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Maximal Characterization of Locally Summable Functions

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Abstract. We prove a characterization of locally summable functions with bounded Stepanoff norm through the maximal function

$$M_\phi f(x) = \sup_{t>0} |(f * \phi_t)(x)|,$$

where ϕ is a suitable function in the class of Schwartz.

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1. Introduction

Given ϕ in the Schwartz class $\mathcal{S}$, the maximal function M_ϕ of a distribution f is

$$M_\phi f(x) = \sup_{t>0} |(f * \phi_t)(x)|,$$

where $\phi_t(x) = \frac{1}{t}\phi(\frac{x}{t})$. The following maximal characterization for $L^p(\mathbb{R})$ is well known (cfr. [5]).

Theorem 1.1. *Let $1 < p \leq +\infty$. If f is a distribution, then:*

$$f \in L^p(\mathbb{R}) \iff \exists \phi \in \mathcal{S}, \text{ with } \int \phi \, dx \neq 0, \text{ so that } M_\phi f \in L^p(\mathbb{R}).$$

It is interesting to consider maximal characterizations of spaces of functions which are only locally summable, that is, "big" at infinity. This problem has been suggested by A. Pankov in a seminar given at the Department of Metodi e Modelli Matematici per le Scienze Applicate at Università di Roma "La Sapienza".

First we need to introduce an appropriate Banach space structure on locally L^p functions. Let us define the spaces $\mathrm{BS}^p(\mathbb{R})$ of Stepanoff bounded functions.

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Definition 1.2. Let $1 \leq p < +\infty$. Then $f \in \mathrm{BS}^p(\mathbb{R})$ if

1. $f \in L^p_{loc}(\mathbb{R})$;
2. $\sup_{x\in\mathbb{R}} \int_x^{x+1} |f(t)|^p \, dt \leq c$, for $c \in \mathbb{R}$.

Note 1.3.

1. $\|f\|_{S^p} = \sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |f(t)|^p \, dt \right)^{\frac{1}{p}}$ is a norm; such norm is equivalent to the norm

$$\|f\|_{S^p_l} = \sup_{x\in\mathbb{R}} \left(\frac{1}{l} \int_x^{x+l} |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

where $l \in \mathbb{R}_+$ (cfr. [1]).

2. The space $\mathrm{BS}^p(\mathbb{R})$ contains the space of Stepanoff almost-periodic functions $S^p(\mathbb{R})$, i.e., the space of functions that can be approximated by trigonometric polynomials in the Stepanoff norm $\|\cdot\|_{S^p}$ defined above (cfr. [1] and [4]).

We prove the following maximal characterization of $\mathrm{BS}^p(\mathbb{R})$:

Theorem 1.4. *Let $1 < p < +\infty$. If f is a distribution, then:*

$$f \in \mathrm{BS}^p(\mathbb{R}) \iff \exists \phi \in \mathcal{S}, \text{ with } \int \phi \, dx \neq 0, \text{ so that } M_\phi f \in \mathrm{BS}^p(\mathbb{R}).$$

For this we need to prove an analogue, for $\mathrm{BS}^p(\mathbb{R})$, of the Hardy-Littlewood maximal theorem. As already observed, a similar result is known for L^p. The proof on L^p, however, does not readly extend to this more general framework, because, in the case of $L^p(\mathbb{R})$, one uses the weak compactness of $L^p(\mathbb{R})$ to prove the sufficient part of the maximal characterization. The dual space of $\mathrm{BS}^p(\mathbb{R})$ is not known and therefore we do not have a weak convergence result in such spaces.

2. The maximal characterization for $\mathrm{BS}^p(\mathbb{R})$

In this section we prove Theorem 1.4. The necessary part follows from the Hardy-Littlewood maximal theorem for $\mathrm{BS}^p(\mathbb{R})$, that we prove separately.

Let Mf be the maximal function of f, defined by

$$Mf(x) = \sup_{x\in I} \frac{1}{m(I)} \int_I |f(t)| \, dt,$$

where the supremum is taken over all intervals I containing x. Here m denotes the Lebesgue measure.

Note 2.1. $M_\phi f(x) \leq c Mf(x)$, where c is a constant (for the proof of this inequality see [5, Chapter 2, Section 2.1]).

Let $l > 0$ and $M_{\nu,l}f(x)$ be the maximal function of f restricted to $[\nu, \nu+l]$, $\nu \in \mathbb{R}$, i.e., let

$$M_{\nu,l}f(x) = \sup_{x\in I\subseteq[\nu,\nu+l]} \frac{1}{m(I)} \int_I |f(t)|\,dt,$$

if $x \in [\nu, \nu+l]$, and $M_{\nu,l}f(x) = 0$, if $x \notin [\nu, \nu+l]$.

