

Two-Scale Convergence of First-Order Operators

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Abstract. Nguetseng's notion of *two-scale convergence* and some of its main properties are first shortly reviewed. The (weak) two-scale limit of the gradient of bounded sequences of $W^{1,p}(\mathbb{R}^N)$ is then studied: if $u_\varepsilon \rightarrow u$ weakly in $W^{1,p}(\mathbb{R}^N)$, a sequence $\{u_{1\varepsilon}\}$ is constructed such that $u_{1\varepsilon}(x) \rightarrow u_1(x, y)$ and $\nabla u_\varepsilon(x) \rightarrow \nabla u(x) + \nabla_y u_1(x, y)$ weakly two-scale. Analogous constructions are introduced for the weak two-scale limit of derivatives in the spaces $W^{1,p}(\mathbb{R}^N)^N$, $L^2_{\text{rot}}(\mathbb{R}^3)^3$, $L^2_{\text{div}}(\mathbb{R}^N)^N$, $L^2_{\text{div}}(\mathbb{R}^N)^{N^2}$. The application to the two-scale limit of some classical equations of electromagnetism and continuum mechanics is outlined. These results are then applied to the homogenization of quasilinear elliptic equations like $\nabla \times [A(u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot \nabla \times u_\varepsilon] = f$.

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Introduction

The following notion was introduced by Nguetseng [27], and then developed by Allaire [1] and others: a bounded sequence $\{u_\varepsilon\}$ of $L^2(\mathbb{R}^N)$ is said (*weakly*) *two-scale convergent* to $u \in L^2(\mathbb{R}^N \times Y)$ (with $Y := [0, 1]^N$) if and only if

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} u_\varepsilon(x) \psi(x, \frac{x}{\varepsilon}) dx = \iint_{\mathbb{R}^N \times Y} u(x, y) \psi(x, y) dx dy, \quad (1)$$

for any smooth function $\psi : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ that is Y -periodic w.r.t. the second argument and such that $\psi \in L^2(\mathbb{R}^N \times Y)$. (This is easily extended to L^p for any $p \in [1, +\infty[$.) It should be noticed that the converging functions u_ε only depend on x , whereas the limit function u may also depend on the variable y ; x and y may respectively be regarded as *coarse-scale* and *fine-scale* variables. Two-scale convergence can thus account for occurrence of a fine-scale periodic

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structure, and indeed has been applied to a number of homogenization problems, see e.g. [1, 4, 8, 12, 15, 18, 28, 42]. For periodic homogenization, this method is indeed alternative to the classical *energy method* (or *method of oscillating test functions*) due to Tartar, see e.g. [2, 6, 14, 19, 23, 26], [29]–[33].

In this paper we deal with the two-scale convergence of some first-order linear differential operators (with constant coefficients). First we briefly review some basic definitions and results, along the lines of [13, 34, 35] and of other works. We then revisit a known result: if $\{u_\varepsilon\}$ is a sequence of $W^{1,p}(\mathbb{R}^N)$ ($1 < p < +\infty$) and $u_\varepsilon \rightarrow u$ weakly in this space, then there exists a (possibly nonunique) function $u_1 \in L^p(\mathbb{R}^N; W^{1,p}(\mathcal{Y}))$ (\mathcal{Y} being the N -dimensional unit torus) such that, up to a subsequence,

$$\nabla u_\varepsilon \rightarrow \nabla u + \nabla_y u_1 \quad \text{weakly two-scale in } L^p(\mathbb{R}^N \times \mathcal{Y})^N. \quad (2)$$

This result was already proved in [27] and [1]. Here we investigate how u_1 may be related to the sequence $\{u_\varepsilon\}$: in Theorem 2.2 we construct a sequence $\{u_{1\varepsilon}\}$ such that, up to a subsequence,

$$\begin{aligned} u_{1\varepsilon} &\rightarrow u_1 && \text{weakly two-scale in } L^p(\mathbb{R}^N \times \mathcal{Y}) \\ \varepsilon \nabla u_{1\varepsilon} &\rightarrow \nabla_y u_1 && \text{weakly two-scale in } L^p(\mathbb{R}^N \times \mathcal{Y})^N. \end{aligned} \quad (3)$$

For any ε , the function $u_{1\varepsilon}$ solves an elliptic periodic problem on each cell $\varepsilon(m + Y)$, with $m \in \mathbb{Z}^N$. This result may be compared with Theorem 1 of [13], which outlines a different approximation of the function u_1 . In addition we show that conversely for any pair (u, u_1) as above there exists a sequence $\{u_\varepsilon\}$ for which (2) is fulfilled, here with strong two-scale convergence. The stated regularity of u and u_1 may thus be regarded as optimal.

There are analogies between our construction of $u_{1\varepsilon}$ and the so-called *cell problem*, that is at the basis of the homogenization of the elliptic equation $-\sum_{i,j=1}^N a_{ij}(\frac{x}{\varepsilon}) D_i D_j u_\varepsilon = f$ in \mathbb{R}^N ($D_j := \frac{\partial}{\partial x_j}$), the matrix function $\{a_{ij}\}$ being Y -periodic and positive-definite. Elliptic equations like this have extensively been studied in a large literature, via asymptotic expansions and Tartar's energy method; see e.g. the abovementioned monographs. There is however a major difference between the two settings: the classical cell problem involves the elliptic operator, whereas here it only depends on the sequence $\{u_\varepsilon\}$, and is not related to any specific problem.

The main aim of this paper is to derive analogous results for other first-order linear differential operators (with constant coefficients), by extending the construction of Section 2. Arguments relative to different operators exhibit similarities but also several differences; we then develop them in detail. In Section 3 for any fixed $p \in]1, +\infty[$ we deal with sequences of the space

$$\{v \in L^p(\mathbb{R}^N)^N : (\nabla^s u)_{ij} := \frac{1}{2}(D_j v_i + D_i v_j) \in L^p(\mathbb{R}^N), i, j = 1, \dots, N\},$$

that we equip with the graph norm. By the classical Korn inequality this space coincides with $W^{1,p}(\mathbb{R}^N)^N$. Given a weakly convergent sequence in this space, we construct a function $u_1 \in L^p(\mathbb{R}^N; W^{1,p}(\mathcal{Y})^N)$, a two-scale approximating sequence $\{u_{1\varepsilon}\}$, and derive statements like (2) and (3), with the symmetrized gradient, ∇^s , in place of the gradient, ∇ . A converse statement for strong two-scale convergence holds for this as well as for the other extensions that follow. This theorem looks prone to applications to continuum mechanics, for the linearized strain is the symmetrized gradient of the displacement.

In the remainder of this paper we assume that $p = 2$. In view of applications to electromagnetism, in Sections 4 and 5 we deal with sequences of $L^2_{\text{rot}}(\mathbb{R}^3)^3$ and of $L^2_{\text{div}}(\mathbb{R}^N)^N$ (respectively the domain of the operators curl and divergence in L^2), and derive formulas analogous to (2) and (3). Dealing with L^2_{div} we assume that N is any positive integer. On the other hand we study convergence in L^2_{rot} just for $N = 3$, which is the case of main applicative interest, for the extension to $N > 3$ would be slightly cumbersome. In Section 6 we prove similar results for sequences of the space

$$L^2_{\text{div}}(\mathbb{R}^3)^9 := \{v \in L^2(\mathbb{R}^3)^9_s : \sum_{j=1}^3 D_j v_{ij} \in L^2(\mathbb{R}^3) \ (i = 1, 2, 3)\},$$

that we equip with the graph norm (the index s restricts the space to symmetric tensors). The latter setting is of potential interest for applications to continuum mechanics, for the stress tensor is a typical element of that space, cf. [37],[38].

Despite of several analogies, Sections 4, 5, 6 differ from Sections 2, 3 in the following respect. Weak convergence in $W^{1,p}(\mathbb{R}^N)$ entails strong convergence in $L^p_{\text{loc}}(\mathbb{R}^N)$; hence the weak two-scale limit does not depend on y and coincides with the strong one-scale limit. On the other hand, weak convergence in $L^2_{\text{rot}}(\mathbb{R}^3)^3$ does not entail strong convergence in $L^2_{\text{loc}}(\mathbb{R}^3)^3$; hence the weak two-scale limit u may also depend on the fine-scale variable y , and thus differ from the weak one-scale limit $\bar{u} := \int_{\mathcal{Y}} u(\cdot, y) dy$. This raises the question whether, assuming that $u_\varepsilon \rightarrow u$ weakly two-scale in $L^2(\mathbb{R}^3 \times \mathcal{Y})^3$, the weak two-scale limit of $\nabla \times u_\varepsilon$ ($\nabla \times := \text{curl}$) should read

$$\nabla \times \bar{u}(x) + \nabla_y \times u_1(x, y) \quad \text{or} \quad \nabla_x \times u(x, y) + \nabla_y \times \tilde{u}_1(x, y), \quad (4)$$

We derive the former expression; this looks fairly natural, for it confines any dependence on the fine-scale variable y to the correcting term $\nabla_y \times u_1$. Anyway this is compatible with the second expression, which indeed has recently been studied by Wellander and Kristensson [41] and used in [7] (analogous results might be proved for the other first-order differential operators we deal with in this paper). A similar remark applies to the results of Sections 5 and 6, dealing with the divergence operator; also in this case the formula that we prove includes the derivatives of \bar{u} , and the two-scale limit reads $\nabla \cdot \bar{u} + \nabla_y \cdot u_1$ ($\nabla \cdot := \text{div}$). It may also be noticed that in all of these cases (for $p = 2$) we represent the weak

two-scale limit of the differential operator as the sum of two orthogonal terms of $L^2(\mathbb{R}^N \times \mathcal{Y})^M$ (M depending on the specific setting).

In Section 7 we illustrate how these results may be applied to the study of the two-scale limit of the classical equations of Ampère and Gauss of electromagnetism, and of the balance of momentum of continuum mechanics.

The above developments may be used in the homogenization of several second-order elliptic equations, for the corresponding a priori estimates are typically expressed in terms of first-order operators like gradient, curl or divergence. In Section 8 we deal with the asymptotic behaviour of the system

$$\nabla \times [A(u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot (\nabla \times u_\varepsilon + f)] = 0, \quad \nabla \cdot u_\varepsilon = 0 \quad \text{in } \Omega, \quad (5)$$

coupled with homogeneous Dirichlet conditions. This may represent equilibrium in a heterogeneous and anisotropic electric conductor with a magnetic-field-dependent resistance A . Here we prove convergence to a two-scale solution. Analogous conclusions might be reached for instance for the equation $-\nabla \cdot [A(u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot \nabla u_\varepsilon] = f$. The homogenization of quasilinear elliptic equations in divergence form has been studied in a large number of papers, see e.g. [9, 16] and references therein; two-scale convergence has been applied e.g. in [1, 10, 13, 27]. But apparently so far less attention has been paid to equations in curl form, and to corresponding variational problems.

This work is part of a research on two-scale convergence. Some of these results have been announced in [34]; see also [35]. This author intends to apply them to the homogenization of nonlinear partial differential equations issued from electromagnetism, phase transitions and continuum mechanics [36]–[39]. Further research will deal with the two-scale homogenization of elliptic operators.

1. Two-scale convergence of the gradient of a scalar field

Two-scale decomposition. In this section we briefly review some properties of two-scale convergence, along the lines of Nguetseng [27] and Allaire [1]. Here we use the formulation of [34, 35], that is based on a technique that has already been investigated in several papers, cf. e.g. [3, 4, 8, 12, 13, 21, 22].

