \in \mathcal{N}$,

$$\sum_{i \neq j} N_i(p_i(k\,|x(i)|)) + N_j(p_j^-(k\,|x(j)|)) = \infty.$$

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