Furthermore let

$$m_f(\alpha, \nu, l) = m\big(\{x \in [\nu, \nu+l] : M_{\nu,l}f(x) > \alpha\}\big),$$

with $\alpha, l \in \mathbb{R}_+$, $\nu \in \mathbb{R}$.

Then we have the following

Lemma 2.2. *For any $\nu \in \mathbb{R}$ and $l > 0$, $\alpha > 0$, and $f \in \mathrm{BS}^p(\mathbb{R})$, $p > 1$,*

$$m_f(\alpha, \nu, l) \leq \frac{4}{\alpha} \int_{\{x\in[\nu,\nu+l]:M_{\nu,l}f(x)>\frac{\alpha}{2}\}} |f(t)|\,dt$$

and

$$\frac{1}{l}\int_\nu^{\nu+l} |M_{\nu,l}f(t)|^p\,dt \leq \frac{2^{p+1}p}{p-1}\frac{1}{l}\int_\nu^{\nu+l} |f(t)|^p\,dt.$$

The proof is analogous to the proof of Theorem 4.3 in [3, Chapter 1].

Theorem 2.3 (Hardy-Littlewood maximal theorem for $\mathrm{BS}^p(\mathbb{R})$). *Let $p > 1$, $f \in \mathrm{BS}^p(\mathbb{R})$, and let $l > 0$. There exists $c > 0$ such that*

$$\|Mf\|_{S_l^p} \leq c\|f\|_{S_l^p}.$$

Proof. Let $\nu \in \mathbb{R}$ and $l > 0$. Set

$$\Pi_1 = \big\{I : I \subseteq [\nu - l, (\nu - l) + 3l]\big\}$$

and

$$\Pi_2 = \big\{J : J \cap (\mathbb{R} \setminus [\nu - l, (\nu - l) + 3l]) \neq \emptyset\big\},$$

where I, J are intervals. Furthermore set

$$N_{\nu-l,3l}f(x) = \sup_{x\in J\in\Pi_2} \frac{1}{m(J)} \int_J |f(t)|\,dt.$$

Then, for all $x \in \mathbb{R}$, $Mf(x) = \max\{M_{\nu-l,3l}f(x), N_{\nu-l,3l}f(x)\}$. By Lemma 2.2, it suffices to prove that $N_{\nu-l,3l}f(x) \leq c\|f\|_{S_l^p}$, for $x \in [\nu, \nu+l]$.

Let $J = [a, b] \in \Pi_2$. Since $x \in [\nu, \nu+l]$,

$$a < \nu - l \quad \text{and} \quad b > \nu$$

or else

$$\nu - l < a < \nu + l \quad \text{and} \quad b > (\nu - l) + 3l$$

and hence, in both cases, we have that $l' = b - a > l$. We can write

$$\begin{aligned}\frac{1}{m(J)} \int_J |f(t)|\, dt &= \frac{1}{b-a} \int_a^b |f(t)|\, dt \\ &= \frac{1}{b-a} \int_a^{a+(b-a)} |f(t)|\, dt \\ &= \frac{1}{l'} \int_a^{a+l'} |f(t)|\, dt.\end{aligned}$$

Since $l' > l$, we may write $l' = nl + \vartheta l$, with $n \in \mathbb{N}$ and $0 < \vartheta < 1$. Hence

$$\begin{aligned}\frac{1}{l'} \int_a^{a+l'} |f(t)|\, dt &= \frac{1}{nl + \vartheta l} \int_a^{a+nl+\vartheta l} |f(t)|\, dt \\ &< \frac{1}{nl} \int_a^{a+(n+1)l} |f(t)|\, dt \\ &\le \frac{1}{nl} \{ \int_a^{a+l} |f(t)|\, dt + \cdots + \int_{a+nl}^{a+(n+1)l} |f(t)|\, dt \} \\ &\le \frac{n+1}{n} \sup_{a \in \mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t)|\, dt \\ &\le 2 \sup_{a \in \mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t)|\, dt \le 2 \|f\|_{S_l^p}\end{aligned}$$

and therefore $N_{\nu - l, 3l} f(x) \le 2\|f\|_{S_l^p}$, for $x \in [\nu, \nu + l]$, and hence the thesis of the theorem is proved. □

In the proof of Theorem 1.4 we use a result due to R. Doss (cfr. [2]). For completeness we state that theorem:

Theorem 2.4. *Let $\{\sigma_m(x)\}$ be a sequence of functions summable in every finite interval and verifying the following condition: to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that, for every set E of diameter less than or equal to 1 and of measure less than or equal to δ,*

$$\int_E |\sigma_m(x)|\, dx \le \epsilon, \quad \forall m.$$

Then there exists a summable function $\sigma(x)$ and a subsequence $\{\sigma_{m_k}\}$ such that, for every bounded function $f(x)$ and every finite interval (a, b),

$$\lim_{k \to \infty} \int_a^b f(x) \sigma_{m_k}(x)\, dx = \int_a^b f(x) \sigma(x)\, dx.$$

Proof of Theorem 1.4. We first prove that $M_\phi f \in \mathrm{BS}^p$, for all $f \in \mathrm{BS}^p$. Let $f \in \mathrm{BS}^p(\mathbb{R})$. By Theorem 2.3, we have that there exists $c > 0$ such that $\|Mf\|_{S^p} \le c\|f\|_{S^p}$. Let $\phi \in \mathcal{S}$ and $M_\phi f(x) = \sup_{t>0} |(f * \phi_t)(x)|$. Then there exists $c' > 0$ such that $M_\phi f(x) \le c' Mf(x)$ (see Note 2.1). Hence

$$\|M_\phi f\|_{S^p} \le c'\|Mf\|_{S^p} \le c'c\|f\|_{S^p}$$

and $M_\phi f \in \mathrm{BS}^p(\mathbb{R})$.

Viceversa, suppose that $M_\phi f \in \mathrm{BS}^p(\mathbb{R})$, with $\phi \in \mathcal{S}$ such that $\int \phi = 1$. We want to show that $f \in \mathrm{BS}^p$. Let us consider the sequence $\big(f * \phi_{\frac{1}{n}}\big)(x)$. We have that

$$\|f * \phi_{\frac{1}{n}}\|_{S^p} = \sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |(f * \phi_{\frac{1}{n}})|^p \, dt \right)^{\frac{1}{p}}.$$

Since $M_\phi f \in \mathrm{BS}^p(\mathbb{R})$,

$$\sup_{x\in\mathbb{R}} \int_x^{x+1} |(f * \phi_{\frac{1}{n}})|^p \, dt \le \sup_{x\in\mathbb{R}} \int_x^{x+1} \left(\sup_{s>0} |(f * \phi_s)(t)| \right)^p dt \le B^p < +\infty,$$

where B is a constant, and hence $\|f * \phi_{\frac{1}{n}}\|_{S^p} \le B < +\infty$, i.e., $f * \phi_{\frac{1}{n}}$ is a bounded sequence in $\mathrm{BS}^p(\mathbb{R})$.

Set $h_n = f * \phi_{\frac{1}{n}}$. We want to show that there exists a subsequence $\{h_{n_j}\}_{j\in\mathbb{N}}$ and a function $f_o \in \mathrm{BS}^p(\mathbb{R})$ such that for any measurable and bounded function φ and for any bounded interval $(a,b) \subset \mathbb{R}$, one has

$$\lim_{j\to+\infty} \int_a^b \varphi(t) h_{n_j}(t) \, dt = \int_a^b \varphi(t) f_o(t) \, dt. \tag{1}$$

We apply Theorem 2.4 (cfr. [2]) in order to get that there exists a function $f_o \in L^1_{loc}(\mathbb{R})$ verifying (1) for any measurable bounded function φ and for any bounded interval (a,b) in $\mathbb{R}$. In order to do this, we need to prove that, if E is any measurable set such that $m(E) \to 0$, then $\int_E |h_n(t)| \, dt \to 0$ uniformly with respect to $n \in \mathbb{N}$.

Let E be measurable such that $m(E) \to 0$. The diameter of E is therefore less than 1 and hence $E \subset (x, x+1)$, for $x \in \mathbb{R}$ suitably chosen. Therefore

$$\begin{aligned}
\int_E |h_n(t)| \, dt &= \int_x^{x+1} \chi_E(t) |h_n(t)| \, dt \\
&\le \left(\int_x^{x+1} |h_n(t)|^p \, dt \right)^{\frac{1}{p}} \left(\int_x^{x+1} \chi_E(t) \, dt \right)^{\frac{1}{q}} \\
&\le \sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |h_n(t)|^p \, dt \right)^{\frac{1}{p}} [m(E)]^{\frac{1}{q}} \\
&\le B[m(E)]^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Hence $\int_E |h_n(t)|\,dt \to 0$, if $m(E)\to 0$, uniformly with respect to $n\in\mathbb{N}$. The hypotheses of Theorem 2.4 are satisfied and therefore (1) holds with $f_o \in L^1_{loc}(\mathbb{R})$.