We set $Y = [0, 1]^N$, denote by \mathcal{Y} the same set equipped with the topological and differential structure of the N -dimensional unit torus. (The case of a cell of the form $[0, a[\times [0, b[\times [0, c[$ might be dealt similarly.) We identify any function on \mathcal{Y} with its Y -periodic extension to \mathbb{R}^N . For any $\varepsilon > 0$ we set

$$\begin{aligned} \hat{n}(x) &:= \max\{n \in \mathbb{Z} : n \leq x\}, \quad \hat{r}(x) := x - \hat{n}(x) \in [0, 1[\quad \forall x \in \mathbb{R} \\ \mathcal{N}(x) &:= (\hat{n}(x_1), \dots, \hat{n}(x_N)) \in \mathbb{Z}^N, \quad \mathcal{R}(x) := x - \mathcal{N}(x) \in \mathcal{Y} \quad \forall x \in \mathbb{R}^N; \end{aligned} \quad (1.1)$$

this yields the *two-scale decomposition* (also named *periodic unfolding* in [13]): $x = \varepsilon[\mathcal{N}(\frac{x}{\varepsilon}) + \mathcal{R}(\frac{x}{\varepsilon})]$ for any $x \in \mathbb{R}^N$; $\varepsilon\mathcal{N}(\frac{x}{\varepsilon})$ and $\mathcal{R}(\frac{x}{\varepsilon})$ may respectively be regarded as coarse-scale and fine-scale variables w.r.t. the scale ε . We also set

$$S_\varepsilon(x, y) := \varepsilon\mathcal{N}(\frac{x}{\varepsilon}) + \varepsilon y \quad \forall (x, y) \in \mathbb{R}^N \times \mathcal{Y}, \quad \forall \varepsilon > 0. \quad (1.2)$$

The next statement can easily be proved via a variable transformation. Let us first denote by $\mathcal{L}(\mathbb{R}^N)$ ($\mathcal{B}(\mathbb{R}^N)$, resp.) the σ -algebra of Lebesgue- (Borel-, resp.) measurable subsets of \mathbb{R}^N , and define $\mathcal{L}(\mathcal{Y})$ and $\mathcal{B}(\mathcal{Y})$ similarly.

Lemma 1.1 ([35]). *Let $f : \mathbb{R}^N \times \mathcal{Y} \rightarrow \mathbb{R}$ be such that*

$$\begin{aligned} f \text{ is measurable either w.r.t. the } \sigma\text{-algebra generated by } \mathcal{B}(\mathbb{R}^N) \times \mathcal{L}(\mathcal{Y}), \\ \text{or w.r.t. that generated by } \mathcal{L}(\mathbb{R}^N) \times \mathcal{B}(\mathcal{Y}), \end{aligned} \quad (1.3)$$

$$\begin{aligned} \text{either } \|f\|_{L^\infty(\mathbb{R}^N)} \in L^1(\mathcal{Y}) \text{ and has compact support,} \\ \text{or } \|f\|_{L^\infty(\mathcal{Y})} \in L^1(\mathbb{R}^N). \end{aligned} \quad (1.4)$$

Let us extend $f(x, \cdot)$ by Y -periodicity to \mathbb{R}^N for a.a. $x \in \mathbb{R}^N$. Then, for any $\varepsilon > 0$, the functions $\mathbb{R}^N \rightarrow \mathbb{R} : x \mapsto f(x, \frac{x}{\varepsilon})$ and $\mathbb{R}^N \times \mathcal{Y} \rightarrow \mathbb{R} : (x, y) \mapsto f(S_\varepsilon(x, y), y)$ are integrable, and

$$\begin{aligned} \int_{\mathbb{R}^N} f(x, \frac{x}{\varepsilon}) dx &= \varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} f(\varepsilon[m + y], y) dy \\ &= \iint_{\mathbb{R}^N \times \mathcal{Y}} f(S_\varepsilon(x, y), y) dx dy \quad \forall \varepsilon > 0. \end{aligned} \quad (1.5)$$

For any $p \in [1, +\infty]$ and $\varepsilon > 0$, the operator $g \mapsto g \circ S_\varepsilon$ is then a (nonsurjective) linear isometry $L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N \times \mathcal{Y})$.

Two-Scale Convergence in $L^p(\mathbb{R}^N \times \mathcal{Y})$. In this paper, by ε we shall denote the generic element of an arbitrary but prescribed, vanishing sequence of positive real numbers; e.g. $\varepsilon = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$. Let $p \in [1, +\infty]$; for any sequence $\{u_\varepsilon\}$ of $L^p(\mathbb{R}^N)$ and any $u \in L^p(\mathbb{R}^N \times \mathcal{Y})$, we say that u_ε strongly *two-scale converges* to u in $L^p(\mathbb{R}^N \times \mathcal{Y})$, whenever $u_\varepsilon \circ S_\varepsilon \rightarrow u$ strongly in the latter space. We similarly define weak (weak star for $p = \infty$) two-scale convergence. We denote these convergences by $u_\varepsilon \xrightarrow{2} u$, $u_\varepsilon \rightharpoonup_2 u$, $u_\varepsilon \xrightarrow{*}_2 u$, resp., and reserve the symbols \rightarrow , \rightharpoonup , $\xrightarrow{*}$ for the ordinary strong, weak, weak star convergence (that we refer to as *one-scale convergence*). Thus for any $p \in [1, +\infty]$

$$\begin{aligned} u_\varepsilon \xrightarrow{2} u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}) &\Leftrightarrow u_\varepsilon \circ S_\varepsilon \rightarrow u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}) \\ u_\varepsilon \rightharpoonup_2 u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}) &\Leftrightarrow u_\varepsilon \circ S_\varepsilon \rightharpoonup u \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}) \\ u_\varepsilon \xrightarrow{*}_2 u \text{ in } L^\infty(\mathbb{R}^N \times \mathcal{Y}) &\Leftrightarrow u_\varepsilon \circ S_\varepsilon \xrightarrow{*} u \text{ in } L^\infty(\mathbb{R}^N \times \mathcal{Y}). \end{aligned}$$

For any domain $\Omega \subset \mathbb{R}^N$, we then define two-scale convergence in $L^p(\Omega \times \mathcal{Y})$ by extending functions to $\mathbb{R}^N \setminus \Omega$ with vanishing value. These definitions slightly extend the original ones of Nguetseng [27] and Allaire [1]; see [35]. The next result is a simple consequence of the above construction.

Proposition 1.2 ([1, 27]). *Let $p \in]1, +\infty[$. For any bounded sequence $\{u_\varepsilon\}$ of $L^p(\mathbb{R}^N)$ there exists $u \in L^p(\mathbb{R}^N \times \mathcal{Y})$ such that, up to a subsequence, $u_\varepsilon \rightharpoonup_2 u$ in $L^p(\mathbb{R}^N)$ ($u_\varepsilon \xrightarrow{*} u$ if $p = \infty$).*

In view of the next statement let us fix a radial function ρ such that $\rho \in C^1(\mathbb{R}^N)$, $\rho \geq 0$, $\rho(x) = 0$ if $|x| \geq 1$, $\int_{\mathbb{R}^N} \rho(x) dx = 1$, and set

$$\rho_\varepsilon(x) := \varepsilon^{-\frac{N}{2}} \rho(\varepsilon^{-\frac{1}{2}}x) \quad \forall x \in \mathbb{R}^N. \tag{1.6}$$

Hence $\nabla \rho_\varepsilon(x) = \varepsilon^{-\frac{N+1}{2}} (\nabla \rho)(\varepsilon^{-\frac{1}{2}}x)$ and $\rho_\varepsilon \xrightarrow{*} \delta_0$ (the Dirac measure) in $C^0(\mathbb{R}^N)'$. For any function $u = u(x, y)$ henceforth we set

$$\bar{u} := \int_{\mathcal{Y}} u(\cdot, y) dy, \quad \tilde{u} := u - \bar{u} \quad \text{a.e. in } \mathbb{R}^N \times \mathcal{Y}. \tag{1.7}$$

Proposition 1.3. *Let $p \in [1, +\infty[$ and $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbb{R}^N)$ such that $u_\varepsilon \rightharpoonup_2 u$ in $L^p(\mathbb{R}^N \times \mathcal{Y})$. Then*

$$u_\varepsilon - \int_{\mathbb{R}^N} \tilde{u}(\xi, \frac{x}{\varepsilon}) \rho_\varepsilon(x - \xi) d\xi \rightarrow \bar{u} \quad \text{in } L^p(\mathbb{R}^N). \tag{1.8}$$

Proof. By Proposition 2.13 of [35] $\int_{\mathbb{R}^N} \tilde{u}(\xi, \frac{x}{\varepsilon}) \rho_\varepsilon(x - \xi) d\xi \rightharpoonup_2 \tilde{u}$ in $L^p(\mathbb{R}^N \times \mathcal{Y})$. By the hypotheses the same then applies to the sequence of (1.8). It then suffices to notice that, for any sequence $\{w_\varepsilon\}$ in $L^p(\mathbb{R}^N)$ and any $w \in L^p(\mathbb{R}^N)$, $w_\varepsilon \rightarrow w$ in $L^p(\mathbb{R}^N)$ if and only if $w_\varepsilon \rightharpoonup_2 w$ in $L^p(\mathbb{R}^N \times \mathcal{Y})$. \square

Further Notation. By appending the index $*$ to a space of functions over \mathcal{Y} , we shall denote the subspace of functions having vanishing mean. For any function of $(x, y) \in \mathbb{R}^N \times \mathcal{Y}$, we shall denote by ∇_x (∇_y , resp.) the gradient w.r.t. to the first (second, resp.) vector argument. Finally expressions like $\nabla u v$ should be interpreted as $(\nabla u) v$ (this will make formulas look a bit lighter).

2. Two-scale convergence of the gradient of a scalar field

In this section we deal with the two-scale limit of the gradient of weakly convergent sequences of $W^{1,p}(\mathbb{R}^N)$, for any $p \in]1, +\infty[$ and $N \in \mathbb{N}$. Trivial examples show that $u_\varepsilon \rightharpoonup_2 u$ in $W^{1,p}(\mathbb{R}^N)$ does not entail $\nabla u_\varepsilon \rightharpoonup_2 \nabla_x u$ in $L^p(\mathbb{R}^N \times \mathcal{Y})^N$: in general the weak two-scale limit of the gradient need not coincide either with the weak one-scale limit of the gradient or with the gradient of the weak one-scale limit.

In view of the next statement, for any Banach space B let us denote by $\ell^p(B)$ the Banach space of p -summable functions $\mathbb{Z}^N \rightarrow B$.

Lemma 2.1 (Cell Projection). *Let $p \in]1, +\infty[$ and $\varepsilon > 0$. For any $w \in W^{1,p}(\mathbb{R}^N)$ there exists a unique $w^* = w^*(m, y) \in \ell^p(W_*^{1,p}(\mathcal{Y}))$ such that*

$$\int_{\mathcal{Y}} [\nabla w^*(m, y) - (\nabla w)(\varepsilon(m + y))] \cdot \nabla \zeta \, dy = 0 \quad (2.1)$$

$$\forall \zeta \in W^{1,p'}(\mathcal{Y}) \quad (p' := \frac{p}{p-1}), \quad \forall m \in \mathbb{Z}^N.$$

Moreover there exists a constant $C_1 > 0$ (independent of w and ε) such that

$$\varepsilon^{\frac{N}{p}} \|\nabla w^*\|_{\ell^p(L^p(\mathcal{Y})^N)} \left(= \left(\varepsilon^N \sum_{m \in \mathbb{Z}^N} \|\nabla w^*(m, \cdot)\|_{L^p(\mathcal{Y})^N}^p \right)^{\frac{1}{p}} \right) \quad (2.2)$$

$$\leq \|\nabla w\|_{L^p(\mathbb{R}^N)^N}$$

$$\varepsilon^{\frac{N}{p}} \|w^*\|_{\ell^p(L^p(\mathcal{Y}))} \leq C_1 \|\nabla w\|_{L^p(\mathbb{R}^N)^N}. \quad (2.3)$$

Notice that $(\nabla w)(\varepsilon(m + y)) := \nabla w(x)|_{x=\varepsilon(m+y)} = \varepsilon^{-1} \nabla_y [w(\varepsilon(m + y))]$. Here $\nabla w^*(m, \cdot)$ (∇w , resp.) is the gradient in the sense of $\mathcal{D}'(\mathcal{Y})^N$ (of $\mathcal{D}'(\mathbb{R}^N)^N$, resp.). Having identified Y to \mathcal{Y} , $L^p(Y)^N = L^p(\mathcal{Y})^N$, but $W^{1,p}(Y) \neq W_*^{1,p}(\mathcal{Y})$.

Leaving aside translations and rescalings, the equation $\int_{\mathcal{Y}} (\nabla z - \nabla v) \cdot \nabla \zeta \, dy = 0$ (for any $\zeta \in W^{1,p'}(\mathcal{Y})$) defines a continuous projection (namely a continuous, linear and idempotent operator) $L : W^{1,p}(Y) \mapsto W_*^{1,p}(\mathcal{Y}) : v \mapsto z$.

Proof. Let us fix any $m \in \mathbb{Z}^N$ and set $g := (\nabla w)(\varepsilon(m + \cdot)) \in L^p(Y)^N$. For $p = 2$ the equation (2.1) is equivalent to minimizing the functional

$$H_*^1(\mathcal{Y}) \rightarrow \mathbb{R} : v \mapsto \int_{\mathcal{Y}} \left(\frac{1}{2} |\nabla v|^2 - g \cdot \nabla v \right) dy;$$

existence and uniqueness of $w_m^* := w^*(m, \cdot)$ are then straightforward. For $p \neq 2$ this setting does not fit the standard framework because of periodicity; we then provide a direct argument for existence and derive a uniform estimate.

If $g \in \mathcal{D}(\mathcal{Y})$, then there exists a unique $w_m^* \in H_*^1(\mathcal{Y})$ such that

$$\int_{\mathcal{Y}} (\nabla w_m^* - g) \cdot \nabla v \, dy = 0 \quad \forall v \in H^1(\mathcal{Y}); \quad (2.4)$$

actually $w_m^* \in \mathcal{D}_*(\mathcal{Y})$ by a known result of local regularity and by the compactness of \mathcal{Y} . Similarly there exists a unique $\psi \in \mathcal{D}_*(\mathcal{Y})$ such that

$$\int_{\mathcal{Y}} (\nabla \psi - |\nabla w_m^*|^{p-2} \nabla w_m^*) \cdot \nabla v \, dy = 0 \quad \forall v \in H^1(\mathcal{Y}).$$

By taking first $v = w_m^*$, then $v = \nabla w(\varepsilon(m + \cdot))$ in the latter formula, and then $v = \psi$ in (2.4), we get

$$\begin{aligned} \int_{\mathcal{Y}} \nabla \psi \cdot \nabla w_m^* dy &= \int_{\mathcal{Y}} |\nabla w_m^*|^p dy \\ \int_{\mathcal{Y}} \nabla \psi \cdot g dy &= \int_{\mathcal{Y}} |\nabla w_m^*|^{p-2} \nabla w_m^* \cdot g dy \\ \int_{\mathcal{Y}} \nabla w_m^* \cdot \nabla \psi dy &= \int_{\mathcal{Y}} g \cdot \nabla \psi dy. \end{aligned}$$

The three latter formulas yield

$$\int_{\mathcal{Y}} |\nabla w_m^*|^p dy = \int_{\mathcal{Y}} |\nabla w_m^*|^{p-2} \nabla w_m^* \cdot g dy \leq \|\nabla w_m^*\|_{L^p(\mathcal{Y})^N}^{p-1} \|g\|_{L^p(\mathcal{Y})^N},$$

whence

$$\|\nabla w_m^*\|_{L^p(\mathcal{Y})^N} \leq \|g\|_{L^p(\mathcal{Y})^N} = \|(\nabla w)(\varepsilon(m + \cdot))\|_{L^p(\mathcal{Y})^N} \quad \forall m \in \mathbb{Z}^N. \quad (2.5)$$

The restriction to $\mathcal{D}(\mathcal{Y})$ of the linear mapping $w(\varepsilon(m + \cdot)) \mapsto w_m^*$ is thus non-expansive w.r.t. the $W_*^{1,p}$ -seminorm. It can then be extended by density to a unique mapping $W^{1,p}(\mathcal{Y}) \rightarrow W_*^{1,p}(\mathcal{Y})$. By (2.5) Lemma 1.1 yields

$$\begin{aligned} \varepsilon^{\frac{N}{p}} \|\nabla w^*\|_{\ell^p(L^p(\mathcal{Y})^N)} &= \left(\varepsilon^N \sum_{m \in \mathbb{Z}^N} \|\nabla w^*(m, \cdot)\|_{L^p(\mathcal{Y})^N}^p \right)^{\frac{1}{p}} \\ &\leq \left(\varepsilon^N \sum_{m \in \mathbb{Z}^N} \|(\nabla w)(\varepsilon(m + \cdot))\|_{L^p(\mathcal{Y})^N}^p \right)^{\frac{1}{p}} \\ &= \|\nabla w\|_{L^p(\mathbb{R}^N)^N}, \end{aligned} \quad (2.6)$$

namely (2.2). By the classical Poincaré inequality, there exists a constant $C_1 > 0$ independent of w and ε such that $\int_{\mathcal{Y}} |w_m^*|^p dy \leq C_1 \int_{\mathcal{Y}} |\nabla w_m^*|^p dy$. (2.5) then yields (2.3). The uniqueness of the solution of (2.1) follows from (2.3). \square

The first part of the next result may be compared with analogous theorems of [1, 27] for $p = 2$; more recently a different argument based on the approximation of the function u_1 (cf. (2.10)) was outlined in [13]. Here the main novelty is that we construct an approximation of that function; an analogous procedure will be applied to other first-order differential operators in the next sections. We also show that conversely any expression of the form $\nabla u(x) + \nabla_y u_1(x, y)$ is the two-scale limit of ∇u_ε for some bounded sequence $\{u_\varepsilon\}$ of $W^{1,p}(\mathbb{R}^N)$.

Theorem 2.2. (i) *Let $p \in]1, +\infty[$ and $\{u_\varepsilon\}$ be a sequence such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$. For any ε there exists a unique $u_{1\varepsilon}^* \in \ell^p(W_*^{1,p}(\mathcal{Y}))$ such that*

$$\begin{aligned} \int_{\mathcal{Y}} [\nabla u_{1\varepsilon}^*(m, y) - (\nabla u_\varepsilon)(\varepsilon(m + y))] \cdot \nabla \zeta dy &= 0 \\ \forall \zeta \in W^{1,p'}(\mathcal{Y}), \forall m \in \mathbb{Z}^N. \end{aligned} \quad (2.7)$$

Moreover there exists $u_1 \in L^p(\mathbb{R}^N; W_*^{1,p}(\mathcal{Y}))$ such that setting

$$\begin{aligned} u_{1\varepsilon}(\varepsilon(m+y)) &:= u_{1\varepsilon}^*(m, y) \\ z_\varepsilon(\varepsilon(m+y)) &:= \nabla u_{1\varepsilon}^*(m, y) \end{aligned} \quad \text{for a.a. } y \in Y, \forall m \in \mathbb{Z}^N, \quad (2.8)$$

as $\varepsilon \rightarrow 0$ along a suitable subsequence,

$$u_{1\varepsilon} \xrightarrow{2} u_1 \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y}), \quad z_\varepsilon \xrightarrow{2} \nabla_y u_1 \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y})^N. \quad (2.9)$$

This entails that, as $\varepsilon \rightarrow 0$ along the extracted subsequence,

$$\nabla u_\varepsilon \xrightarrow{2} \nabla u + \nabla_y u_1 \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y})^N. \quad (2.10)$$

(ii) Conversely, for any $u \in W^{1,p}(\mathbb{R}^N)$ and any $u_1 \in L^p(\mathbb{R}^N; W_*^{1,p}(\mathcal{Y}))$ there exists a sequence $\{u_\varepsilon\}$ of $W^{1,p}(\mathbb{R}^N)$ such that

$$u_\varepsilon \rightarrow u \text{ in } L^p(\mathbb{R}^N), \quad \nabla u_\varepsilon \xrightarrow{2} \nabla u + \nabla_y u_1 \text{ in } L^p(\mathbb{R}^N \times \mathcal{Y})^N. \quad (2.11)$$

(Notice that z_ε need not coincide with $\nabla u_{1\varepsilon}$ a.e. in \mathbb{R}^N .)

Proof. (i) By Lemma 2.1, $u_{1\varepsilon}^*$ exists and

$$\|z_\varepsilon\|_{L^p(\mathbb{R}^N)^N}, \|u_{1\varepsilon}\|_{L^p(\mathbb{R}^N)} \leq \text{Constant (independent of } \varepsilon).$$

By Proposition 1.2 then there exists $u_1 \in L^p(\mathbb{R}^N \times \mathcal{Y})$ such that, as $\varepsilon \rightarrow 0$ along a suitable subsequence, $u_{1\varepsilon} \xrightarrow{2} u_1$ in $L^p(\mathbb{R}^N \times \mathcal{Y})$; hence $u_{1\varepsilon} \xrightarrow{2} u_1$ in $L^p(\mathbb{R}^N \times \mathcal{Y})$.

For any $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and $\psi \in \mathcal{D}(\mathcal{Y})^N$ we have

$$\begin{aligned} \int_{\mathbb{R}^N} z_\varepsilon(x) \cdot \psi\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) dx &= \varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} \nabla u_{1\varepsilon}^*(m, y) \cdot \psi(y) \varphi(\varepsilon m) dy \\ &= -\varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} u_{1\varepsilon}^*(m, y) \nabla \cdot \psi(y) \varphi(\varepsilon m) dy \\ &= -\varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} u_{1\varepsilon}(\varepsilon(m+y)) \nabla \cdot \psi(y) \varphi(\varepsilon m) dy \\ &\rightarrow - \iint_{\mathbb{R}^N \times \mathcal{Y}} u_1(x, y) \nabla \cdot \psi(y) \varphi(x) dx dy. \end{aligned}$$

As z_ε is uniformly bounded in $L^p(\mathbb{R}^N)$ and $\varphi(\varepsilon \mathcal{N}(\frac{x}{\varepsilon})) \rightarrow \varphi(x)$ uniformly for $x \in \mathbb{R}^N$, the latter convergence also holds if in the first integral $\varphi(\varepsilon \mathcal{N}(\frac{x}{\varepsilon}))$ is replaced by $\varphi(x)$. Thus $u_1 \in L^p(\mathbb{R}^N; W_*^{1,p}(\mathcal{Y}))$ and (2.9) is fulfilled.

(ii) In view of the derivation of (2.10), let us still fix any $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and any $\psi \in \mathcal{D}(\mathcal{Y})^N$, and notice that there exists a (unique) $\zeta \in H_*^1(\mathcal{Y})$ such that

$$\int_{\mathcal{Y}} \nabla \zeta \cdot \nabla v dy = \int_{\mathcal{Y}} \psi \cdot \nabla v dy \quad \forall v \in H_*^1(\mathcal{Y}).$$

Hence (setting $\nabla \cdot := \operatorname{div}$) $\Delta \zeta = \nabla \cdot \psi + C$ in \mathcal{Y} for some constant $C \in \mathbb{R}$; integrating over \mathcal{Y} , by the Y -periodicity of ζ and ψ we get $C = 0$. By the regularity of ψ we infer that $\zeta \in \mathcal{D}(\mathcal{Y})$. By setting $\tilde{\psi} := \nabla \zeta$ and $\hat{\psi} := \psi - \tilde{\psi}$ in \mathcal{Y} , we thus have

$$\tilde{\psi}, \hat{\psi} \in \mathcal{D}(\mathcal{Y})^N, \quad \nabla \cdot \tilde{\psi} = \nabla \cdot \psi, \quad \nabla \cdot \hat{\psi} = 0 \quad \text{in } \mathcal{Y}.$$

By Lemma 1.1, (2.7) and (2.9) we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla u_\varepsilon(x) \cdot \tilde{\psi}\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) dx \\ &= \int_{\mathbb{R}^N} dx \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) \int_{\mathcal{Y}} (\nabla u_\varepsilon)\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right) + \varepsilon y\right) \cdot \nabla \zeta(y) dy \\ &= \int_{\mathbb{R}^N} dx \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) \int_{\mathcal{Y}} \nabla u_{1\varepsilon}^*\left(\mathcal{N}\left(\frac{x}{\varepsilon}\right), y\right) \cdot \nabla \zeta(y) dy \\ &\rightarrow \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla_y u_1(x, y) \cdot \tilde{\psi}(y) \varphi(x) dx dy. \end{aligned} \tag{2.12}$$

Moreover, as $\nabla \cdot \hat{\psi} = 0$ a.e. in \mathcal{Y} and $\int_{\mathcal{Y}} \tilde{\psi}(y) dy = \int_{\mathcal{Y}} \nabla \zeta(y) dy = 0$,

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla u_\varepsilon(x) \cdot \hat{\psi}\left(\frac{x}{\varepsilon}\right) \varphi(x) dx = - \int_{\mathbb{R}^N} u_\varepsilon(x) \hat{\psi}\left(\frac{x}{\varepsilon}\right) \cdot \nabla \varphi(x) dx \rightarrow \\ & - \iint_{\mathbb{R}^N \times \mathcal{Y}} u(x) \hat{\psi}(y) \cdot \nabla \varphi(x) dx dy = \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla u(x) \cdot \hat{\psi}(y) \varphi(x) dx dy \\ & \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla_y u_1(x, y) \cdot \hat{\psi}(y) \varphi(x) dx dy = - \iint_{\mathbb{R}^N \times \mathcal{Y}} u_1(x, y) \nabla \cdot \hat{\psi}(y) \varphi(x) dx dy = 0, \\ & \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla u(x) \cdot \tilde{\psi}(y) \varphi(x) dx dy = \int_{\mathbb{R}^N} \nabla u(x) \varphi(x) dx \cdot \int_{\mathcal{Y}} \tilde{\psi}(y) dy = 0. \end{aligned}$$

As $\|\nabla u_\varepsilon\|_{L^p(\mathbb{R}^N)^N}$ is uniformly bounded and $\varphi(\varepsilon \mathcal{N}(\frac{x}{\varepsilon})) \rightarrow \varphi(x)$ uniformly in \mathbb{R}^N , cf. (1.1),

$$\int_{\mathbb{R}^N} \nabla u_\varepsilon(x) \cdot \psi\left(\frac{x}{\varepsilon}\right) \varphi(x) dx - \int_{\mathbb{R}^N} \nabla u_\varepsilon(x) \cdot \psi\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) dx \rightarrow 0;$$

a similar statement holds for the first member of (2.12). Assembling the five latter formulas, we get

$$\int_{\mathbb{R}^N} \nabla u_\varepsilon(x) \cdot \psi\left(\frac{x}{\varepsilon}\right) \varphi(x) dx \rightarrow \iint_{\mathbb{R}^N \times \mathcal{Y}} [\nabla u(x) + \nabla_y u_1(x, y)] \cdot \psi(y) \varphi(x) dx dy,$$

that is (2.10).