We need to prove that $f_o \in \mathrm{BS}^p(\mathbb{R})$. To see this, let $\delta\in(0,1)$ and set, for $n\in\mathbb{N}$,

$$f_o^{(\delta)}(x)=\frac{1}{\delta}\int_x^{x+\delta} f_o(t)\,dt \quad\text{and}\quad h_n^{(\delta)}(x)=\frac{1}{\delta}\int_x^{x+\delta} h_n(t)\,dt\,.$$

Since the integral function can be derivated a.e., using Lebesgue's theorem we get

$$\lim_{\delta\to 0} f_o^{(\delta)}(x)=f_o(x),\qquad x\in\mathbb{R}\quad\text{a.e.}$$
$$\lim_{\delta\to 0} h_n^{(\delta)}(x)=h_n(x),\qquad x\in\mathbb{R}\quad\text{a.e..}$$

Furthermore, by (1), we get that

$$\lim_{j\to+\infty} h_{n_j}^{(\delta)}(x)=f_o^{(\delta)}(x),\quad \forall x\in\mathbb{R},\ \forall\delta\in(0,1).$$

Hence, by Fatou's lemma

$$\int_x^{x+1}|f_o^{(\delta)}(t)|^p\,dt\le\liminf_{j\to+\infty}\int_x^{x+1}|h_{n_j}^{(\delta)}(t)|^p\,dt.$$

In order to prove that $f_o\in\mathrm{BS}^p(\mathbb{R})$, we need to show first that $\int_x^{x+1}|f_o^{(\delta)}(t)|^p\,dt$ is bounded independently of x. To see this, consider

$$\begin{aligned}\int_x^{x+1}|h_{n_j}^{(\delta)}(t)|^p\,dt&=\int_x^{x+1}\left(\frac{1}{\delta}\Big|\int_t^{t+\delta}h_{n_j}(s)\,ds\Big|\right)^p dt\\&\le\int_x^{x+1}\left(\frac{1}{\delta}\int_t^{t+\delta}|h_{n_j}(s)|\,ds\right)^p dt.\end{aligned}$$

Consider now

$$\frac{1}{\delta}\int_t^{t+\delta}|h_{n_j}(s)|\,ds\le\frac{1}{\delta}\left(\int_t^{t+\delta}|h_{n_j}(s)|^p\,ds\right)^{\frac{1}{p}}\delta^{\frac{1}{q}}=\left(\frac{1}{\delta}\int_t^{t+\delta}|h_{n_j}(s)|^p\,ds\right)^{\frac{1}{p}}.$$

Hence

$$\begin{aligned}\liminf_{j\to+\infty}\int_x^{x+1}|h_{n_j}^{(\delta)}(t)|^p\,dt&\le\liminf_{j\to+\infty}\int_x^{x+1}\frac{1}{\delta}\int_t^{t+\delta}|h_{n_j}(s)|^p\,ds\,dt\\&\le\liminf_{j\to+\infty}\left(\int_x^{x+1+\delta}|h_{n_j}(s)|^p\,ds\right)\left(\frac{1}{\delta}\int_{s-\delta}^{s}dt\right)\\&=\liminf_{j\to+\infty}\int_x^{x+1+\delta}|h_{n_j}(s)|^p\,ds\\&\le\liminf_{j\to+\infty}\int_x^{x+2}|h_{n_j}(s)|^p\,ds\\&\le 2B^p<+\infty,\end{aligned}$$

and so $\int_x^{x+1} |f_o^{(\delta)}(t)|^p \, dt \le 2B^p$. Applying once more Fatou's lemma, we get

$$\int_x^{x+1} |f_o(t)|^p \, dt \le \liminf_{\delta\to 0} \int_x^{x+1} |f_o^{(\delta)}(t)|^p \, dt \le 2B^p.$$

Hence

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |f_o(t)|^p \, dt \right)^{\frac{1}{p}} \le 2^{\frac{1}{p}} B < +\infty$$

and $f_o \in \mathrm{BS}^p(\mathbb{R})$.

We have shown that there exists $f_o \in \mathrm{BS}^p(\mathbb{R})$ such that

$$\lim_{j\to+\infty} \int_a^b \varphi(x) h_{n_j}(x) \, dx = \lim_{j\to+\infty} \int_a^b \varphi(x)(f * \phi_{\frac{1}{n_j}})(x) \, dx = \int_a^b \varphi(x) f_o(x) \, dx,$$

for any measurable bounded function φ and for any bounded interval $(a, b) \subset \mathbb{R}$.

On the other hand $f * \phi_{\frac{1}{n_j}} \to f$ as $j \to +\infty$ in the sense of distributions, and so $f = f_o \in \mathrm{BS}^p(\mathbb{R})$. $\square$

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