(iii) Let us now come to part (ii) of the thesis. Let us fix any $u \in W^{1,p}(\mathbb{R}^N)$ and any $u_1 \in L^p(\mathbb{R}^N; W_*^{1,p}(\mathcal{Y}))$, define ρ as in (1.6), and set

$$u_\varepsilon(x) := u(x) + \varepsilon \int_{\mathbb{R}^N} u_1(\xi, \frac{x}{\varepsilon}) \rho_\varepsilon(x - \xi) d\xi \quad \text{for a.a. } x \in \mathbb{R}^N; \quad (2.13)$$

hence $u_\varepsilon \rightarrow u$ in $L^p(\mathbb{R}^N)$ and

$$\nabla u_\varepsilon(x) = \nabla u(x) + \varepsilon \int_{\mathbb{R}^N} u_1(\xi, \frac{x}{\varepsilon}) \nabla \rho_\varepsilon(x - \xi) d\xi + \int_{\mathbb{R}^N} \nabla_y u_1(\xi, \frac{x}{\varepsilon}) \rho_\varepsilon(x - \xi) d\xi.$$

We claim that

$$\varepsilon \int_{\mathbb{R}^N} u_1(\xi, \frac{x}{\varepsilon}) \nabla \rho_\varepsilon(x - \xi) d\xi \rightarrow 0 \quad (2.14)$$

$$g_\varepsilon(x) := \int_{\mathbb{R}^N} \nabla_y u_1(\xi, \frac{x}{\varepsilon}) \rho_\varepsilon(x - \xi) d\xi \xrightarrow{\frac{1}{2}} \nabla_y u_1 \quad \text{in } L^p(\mathbb{R}^N \times \mathcal{Y})^N, \quad (2.15)$$

which suffices to infer (2.11). As $\varepsilon^{\frac{1}{2}} \int_{\mathbb{R}^N} \nabla \rho_\varepsilon(x) dx$ is independent of ε , (2.14) is easily checked. In view of proving (2.15), notice that by Lemma 1.1

$$\begin{aligned} & |g_\varepsilon(S_\varepsilon(x, y)) - \nabla_y u_1(S_\varepsilon(x, y), y)|^p \\ &= \left| \int_{\mathbb{R}^N} \nabla_y u_1(\xi, y) \rho_\varepsilon(S_\varepsilon(x, y) - \xi) d\xi - \nabla_y u_1(S_\varepsilon(x, y), y) \right|^p \quad \forall \varepsilon, \end{aligned}$$

and as $\varepsilon \rightarrow 0$ the latter function vanishes a.e. in $\mathbb{R}^N \times \mathcal{Y}$, for $S_\varepsilon(x, y) \rightarrow x$ uniformly in $\mathbb{R}^N \times \mathcal{Y}$ by (1.2). Thus $g_\varepsilon \xrightarrow{\frac{1}{2}} \nabla_y u_1$ a.e. in $\mathbb{R}^N \times \mathcal{Y}$. By the equi- L^p -integrability of g_ε , and by Lemma 2.3 below, we then infer (2.15). \square

Lemma 2.3 (Two-Scale Vitali's Theorem [35]). *Let $p \in [1, +\infty[$, $\{u_\varepsilon\}$ be a sequence in $L^p(\mathbb{R}^N)$ such that $\sup_\varepsilon \int_{\mathbb{R}^N \setminus B(0, R)} |u_\varepsilon(x)|^p dx \rightarrow 0$ as $R \rightarrow +\infty$, and $u_\varepsilon \xrightarrow{\frac{1}{2}} u$ a.e. in $\mathbb{R}^N \times \mathcal{Y}$. Then*

$$u \in L^p(\mathbb{R}^N \times \mathcal{Y}), \quad u_\varepsilon \xrightarrow{\frac{1}{2}} u \quad \text{in } L^p(\mathbb{R}^N \times \mathcal{Y}) \quad (2.16)$$

iff $\{|u_\varepsilon|^p\}$ is equi-integrable, in the sense that $\lim_{n \rightarrow \infty} \sup_\varepsilon \int_{A_n} |u_\varepsilon(x)|^p dx = 0$ for any sequence $\{A_n\}$ of measurable subsets of \mathbb{R}^N such that $A_n \searrow \emptyset$.

Remarks. (i) Trivial counterexamples show that the function u_1 need not be uniquely determined by the sequence $\{u_\varepsilon\}$.

(ii) Denoting the weak one-scale (two-scale, resp.) limit by $\lim_{\varepsilon \rightarrow 0}^{(1)}$ (by $\lim_{\varepsilon \rightarrow 0}^{(2)}$, resp.), (2.10) also reads

$$\lim_{\varepsilon \rightarrow 0}^{(2)} \nabla u_\varepsilon = \lim_{\varepsilon \rightarrow 0}^{(1)} \nabla u_\varepsilon + \nabla_y u_1 (= \nabla \lim_{\varepsilon \rightarrow 0}^{(1)} u_\varepsilon + \nabla_y u_1) \quad \text{a.e. in } \mathbb{R}^N \times \mathcal{Y}. \quad (2.17)$$

This illustrates the relation between the one-scale and two-scale limit of the gradient. For $p = 2$ the decomposition (2.17) is orthogonal in $L^2(\mathbb{R}^N \times \mathcal{Y})^N$.

For $N = 1$ the proof of Theorem 2.2 might be simplified, by decomposing $u_\varepsilon \rightarrow u$ into its piecewise linear interpolate $\{u_\varepsilon(m\varepsilon) : m \in \mathbb{Z}\}$ plus a remainder.

By the argument of Theorem 2.2 one can also study the limit of sequences $\{u_\varepsilon\}$ that are bounded in $L^p(\mathbb{R}^N)$ jointly with a fixed directional derivative, $\{D_j u_\varepsilon\}$ say. In this case the variables x_i with $i \neq j$ are reduced to parameters.

3. Two-scale convergence of the symmetrized gradient of a vector field

In this section we assume that $N \in \mathbb{N}$, $p \in]1, +\infty[$, and define the *symmetrized gradient*

$$(\nabla^s v)_{ij} := \frac{1}{2}(D_j v_i + D_i v_j) \quad \text{for } i, j = 1, \dots, N, \quad \forall v \in W^{1,p}(\mathbb{R}^N)^N. \quad (3.1)$$

Throughout this paper we shall label spaces of symmetric tensors by the index “ s ”. We also denote by “ \cdot ” the scalar product in \mathbb{R}^{N^2} (i.e. the contraction w.r.t. two indices); thus $A:B := \sum_{i,j=1}^N a_{ij}b_{ij}$ for any $A := \{a_{ij}\}, B := \{b_{ij}\} \in \mathbb{R}^{N^2}$.

The classical Korn inequality (cf. e.g. [17, Sect. VII.2]) can be extended to any N . Thus $\{v \in L^p(\mathcal{Y})^N : \nabla^s v \in L^p(\mathcal{Y})^{N^2}\}$ equipped with the graph norm coincides with $W^{1,p}(\mathcal{Y})^N$. In the Poincaré inequality ∇v may be replaced by $\nabla^s v$, and $W_*^{1,p}(\mathcal{Y})^N$ may be equipped with the equivalent norm $\|\nabla^s v\|_{L^p(\mathcal{Y})^{N^2}}$.

Lemma 3.1 (Cell Projection). *Let $p \in]1, +\infty[$ and $\varepsilon > 0$. For any $w \in W^{1,p}(\mathbb{R}^N)^N$ there exists a unique $w^* = w^*(m, y) \in \ell^p(W_*^{1,p}(\mathcal{Y})^N)$ such that*

$$\begin{aligned} \int_{\mathcal{Y}} [\nabla^s w^*(m, y) - (\nabla^s w)(\varepsilon(m + y))] : \nabla^s \zeta \, dy &= 0 \\ \forall \zeta \in W^{1,p'}(\mathcal{Y})^N \quad (p' := \frac{p}{p-1}), \quad \forall m \in \mathbb{Z}^N. \end{aligned} \quad (3.2)$$

Moreover there exists a constant $C_2 > 0$ (independent of w and ε) such that

$$\begin{aligned} \varepsilon^{\frac{N}{p}} \|\nabla^s w^*\|_{\ell^p(L^p(\mathcal{Y})^{N^2})} &= \left(\varepsilon^N \sum_{m \in \mathbb{Z}^N} \|\nabla^s w^*(m, \cdot)\|_{L^p(\mathcal{Y})^{N^2}}^p \right)^{\frac{1}{p}} \\ &\leq \|\nabla^s w\|_{L^p(\mathbb{R}^N)^{N^2}} \end{aligned} \quad (3.3)$$

$$\varepsilon^{\frac{N}{p}} \|w^*\|_{\ell^p(L^p(\mathcal{Y})^N)} \leq C_2 \|\nabla^s w\|_{L^p(\mathbb{R}^N)^{N^2}}. \quad (3.4)$$

The argument mimics that of Lemma 2.1, and is omitted. Here the Poincaré inequality is replaced by the Korn inequality.

Theorem 3.2. (i) Let $\{u_\varepsilon\}$ be a sequence such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)^N$. For any ε there exists a unique $u_{1\varepsilon}^* \in \ell^p(W_*^{1,p}(\mathcal{Y})^N)$ such that

$$\int_{\mathcal{Y}} [\nabla^s u_{1\varepsilon}^*(m, y) - (\nabla^s u_\varepsilon)(\varepsilon(m + y))] \cdot \nabla^s \zeta \, dy = 0 \quad (3.5)$$

$$\forall \zeta \in W^{1,p'}(\mathcal{Y})^N, \forall m \in \mathbb{Z}^N.$$

Moreover there exists $u_1 \in L^p(\mathbb{R}^N; W_*^{1,p}(\mathcal{Y})^N)$ such that, setting

$$\begin{aligned} u_{1\varepsilon}(\varepsilon(m + y)) &:= u_{1\varepsilon}^*(m, y) \\ z_\varepsilon(\varepsilon(m + y)) &:= \nabla^s u_{1\varepsilon}^*(m, y) \end{aligned} \quad \text{for a.a. } y \in \mathcal{Y}, \forall m \in \mathbb{Z}^N, \quad (3.6)$$

as $\varepsilon \rightarrow 0$ along a suitable subsequence

$$u_{1\varepsilon} \xrightarrow{2} u_1 \quad \text{in } L^p(\mathbb{R}^N \times \mathcal{Y})^N, \quad z_\varepsilon \xrightarrow{2} \nabla_y^s u_1 \quad \text{in } L^p(\mathbb{R}^N \times \mathcal{Y})^{N^2}. \quad (3.7)$$

This entails that, as $\varepsilon \rightarrow 0$ along the extracted subsequence,

$$\nabla^s u_\varepsilon \xrightarrow{2} \nabla^s u + \nabla_y^s u_1 \quad \text{in } L^p(\mathbb{R}^N \times \mathcal{Y})^{N^2}. \quad (3.8)$$

(ii) Conversely, for any $u \in W^{1,p}(\mathbb{R}^N)^N$ and any $u_1 \in L^p(\mathbb{R}^N; W_*^{1,p}(\mathcal{Y})^N)$ there exists a sequence $\{u_\varepsilon\}$ of $W^{1,p}(\mathbb{R}^N)^N$ such that

$$u_\varepsilon \rightarrow u \quad \text{in } L^p(\mathbb{R}^N)^N, \quad \nabla^s u_\varepsilon \xrightarrow{2} \nabla^s u + \nabla_y^s u_1 \quad \text{in } L^p(\mathbb{R}^N \times \mathcal{Y})^{N^2}. \quad (3.9)$$

Proof. (i) By Lemma 3.1, $u_{1\varepsilon}$ exists and

$$\|z_\varepsilon\|_{L^p(\mathbb{R}^N)^{N^2}}, \|u_{1\varepsilon}\|_{L^p(\mathbb{R}^N)^N} \leq \text{Constant (independent of } \varepsilon).$$

By Proposition 1.2 then there exists $u_1 \in L^p(\mathbb{R}^N \times \mathcal{Y})^N$ such that, as ε vanishes along a suitable subsequence, $u_{1\varepsilon} \xrightarrow{2} u_1$ in $L^p(\mathbb{R}^N \times \mathcal{Y})^N$. Let us now fix any $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and any $\psi \in \mathcal{D}(\mathcal{Y})^{N^2}$. Setting $(\nabla \cdot \psi)_i := \sum_{j=1}^N D_{y_j} \psi_{ij}$ for $i = 1, \dots, N$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} z_\varepsilon(x) \cdot \psi\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) dx &= \varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} \nabla u_{1\varepsilon}^*(m, y) : \psi(y) \varphi(\varepsilon m) \, dy \\ &= -\varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} u_{1\varepsilon}^*(m, y) \cdot \nabla \cdot \psi(y) \varphi(\varepsilon m) \, dy \\ &= -\varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} u_{1\varepsilon}(\varepsilon(m + y)) \cdot \nabla \cdot \psi(y) \varphi(\varepsilon m) \, dy \\ &\rightarrow - \iint_{\mathbb{R}^N \times \mathcal{Y}} u_1(x, y) \cdot \nabla \cdot \psi(y) \varphi(x) \, dx \, dy. \end{aligned}$$

As z_ε is uniformly bounded in $L^p(\mathbb{R}^N)^{N^2}$ and $\varphi(\varepsilon\mathcal{N}(\frac{x}{\varepsilon})) \rightarrow \varphi(x)$ uniformly for $x \in \mathbb{R}^N$, the latter convergence also holds if in the first integral $\varphi(\varepsilon\mathcal{N}(\frac{x}{\varepsilon}))$ is replaced by $\varphi(x)$. Thus $u_1 \in L^p(\mathbb{R}^N; W_*^{1,p}(\mathcal{Y})^N)$ and (3.7) is fulfilled.

(ii) In view of the derivation of (3.8), let us fix any $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and any $\psi \in \mathcal{D}(\mathcal{Y})^{N^2}$, and notice that there exists a (unique) $\zeta \in W_*^{1,p}(\mathcal{Y})^N$ such that

$$\int_{\mathcal{Y}} \nabla^s \zeta : \nabla^s v \, dy = \int_{\mathcal{Y}} \psi : \nabla^s v \, dy \quad \forall v \in W^{1,p'}(\mathcal{Y})^N.$$

Setting $(\nabla^s \cdot \psi)_i := \frac{1}{2} \sum_{j=1}^N (D_j \psi_{ij} + D_j \psi_{ji})$, we then have $\nabla^s \cdot \nabla^s \zeta = \nabla^s \cdot \psi + C$ in \mathcal{Y} for some constant $C \in \mathbb{R}^N$; integrating over \mathcal{Y} , by the Y -periodicity of ζ and ψ we get $C = 0$. By setting $\tilde{\psi} := \nabla^s \zeta$ and $\hat{\psi} := \psi - \tilde{\psi}$ in \mathcal{Y} , we then have

$$\tilde{\psi}, \hat{\psi} \in W^{1,p}(\mathcal{Y})^{N^2}, \quad \nabla^s \cdot \tilde{\psi} = \nabla^s \cdot \psi, \quad \nabla^s \cdot \hat{\psi} = 0 \quad \text{a.e. in } \mathcal{Y}.$$

By Lemma 1.1, (3.5) and (3.7), we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla^s u_\varepsilon(x) : \tilde{\psi}\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon\mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) dx \\ &= \int_{\mathbb{R}^N} dx \varphi\left(\varepsilon\mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) \int_{\mathcal{Y}} \nabla_y^s u_\varepsilon(\varepsilon[\mathcal{N}\left(\frac{x}{\varepsilon}\right) + y]) : \nabla^s \zeta(y) \, dy \\ &= \int_{\mathbb{R}^N} dx \varphi\left(\varepsilon\mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) \int_{\mathcal{Y}} \nabla^s u_{1\varepsilon}^*(\mathcal{N}\left(\frac{x}{\varepsilon}\right), y) : \nabla^s \zeta(y) \, dy \\ &\rightarrow \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla_y^s u_1(x, y) : \tilde{\psi}(y) \varphi(x) \, dx \, dy. \end{aligned}$$

Moreover, as $\nabla^s \cdot \hat{\psi} = 0$ a.e. in \mathcal{Y} and $\int_{\mathcal{Y}} \tilde{\psi}(y) \, dy = \int_{\mathcal{Y}} \nabla^s \zeta(y) \, dy = 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla^s u_\varepsilon(x) : \hat{\psi}\left(\frac{x}{\varepsilon}\right) \varphi(x) \, dx &= - \int_{\mathbb{R}^N} u_\varepsilon(x) \cdot \hat{\psi}\left(\frac{x}{\varepsilon}\right) \cdot \nabla \varphi(x) \, dx \rightarrow \\ - \iint_{\mathbb{R}^N \times \mathcal{Y}} u(x) \cdot \hat{\psi}(y) \cdot \nabla \varphi(x) \, dx \, dy &= \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla^s u(x) : \hat{\psi}(y) \varphi(x) \, dx \, dy, \\ \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla_y^s u_1(x, y) : \hat{\psi}(y) \varphi(x) \, dx \, dy &= - \iint_{\mathbb{R}^N \times \mathcal{Y}} u_1(x, y) \cdot \nabla^s \cdot \hat{\psi}(y) \varphi(x) \, dx \, dy = 0, \\ \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla^s u(x) : \tilde{\psi}(y) \varphi(x) \, dx \, dy &= \int_{\mathbb{R}^N} \nabla^s u(x) \varphi(x) \, dx : \int_{\mathcal{Y}} \tilde{\psi}(y) \, dy = 0. \end{aligned} \tag{3.10}$$

As $\|\nabla^s u_\varepsilon\|_{L^p(\mathbb{R}^N)^{N^2}}$ is uniformly bounded and $\varphi(\varepsilon\mathcal{N}(\frac{x}{\varepsilon})) \rightarrow \varphi(x)$ uniformly in \mathbb{R}^N , cf. (1.1),

$$\int_{\mathbb{R}^N} \nabla^s u_\varepsilon(x) : \psi\left(\frac{x}{\varepsilon}\right) \varphi(x) \, dx - \int_{\mathbb{R}^N} \nabla^s u_\varepsilon(x) : \psi\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon\mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) \, dx \rightarrow 0;$$

a similar statement holds for the first member of (3.10). By assembling the five latter formulas we end up with

$$\int_{\mathbb{R}^N} \nabla^s u_\varepsilon(x) : \psi\left(\frac{x}{\varepsilon}\right) \varphi(x) dx \rightarrow \iint_{\mathbb{R}^N \times \mathcal{Y}} [\nabla^s u(x) + \nabla_y^s u_1(x, y)] : \psi(y) \varphi(x) dx dy,$$

that is (3.8). The proof of part (iii) of the thesis mimics that of Theorem 3.2, and is then omitted. \square

Remarks. (i) The function u_1 need not be uniquely determined by the sequence $\{u_\varepsilon\}$.

(ii) Defining $\lim_{\varepsilon \rightarrow 0}^{(1)}$ and $\lim_{\varepsilon \rightarrow 0}^{(2)}$ as in Section 2, (3.8) also reads

$$\lim_{\varepsilon \rightarrow 0}^{(2)} \nabla^s u_\varepsilon = \nabla^s \lim_{\varepsilon \rightarrow 0}^{(1)} u_\varepsilon + \nabla_y^s u_1 = \lim_{\varepsilon \rightarrow 0}^{(1)} \nabla^s u_\varepsilon + \nabla_y^s u_1 \quad (3.11)$$

a.e. in $\mathbb{R}^N \times \mathcal{Y}$. If $p = 2$ this decomposition is orthogonal in $L^2(\mathbb{R}^N \times \mathcal{Y})^{N^2}$.

4. Two-scale convergence of the curl of a vector field

In this section we assume that $N = 3$ and $p = 2$. We remind the reader that $L_{\text{rot}}^2(Y)^3 := \{v \in L^2(Y)^3 : \nabla \times v \in L^2(Y)^3\}$ ($\nabla \times := \text{curl}$) is a Hilbert space equipped with the graph norm.

The same applies to $L_{\text{rot}}^2(\mathbb{R}^3)^3$ and $L_{\text{rot}}^2(\mathcal{Y})^3$; notice that the tangential components of the functions of the latter space coincide on opposite faces of Y .

Lemma 4.1 (Cell Projection). *Let $\varepsilon > 0$. For any $w \in L_{\text{rot}}^2(\mathbb{R}^3)^3$ there exists a unique $w^* = w^*(m, y) \in \ell^2(H_*^1(\mathcal{Y})^3)$ such that for any $m \in \mathbb{Z}^3$*

$$\begin{aligned} \nabla \cdot w^*(m, y) &= 0 \quad \text{for a.a. } y \in \mathcal{Y} \quad (\nabla \cdot := \text{div}) \\ \int_{\mathcal{Y}} [\nabla \times w^*(m, y) - (\nabla \times w)(\varepsilon(m + y))] \cdot \nabla \times \zeta dy &= 0 \quad \forall \zeta \in H^1(\mathcal{Y})^3. \end{aligned} \quad (4.1)$$

This entails that there exists a constant $C_3 > 0$ (independent of w and ε) such that

$$\varepsilon^{\frac{3}{2}} \|\nabla \times w^*\|_{\ell^2(L^2(\mathcal{Y})^3)} \left(= \left(\varepsilon^3 \sum_{m \in \mathbb{Z}^3} \|\nabla \times w^*(m, \cdot)\|_{L^2(\mathcal{Y})^3}^2 \right)^{\frac{1}{2}} \right) \quad (4.2)$$

$$\begin{aligned} &\leq \|\nabla \times w\|_{L^2(\mathbb{R}^3)^3} \\ \varepsilon^{\frac{3}{2}} \|w^*\|_{\ell^2(L^2(\mathcal{Y})^3)} &\leq C_3 \|\nabla \times w\|_{L^2(\mathbb{R}^3)^3}. \end{aligned} \quad (4.3)$$

Proof. Let us fix any $m \in \mathbb{Z}^3$, define the Hilbert space $V_m := \{v \in H_*^1(\mathcal{Y})^3 : \nabla \cdot v = 0\}$, and consider the quadratic functional

$$V_m \rightarrow \mathbb{R} : v \mapsto \int_{\mathcal{Y}} \left(\frac{1}{2} |\nabla \times v|^2 - (\nabla \times w)(\varepsilon(m + y)) \cdot \nabla \times v \right) dy.$$

By the Poincaré inequality this functional is coercive on V_m ; hence it has a (unique) minimizer, and this is the unique solution of (4.1). This yields (4.2). It is not restrictive to assume that w is divergence-free, for w^* only depends on $\nabla \times w$; (2.7) then yields (4.3). The uniqueness of the solution of (4.1) is a simple consequence of (4.3). \square

Theorem 4.2. (i) *Let $\{u_\varepsilon\}$ be a bounded sequence in the space $L_{\text{rot}}^2(\mathbb{R}^3)^3$ such that $u_\varepsilon \rightharpoonup u$ in $L^2(\mathbb{R}^3 \times \mathcal{Y})^3$. For any ε there exists a unique $u_{1\varepsilon}^* \in \ell^2(H_*^1(\mathcal{Y})^3)$ such that, for any $m \in \mathbb{Z}^3$,*

$$\begin{aligned} \nabla \cdot u_{1\varepsilon}^* &= 0 \quad \text{for a.a. } y \in \mathcal{Y} \\ \int_{\mathcal{Y}} [\nabla \times u_{1\varepsilon}^*(m, y) - (\nabla \times u_\varepsilon)(\varepsilon(m + y))] \cdot \nabla \times \zeta \, dy &= 0 \quad \forall \zeta \in H^1(\mathcal{Y})^3. \end{aligned} \quad (4.4)$$

Moreover there exists $u_1 \in L^2(\mathbb{R}^3; H_*^1(\mathcal{Y})^3)$ such that $\nabla_y \cdot u_1 = 0$ a.e. in $\mathbb{R}^3 \times \mathcal{Y}$ and, setting

$$\begin{aligned} u_{1\varepsilon}(\varepsilon(m + y)) &:= u_{1\varepsilon}^*(m, y) \\ z_\varepsilon(\varepsilon(m + y)) &:= \nabla \times u_{1\varepsilon}^*(m, y) \end{aligned} \quad \text{for a.a. } y \in Y, \forall m \in \mathbb{Z}^3, \quad (4.5)$$

as $\varepsilon \rightarrow 0$ along a suitable subsequence

$$u_{1\varepsilon} \rightharpoonup u_1, \quad z_\varepsilon \rightharpoonup \nabla_y \times u_1 \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (4.6)$$

This entails that, as $\varepsilon \rightarrow 0$ along the extracted subsequence, defining \bar{u} as in (1.7),

$$\nabla \times u_\varepsilon \rightharpoonup \nabla \times \bar{u} + \nabla_y \times u_1 \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (4.7)$$

Furthermore $u \in L^2(\mathbb{R}^3; L_{\text{rot}}^2(\mathcal{Y})^3)$ and for the whole given sequence

$$\varepsilon \nabla \times u_\varepsilon \rightharpoonup \nabla_y \times u = 0 \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (4.8)$$

(ii) *Conversely, for any $u \in L^2(\mathbb{R}^3 \times \mathcal{Y})^3$ such that $\bar{u} \in L_{\text{rot}}^2(\mathbb{R}^3)^3$ and $\nabla_y \times u = 0$ in $\mathcal{D}'(\mathbb{R}^3 \times \mathcal{Y})^3$ and for any $u_1 \in L^2(\mathbb{R}^3; H_*^1(\mathcal{Y})^3)$, there exists a sequence $\{u_\varepsilon\}$ of $H^1(\mathbb{R}^3)^3$ such that*

$$u_\varepsilon \rightarrow u \quad \text{in } L^2(\mathbb{R}^3)^3, \quad \nabla \times u_\varepsilon \rightharpoonup \nabla \times \bar{u} + \nabla_y \times u_1 \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (4.9)$$

Proof. (i) By Lemma 4.1, $u_{1\varepsilon}$ exists and

$$\|z_\varepsilon\|_{L^2(\mathbb{R}^3)^3}, \|u_{1\varepsilon}\|_{L^2(\mathbb{R}^3)^3} \leq \text{Constant (independent of } \varepsilon).$$

By Proposition 1.2 then there exists $u_1 \in L^2(\mathbb{R}^3 \times \mathcal{Y})^3$ such that, as $\varepsilon \rightarrow 0$ along a suitable subsequence, $u_{1\varepsilon} \rightharpoonup u_1$ in $L^2(\mathbb{R}^3 \times \mathcal{Y})^3$. For any $\varphi \in \mathcal{D}(\mathbb{R}^3)$ and any $\psi \in \mathcal{D}(\mathcal{Y})^3$ we then have

$$\begin{aligned} \int_{\mathbb{R}^3} z_\varepsilon(x) \cdot \psi\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) dx &= \varepsilon^3 \sum_{m \in \mathbb{Z}^3} \int_{\mathcal{Y}} \nabla \times u_{1\varepsilon}^*(m, y) \cdot \psi(y) \varphi(\varepsilon m) dy \\ &= -\varepsilon^3 \sum_{m \in \mathbb{Z}^3} \int_{\mathcal{Y}} u_{1\varepsilon}^*(m, y) \cdot \nabla \times \psi(y) \varphi(\varepsilon m) dy \\ &= -\varepsilon^3 \sum_{m \in \mathbb{Z}^3} \int_{\mathcal{Y}} u_{1\varepsilon}(\varepsilon(m+y)) \cdot \nabla \times \psi(y) \varphi(\varepsilon m) dy \\ &\rightarrow - \iint_{\mathbb{R}^3 \times \mathcal{Y}} u_1(x, y) \cdot \nabla \times \psi(y) \varphi(x) dx dy. \end{aligned}$$

As z_ε is uniformly bounded in $L^2(\mathbb{R}^3)^3$ and $\varphi(\varepsilon \mathcal{N}(\frac{x}{\varepsilon})) \rightarrow \varphi(x)$ uniformly for $x \in \mathbb{R}^3$, the latter convergence also holds if in the first integral $\varphi(\varepsilon \mathcal{N}(\frac{x}{\varepsilon}))$ is replaced by $\varphi(x)$. Hence $\nabla_y \times u_1 \in L^2(\mathbb{R}^3 \times \mathcal{Y})^3$ and (4.6) is fulfilled. By (4.4)₁, we also have $u_1 \in L^2(\mathbb{R}^3; H_*^1(\mathcal{Y})^3)$ and $\nabla_y \cdot u_1 = 0$ a.e. in $\mathbb{R}^3 \times \mathcal{Y}$.

(ii) Let us fix any $\varphi \in \mathcal{D}(\mathbb{R}^3)$ and any $\psi \in \mathcal{D}(\mathcal{Y})^3$. Arguing as for Lemma 4.1, it is easy to see that there exists $\zeta \in L^2_{\text{rot}^*}(\mathcal{Y})^3$ such that

$$\int_{\mathcal{Y}} (\nabla \times \zeta - \psi) \cdot \nabla \times v dy = 0 \quad \forall v \in H^1(\mathcal{Y})^3. \quad (4.10)$$

By setting $\tilde{\psi} := \nabla \times \zeta$ and $\hat{\psi} := \psi - \tilde{\psi}$ in \mathcal{Y} , we then have $\tilde{\psi}, \hat{\psi} \in L^2(\mathcal{Y})^3$, $\nabla \times \hat{\psi} = 0$ in $(L^2_{\text{rot}}(\mathcal{Y})^3)'$. By Lemma 1.1, (4.4) and (4.5), we have

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla \times u_\varepsilon(x) \tilde{\psi}\left(\frac{x}{\varepsilon}\right) \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) dx &= \int_{\mathbb{R}^3} dx \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) \int_{\mathcal{Y}} (\nabla \times u_\varepsilon)(\varepsilon[\mathcal{N}\left(\frac{x}{\varepsilon}\right) + y]) \cdot \nabla \times \zeta(y) dy \\ &= \int_{\mathbb{R}^3} dx \varphi\left(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)\right) \int_{\mathcal{Y}} \nabla \times u_{1\varepsilon}^*\left(\mathcal{N}\left(\frac{x}{\varepsilon}\right), y\right) \cdot \nabla \times \zeta(y) dy \\ &\rightarrow \iint_{\mathbb{R}^3 \times \mathcal{Y}} \nabla_y \times u_1(x, y) \cdot \tilde{\psi}(y) \varphi(x) dx dy. \end{aligned} \quad (4.11)$$

(iii) In general u_ε will not converge strongly in $L^2(\mathbb{R}^3)^3$; we then construct another sequence $\{w_\varepsilon\}$ as follows. Having fixed a test function $\varphi \in \mathcal{D}(\mathbb{R}^3)$, it is

clear that the limit of $\nabla \times u_\varepsilon(x) \cdot \hat{\psi}(\frac{x}{\varepsilon})\varphi(x)$ does not depend on the behaviour of u_ε outside the support S of φ , so that we may assume S to be equal to a ball without loss of generality. Let us then set

$$V_S := \{v \in H^1(\mathbb{R}^3)^3 : v = 0 \text{ a.e. in } \mathbb{R}^3 \setminus S, \nabla \cdot v = 0\}.$$

For any $\varepsilon > 0$, by the Poincaré inequality there exists a (unique) minimizer $w_\varepsilon \in V_S$ of the functional

$$\Psi_\varepsilon(v) := \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla \times v|^2 - \nabla \times u_\varepsilon \cdot \nabla \times v \right) dx \quad \forall v \in V_S.$$

We then have

$$\nabla \cdot w_\varepsilon = 0, \quad \int_{\mathbb{R}^3} (\nabla \times w_\varepsilon - \nabla \times u_\varepsilon) \cdot \nabla \times v \, dx = 0 \quad \forall v \in V_S.$$

As $\nabla \times V_S = \nabla \times \{v \in H^1(\mathbb{R}^3)^3 : v = 0 \text{ a.e. in } \mathbb{R}^3 \setminus S\}$, we conclude that

$$\nabla \times w_\varepsilon = \nabla \times u_\varepsilon \quad \text{a.e. in } S, \quad \forall \varepsilon. \quad (4.12)$$

The sequences $\{w_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)^3$ (at variance with $\{u_\varepsilon\}$); hence there exists $w \in H^1(\mathbb{R}^3)^3$ such that, up to a subsequence,

$$w_\varepsilon \rightharpoonup w \quad \text{in } H^1(\mathbb{R}^3)^3.$$

As $u_\varepsilon \rightharpoonup \bar{u}$ in $L^2_{\text{rot}}(\mathbb{R}^3)^3$, passing to the weak (one-scale) limit in (4.12) we get

$$\nabla \times w = \nabla \times \bar{u} \quad \text{a.e. in } S.$$

(iv) Recalling that $\hat{\psi}$ is curl-free, by the three latter displayed formulas we have

$$\begin{aligned} \int_{\mathbb{R}^3} [\nabla \times u_\varepsilon(x) - \nabla \times w_\varepsilon(x)] \cdot \hat{\psi}(\frac{x}{\varepsilon})\varphi(x) \, dx &= 0 \\ \int_{\mathbb{R}^3} \nabla \times u_\varepsilon(x) \cdot \hat{\psi}(\frac{x}{\varepsilon})\varphi(x) \, dx &= \int_{\mathbb{R}^3} \nabla \times w_\varepsilon(x) \cdot \hat{\psi}(\frac{x}{\varepsilon})\varphi(x) \, dx \\ &= - \int_{\mathbb{R}^3} w_\varepsilon(x) \cdot \hat{\psi}(\frac{x}{\varepsilon}) \times \nabla \varphi(x) \, dx \rightarrow \\ - \iint_{\mathbb{R}^3 \times \mathcal{Y}} w(x) \cdot \hat{\psi}(y) \times \nabla \varphi(x) \, dx \, dy &= \iint_{\mathbb{R}^3 \times \mathcal{Y}} \nabla \times w(x) \cdot \hat{\psi}(y)\varphi(x) \, dx \, dy \\ &= \iint_{\mathbb{R}^3 \times \mathcal{Y}} \nabla \times \bar{u}(x) \cdot \hat{\psi}(y)\varphi(x) \, dx \, dy \\ \iint_{\mathbb{R}^3 \times \mathcal{Y}} \nabla_y \times u_1(x, y) \cdot \hat{\psi}(y)\varphi(x) \, dx \, dy &= \iint_{\mathbb{R}^3 \times \mathcal{Y}} u_1(x, y) \cdot \nabla \times \hat{\psi}(y)\varphi(x) \, dx \, dy = 0 \\ \iint_{\mathbb{R}^3 \times \mathcal{Y}} \nabla \times \bar{u}(x) \cdot \tilde{\psi}(y)\varphi(x) \, dx \, dy &= \int_{\mathbb{R}^3} \nabla \times \bar{u}(x)\varphi(x) \, dx \cdot \int_{\mathcal{Y}} \tilde{\psi}(y) \, dy = 0. \end{aligned}$$

In (4.11) we may replace $\varphi(\varepsilon\mathcal{N}(\frac{x}{\varepsilon}))$ by $\varphi(x)$, since their difference vanishes uniformly. By this and the three latter displayed formulas above we get

$$\int_{\mathbb{R}^3} \nabla \times u_\varepsilon(x) \cdot \psi(\frac{x}{\varepsilon}) \varphi(x) dx \rightarrow \iint_{\mathbb{R}^3 \times \mathcal{Y}} [\nabla \times \bar{u}(x) + \nabla_y \times u_1(x, y)] \cdot \psi(y) \varphi(x) dx dy,$$

i.e., (4.7). (4.8) follows by the argument of part (i).

(v) Part (ii) of the thesis can be proved via an argument analogous to that of part (ii) of Theorem 2.2. However here instead of (2.13) we set

$$u_\varepsilon(x) := \bar{u}(x) + \varepsilon \int_{\mathbb{R}^3} u_1(\xi, \frac{x}{\varepsilon}) \rho_\varepsilon(x - \xi) d\xi \quad \text{for a.a. } x \in \mathbb{R}^3 \quad (4.13)$$

(here with $\bar{u}(x)$ instead of $u(x)$). The argument of part (iii) is also analogous to that of Theorem 2.2; however here the strong convergence of the curl does not entail that of the function. \square

Remarks. (i) The function u_1 need not be uniquely determined by the sequence $\{u_\varepsilon\}$.

(ii) Defining $\lim_{\varepsilon \rightarrow 0}^{(1)}$ and $\lim_{\varepsilon \rightarrow 0}^{(2)}$ as in Sect. 2, (4.7) also reads

$$\lim_{\varepsilon \rightarrow 0}^{(2)} \nabla \times u_\varepsilon = \nabla \times \lim_{\varepsilon \rightarrow 0}^{(1)} u_\varepsilon + \nabla_y \times u_1 = \lim_{\varepsilon \rightarrow 0}^{(1)} \nabla \times u_\varepsilon + \nabla_y \times u_1 \quad (4.14)$$

a.e. in $\mathbb{R}^3 \times \mathcal{Y}$. This decomposition is orthogonal in $L^2(\mathbb{R}^3 \times \mathcal{Y})^3$. Thus

$$\lim_{\varepsilon \rightarrow 0}^{(2)} \nabla \times u_\varepsilon \equiv 0 \quad \Rightarrow \quad \nabla \times \lim_{\varepsilon \rightarrow 0}^{(1)} u_\varepsilon = \nabla_y \times u_1 \equiv 0.$$

5. Two-scale convergence of the divergence of a vector field

In this section we assume that $N \in \mathbb{N}$ and $p = 2$. It is known that $L_{\text{div}}^2(Y)^N := \{v \in L^2(Y)^N : \nabla \cdot v \in L^2(Y)\}$ ($\nabla \cdot := \text{div}$) is a Hilbert space equipped with the graph norm. We similarly define $L_{\text{div}}^2(\mathbb{R}^N)^N$ and $L_{\text{div}}^2(\mathcal{Y})^N$; notice that the normal traces of the functions of the latter space coincide on opposite faces of Y . Dealing with $N \neq 3$, we define the curl as the antisymmetric part of the Hessian matrix:

$$(\nabla \times v)_{ij} := D_j v_i - D_i v_j \quad \forall i, j \in \{1, \dots, N\}, \quad \forall v \in \mathcal{D}'(\mathbb{R}^N)^N. \quad (5.1)$$

Lemma 5.1 (Cell Projection). *Let $\varepsilon > 0$. For any $w \in L_{\text{div}}^2(\mathbb{R}^N)^N$ there exists a unique $w^* = w^*(m, y) \in \ell^2(H_*^1(\mathcal{Y})^N)$ such that for any $m \in \mathbb{Z}^N$*

$$\begin{aligned} \nabla \times w^*(m, y) &= 0 \quad \text{for a.a. } y \in \mathcal{Y} \\ \int_{\mathcal{Y}} [\nabla \cdot w^*(m, y) - (\nabla \cdot w)(\varepsilon(m + y))] \eta dy &= 0 \quad \forall \eta \in L_*^2(\mathcal{Y}). \end{aligned} \quad (5.2)$$

This entails that there exists a constant $C_4 > 0$ (independent of w and ε) such that

$$\varepsilon^{\frac{N}{2}} \|\nabla \cdot w^*\|_{\ell^2(L^2(\mathcal{Y}))} \left(= \left(\varepsilon^N \sum_{m \in \mathbb{Z}^N} \|\nabla \cdot w^*(m, \cdot)\|_{L^2(\mathcal{Y})}^2 \right)^{\frac{1}{2}} \right) \leq \|\nabla \cdot w\|_{L^2(\mathbb{R}^N)} \quad (5.3)$$

$$\varepsilon^{\frac{N}{2}} \|w^*\|_{\ell^2(L^2(\mathcal{Y})^N)} \leq C_4 \|\nabla \cdot w\|_{L^2(\mathbb{R}^N)}. \quad (5.4)$$

Proof. An argument analogous to that of Lemma 4.1, with the divergence in place of the curl operator, yields the existence and uniqueness of w^* .

It is not restrictive to assume that w is curl-free, for w_ε only depends on $\nabla \cdot w$; hence $\int_{\mathcal{Y}} |\nabla w|^2 dx = \int_{\mathcal{Y}} |\nabla \cdot w|^2 dx$, and (5.4) follows from (2.3). \square

Theorem 5.2. (i) Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2_{\text{div}}(\mathbb{R}^N)^N$ such that $u_\varepsilon \rightharpoonup u$ in $L^2(\mathbb{R}^N \times \mathcal{Y})^N$. For any ε there exists a unique $u_{1\varepsilon}^* \in \ell^2(H_*^1(\mathcal{Y})^N)^N$ such that, for any $m \in \mathbb{Z}^N$,

$$\begin{aligned} \nabla \times u_{1\varepsilon}^*(m, y) &= 0 \quad \text{for a.a. } y \in \mathcal{Y} \\ \int_{\mathcal{Y}} [\nabla \cdot u_{1\varepsilon}^*(m, y) - (\nabla \cdot w)(\varepsilon(m + y))]\eta dy &= 0 \quad \forall \eta \in L_*^2(\mathcal{Y}). \end{aligned} \quad (5.5)$$

Moreover there exists $u_1 \in L^2(\mathbb{R}^N; H_*^1(\mathcal{Y})^N)$ such that $\nabla_y \times u_1 = 0$ a.e. in \mathbb{R}^N , and, setting

$$\begin{aligned} u_{1\varepsilon}(\varepsilon(m + y)) &:= u_{1\varepsilon}^*(m, y) \\ z_\varepsilon(\varepsilon(m + y)) &:= \nabla \cdot u_{1\varepsilon}^*(m, y) \end{aligned} \quad \text{for a.a. } y \in \mathcal{Y}, \forall m \in \mathbb{Z}^N, \quad (5.6)$$

as $\varepsilon \rightarrow 0$ along a suitable subsequence

$$u_{1\varepsilon} \rightharpoonup u_1 \quad \text{in } L^2(\mathbb{R}^N \times \mathcal{Y})^N, \quad z_\varepsilon \rightharpoonup \nabla_y \cdot u_1 \quad \text{in } L^2(\mathbb{R}^N \times \mathcal{Y}). \quad (5.7)$$

This entails that, as $\varepsilon \rightarrow 0$ along the extracted subsequence, defining \bar{u} as in (1.7),

$$\nabla \cdot u_\varepsilon \rightharpoonup \nabla \cdot \bar{u} + \nabla_y \cdot u_1 \quad \text{in } L^2(\mathbb{R}^N \times \mathcal{Y}). \quad (5.8)$$

Furthermore $u \in L^2(\mathbb{R}^N; L^2_{\text{div}}(\mathcal{Y})^N)$ and for the whole given sequence

$$\varepsilon \nabla \cdot u_\varepsilon \rightharpoonup \nabla_y \cdot u = 0 \quad \text{in } L^2(\mathbb{R}^N \times \mathcal{Y}). \quad (5.9)$$

(ii) Conversely, for any $u \in L^2(\mathbb{R}^N \times \mathcal{Y})^N$ such that $\bar{u} \in L^2_{\text{div}}(\mathbb{R}^N)^N$ and $\nabla_y \cdot u = 0$ in $\mathcal{D}'(\mathbb{R}^N \times \mathcal{Y})$ and for any $u_1 \in L^2(\mathbb{R}^N; H_*^1(\mathcal{Y})^N)$, there exists a sequence $\{u_\varepsilon\}$ of $H^1(\mathbb{R}^N)^N$ such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^N)^N, \quad \nabla \cdot u_\varepsilon \rightharpoonup \nabla \cdot \bar{u} + \nabla_y \cdot u_1 \quad \text{in } L^2(\mathbb{R}^N \times \mathcal{Y}). \quad (5.10)$$

Proof. (i) By Lemma 5.1, $u_{1\varepsilon}$ exists and

$$\|z_\varepsilon\|_{L^2(\mathbb{R}^N)}, \|u_{1\varepsilon}\|_{L^2(\mathbb{R}^N)^N} \leq \text{Constant (independent of } \varepsilon).$$

By Proposition 1.2 then there exists $u_1 \in L^2(\mathbb{R}^N \times \mathcal{Y})^N$ such that, as $\varepsilon \rightarrow 0$ along a suitable subsequence, $u_{1\varepsilon} \rightharpoonup u_1$ in $L^2(\mathbb{R}^N \times \mathcal{Y})^N$. For any $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and any $\psi \in \mathcal{D}(\mathcal{Y})$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} z_\varepsilon(x) \psi\left(\frac{x}{\varepsilon}\right) \varphi(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)) dx &= \varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} \nabla \cdot u_{1\varepsilon}^*(m, y) \psi(y) \varphi(\varepsilon m) dy \\ &= -\varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} u_{1\varepsilon}^*(m, y) \cdot \nabla \psi(y) \varphi(\varepsilon m) dy \\ &= -\varepsilon^N \sum_{m \in \mathbb{Z}^N} \int_{\mathcal{Y}} u_{1\varepsilon}(\varepsilon(m+y)) \cdot \nabla \psi(y) \varphi(\varepsilon m) dy \\ &\rightarrow - \iint_{\mathbb{R}^N \times \mathcal{Y}} u_1(x, y) \cdot \nabla \psi(y) \varphi(x) dx dy. \end{aligned}$$

As z_ε is uniformly bounded in $L^2(\mathbb{R}^N)^N$ and $\varphi(\varepsilon \mathcal{N}(\frac{x}{\varepsilon})) \rightarrow \varphi(x)$ uniformly for $x \in \mathbb{R}^N$, the latter convergence also holds if in the first integral $\varphi(\varepsilon \mathcal{N}(\frac{x}{\varepsilon}))$ is replaced by $\varphi(x)$. Moreover $\nabla_y \times u_1 = 0$ a.e. in $\mathbb{R}^N \times \mathcal{Y}$, by (5.5)₁. Thus $u_1 \in L^2(\mathbb{R}^N; H_*^1(\mathcal{Y})^N)$ and (5.7) is fulfilled.

(ii) Let us fix any $\varphi \in \mathcal{D}(\mathbb{R}^N)$, any $\psi \in \mathcal{D}(\mathcal{Y})$, and define $\bar{\psi}$ and $\tilde{\psi}$ as in (1.7). By (5.5) and (5.7),

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla \cdot u_\varepsilon \tilde{\psi}\left(\frac{x}{\varepsilon}\right) \varphi(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)) dx &= \int_{\mathbb{R}^N} \varphi(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)) dx \int_{\mathcal{Y}} \nabla \cdot u_\varepsilon(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right) + \varepsilon y) \tilde{\psi}\left(\frac{x}{\varepsilon}\right) dy \\ &= \int_{\mathbb{R}^N} \varphi(\varepsilon \mathcal{N}\left(\frac{x}{\varepsilon}\right)) dx \int_{\mathcal{Y}} \nabla \cdot u_{1\varepsilon}^*(\mathcal{N}\left(\frac{x}{\varepsilon}\right), y) \tilde{\psi}\left(\frac{x}{\varepsilon}\right) dy \\ &\rightarrow \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla_y \cdot u_1(x, y) \tilde{\psi}(y) \varphi(x) dx dy. \end{aligned}$$

As $u_\varepsilon \rightharpoonup \bar{u}$ weakly in $L^2(\mathbb{R}^N)^N$,

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla \cdot u_\varepsilon(x) \bar{\psi} \varphi(x) dx &= - \int_{\mathbb{R}^N} u_\varepsilon(x) \bar{\psi} \cdot \nabla \varphi(x) dx \rightarrow \\ - \int_{\mathbb{R}^N} \bar{u}(x) \bar{\psi} \cdot \nabla \varphi(x) dx &= - \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla \cdot \bar{u}(x) \bar{\psi} \varphi(x) dx dy. \end{aligned}$$

Moreover, as $\int_{\mathcal{Y}} \nabla_y \cdot u_1(x, y) dy = 0$ for a.a. x and $\int_{\mathcal{Y}} \tilde{\psi}(y) dy = 0$,

$$\begin{aligned} \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla_y \cdot u_1(x, y) \bar{\psi} \varphi(x) dx dy &= \int_{\mathcal{Y}} \nabla_y \cdot u_1(x, y) dy \int_{\mathbb{R}^N} \bar{\psi} \varphi(x) dx = 0 \\ \iint_{\mathbb{R}^N \times \mathcal{Y}} \nabla \cdot \bar{u}(x) \tilde{\psi}(y) \varphi(x) dx dy &= \int_{\mathbb{R}^N} \nabla \cdot \bar{u}(x) \varphi(x) dx \int_{\mathcal{Y}} \tilde{\psi}(y) dy = 0. \end{aligned}$$

As $\varphi(\varepsilon\mathcal{N}(\frac{x}{\varepsilon})) \rightarrow \varphi(x)$ uniformly in \mathbb{R}^N , the four latter formulas yield

$$\int_{\mathbb{R}^N} \nabla \cdot u_\varepsilon(x) \psi(\frac{x}{\varepsilon}) \varphi(x) dx \rightarrow \iint_{\mathbb{R}^N \times \mathcal{Y}} [\nabla \cdot \bar{u}(x) + \nabla_y \cdot u_1(x, y)] \psi(y) \varphi(x) dx dy,$$

i.e., (5.8). (5.9) follows by the argument of part (i). Parts (ii) can be proved as in Theorems 2.2 and 4.2, defining u_ε as in (4.13). \square

Remarks. (i) The function u_1 need not be uniquely determined by the sequence $\{u_\varepsilon\}$.

(ii) Defining $\lim_{\varepsilon \rightarrow 0}^{(1)}$ and $\lim_{\varepsilon \rightarrow 0}^{(2)}$ as in Sect. 2, (5.8) also reads

$$\lim_{\varepsilon \rightarrow 0}^{(2)} \nabla \cdot u_\varepsilon = \nabla \cdot \lim_{\varepsilon \rightarrow 0}^{(1)} u_\varepsilon + \nabla_y \cdot u_1 = \lim_{\varepsilon \rightarrow 0}^{(1)} \nabla \cdot u_\varepsilon + \nabla_y \cdot u_1 \quad (5.11)$$

a.e. in $\mathbb{R}^N \times \mathcal{Y}$. This decomposition is orthogonal in $L^2(\mathbb{R}^N \times \mathcal{Y})$. In particular if $\nabla \cdot u_\varepsilon \xrightarrow{2} 0$ in $L^2(\mathbb{R}^N \times \mathcal{Y})$ then $\nabla \cdot \bar{u} = \nabla_y \cdot u_1 \equiv 0$, consistently with part (iii) of Proposition 1.14 of [1].

6. Two-scale convergence of the divergence of a tensor field

In this section we assume that $N = 3, p = 2$, and define the divergence and the curl of a 3×3 -tensor field $v \in L^2(Y)^9$ as follows, setting $v_i := (v_{i1}, v_{i2}, v_{i3})$:

$$(\nabla \cdot v)_i := \nabla \cdot (v_i), \quad (\nabla \times v)_i := \nabla \times (v_i) \quad \text{in } \mathcal{D}'(Y)^3, \text{ for } i = 1, 2, 3. \quad (6.1)$$

Notice that $\|\nabla \times v\|_{L^2(Y)^9}^2 + \|\nabla \cdot v\|_{L^2(Y)^3}^2 = \|\nabla v\|_{L^2(Y)^{27}}^2$ for any $v \in H^1(Y)^9$.

We also set $L_{\text{div}}^2(Y)^9 := \{v \in L^2(Y)^9 : \nabla \cdot v \in L^2(Y)^3\}$; this is a Hilbert subspace of $H^1(Y)^9$ equipped with the graph norm. We similarly define $L_{\text{div}}^2(\mathbb{R}^3)^9$, $L_{\text{div}}^2(\mathcal{Y})^9$, and so on.

Theorem 6.1. (i) Let $\{u_\varepsilon\}$ be a bounded sequence of $L_{\text{div}}^2(\mathbb{R}^3)^9$ such that $u_\varepsilon \xrightarrow{2} u$ in $L^2(\mathbb{R}^3 \times \mathcal{Y})^9$. For any ε there exists a unique $u_{1\varepsilon}^* \in \ell^2(H_*^1(\mathcal{Y})^9)$ such that, for any $m \in \mathbb{Z}^3$,

$$\begin{aligned} \nabla \times u_{1\varepsilon}^*(m, y) &= 0 \quad \text{for a.a. } y \in \mathcal{Y} \\ \int_{\mathcal{Y}} [\nabla \cdot u_{1\varepsilon}^*(m, y) - (\nabla \cdot u_\varepsilon)(\varepsilon(m + y))] \eta dy &= 0 \quad \forall \eta \in L_*^2(\mathcal{Y}). \end{aligned} \quad (6.2)$$

Moreover there exists $u_1 \in L^2(\mathbb{R}^3; H_*^1(\mathcal{Y})^9)$ such that $\nabla_y \times u_1 = 0$ a.e. in \mathcal{Y} for a.a. $x \in \mathbb{R}^3$, and, setting

$$\begin{aligned} u_{1\varepsilon}(\varepsilon(m + y)) &:= u_{1\varepsilon}^*(m, y) \\ z_\varepsilon(\varepsilon(m + y)) &:= \nabla \cdot u_{1\varepsilon}^*(m, y) \end{aligned} \quad \text{for a.a. } y \in Y, \forall m \in \mathbb{Z}^3, \quad (6.3)$$

as $\varepsilon \rightarrow 0$ along a suitable subsequence

$$u_{1\varepsilon} \rightharpoonup_2 u_1 \text{ in } L^2(\mathbb{R}^3 \times \mathcal{Y})^9, \quad z_\varepsilon \rightharpoonup_2 \nabla_y \cdot u_1 \text{ in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (6.4)$$

This entails that, as $\varepsilon \rightarrow 0$ along the extracted subsequence, defining \bar{u} as in (1.7),

$$\nabla \cdot u_\varepsilon \rightharpoonup_2 \nabla \cdot \bar{u} + \nabla_y \cdot u_1 \text{ in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (6.5)$$

Furthermore $u \in L^2(\mathbb{R}^3; L^2_{\text{div}}(\mathcal{Y})^9)$ and for the whole given sequence

$$\varepsilon \nabla \cdot u_\varepsilon \rightharpoonup_2 \nabla_y \cdot u = 0 \text{ in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (6.6)$$

If the tensors u_ε are symmetric, then u_1 is also symmetric.

(ii) Conversely, for any $u \in L^2(\mathbb{R}^3 \times \mathcal{Y})^9$ such that $\bar{u} \in L^2_{\text{div}}(\mathbb{R}^3)^9$ and $\nabla_y \cdot u = 0$ in $\mathcal{D}'(\mathbb{R}^3 \times \mathcal{Y})^3$ and for any $u_1 \in L^2(\mathbb{R}^3; H^1_\star(\mathcal{Y})^9)$, there exists a sequence $\{u_\varepsilon\}$ of $H^1(\mathbb{R}^3)^9$ such that

$$u_\varepsilon \rightarrow u \text{ in } L^2(\mathbb{R}^3)^9, \quad \nabla \cdot u_\varepsilon \rightharpoonup_2 \nabla \cdot \bar{u} + \nabla_y \cdot u_1 \text{ in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (6.7)$$

Proof. It suffices to apply Theorem 5.2 to the sequences of vectors $u_{\varepsilon i} := (u_{\varepsilon i1}, u_{\varepsilon i2}, u_{\varepsilon i3})$, for $i = 1, 2, 3$. Part (ii) can also be proved via the procedure of Theorems 2.2 and 4.2, defining u_ε as in (4.13). \square

Remarks. (i) The function u_1 need not be uniquely determined by the sequence $\{u_\varepsilon\}$.

(ii) Defining $\lim_{\varepsilon \rightarrow 0}^{(1)}$ and $\lim_{\varepsilon \rightarrow 0}^{(2)}$ as in Sect. 2, (6.5) also reads

$$\lim_{\varepsilon \rightarrow 0}^{(2)} \nabla \cdot u_\varepsilon = \nabla \cdot \lim_{\varepsilon \rightarrow 0}^{(1)} u_\varepsilon + \nabla_y \cdot u_1 = \lim_{\varepsilon \rightarrow 0}^{(1)} \nabla \cdot u_\varepsilon + \nabla_y \cdot u_1 \quad (6.8)$$

a.e. in $\mathbb{R}^3 \times \mathcal{Y}$. This decomposition is orthogonal in $L^2(\mathbb{R}^3 \times \mathcal{Y})^3$.

7. Two-scale limit of some classical laws

In this section we illustrate some simple applications of the above results, in view of the application to the homogenization of linear and nonlinear problems in electromagnetism and continuum mechanics, cf. e.g. [36]–[39].

Two-scale limit of the Ampère law. Let us consider the classical Ampère law of magnetostatics, and assume that the magnetic field H and the electric current field J depend on a scale parameter ε :

$$\nabla \times H_\varepsilon = 4\pi J_\varepsilon \quad \text{a.e. in } \mathbb{R}^3. \quad (7.1)$$

Under boundedness hypotheses (here omitted), after Theorem 4.2 as ε vanishes along a suitable sequence

$$J_\varepsilon \xrightarrow{2} J, \quad H_\varepsilon \xrightarrow{2} H, \quad \nabla \times H_\varepsilon \xrightarrow{2} \nabla \times \bar{H} + \nabla_y \times H_1 \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \quad (7.2)$$

Defining \bar{J} and \tilde{J} as in (1.7), we then get the coarse- and fine-scale Ampère laws:

$$\begin{aligned} \nabla \times \bar{H} &= 4\pi \bar{J} && \text{a.e. in } \mathbb{R}^3, \\ \nabla_y \times H &= 0, \quad \nabla_y \times H_1 = 4\pi \tilde{J} && \text{a.e. in } \mathbb{R}^3 \times \mathcal{Y}. \end{aligned} \quad (7.3)$$

Moreover, as J_ε is divergence-free, by Theorem 5.2,

$$\nabla \cdot \bar{J} = 0 \quad \text{a.e. in } \mathbb{R}^3, \quad \nabla_y \cdot \tilde{J} = 0 \quad \text{a.e. in } \mathbb{R}^3 \times \mathcal{Y}. \quad (7.4)$$

Two-scale limit of the Gauss law. Next we consider the classical Gauss law of electrostatics, and assume that the electric displacement D and the electric charge γ depend on a parameter ε :

$$\nabla \cdot D_\varepsilon = 4\pi \gamma_\varepsilon \quad \text{a.e. in } \mathbb{R}^3. \quad (7.5)$$

Under boundedness hypotheses, after Theorem 5.2 as ε vanishes along a suitable sequence

$$\begin{aligned} D_\varepsilon &\xrightarrow{2} D && \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3 \\ \gamma_\varepsilon &\xrightarrow{2} \gamma, \quad \nabla \cdot D_\varepsilon \xrightarrow{2} \nabla \cdot \bar{D} + \nabla_y \cdot D_1 && \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y}). \end{aligned} \quad (7.6)$$

Defining $\bar{\gamma}$ and $\tilde{\gamma}$ as in (1.7), we then get the coarse- and fine-scale Gauss laws:

$$\begin{aligned} \nabla \cdot \bar{D} &= 4\pi \bar{\gamma} && \text{a.e. in } \mathbb{R}^3 \\ \nabla_y \cdot D &= 0, \quad \nabla_y \cdot D_1 = 4\pi \tilde{\gamma} && \text{a.e. in } \mathbb{R}^3 \times \mathcal{Y}. \end{aligned} \quad (7.7)$$

Two-scale limit of the system of linear elasticity. Let us denote the displacement by u , the deformation tensor by e , the Cauchy stress by σ , and an applied load by f in a domain Ω of \mathbb{R}^3 . Assuming dependence on a scale parameter ε , the definition of e and the balance of momentum read

$$\nabla^s u_\varepsilon = e_\varepsilon, \quad \nabla \cdot \sigma_\varepsilon = f \quad \text{a.e. in } \Omega. \quad (7.8)$$

(Of course a further equation is needed to close the problem). Under suitable boundedness hypotheses, after Theorems 3.2 and 6.2 as ε vanishes along a suitable sequence

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}^3)^3, \quad e_\varepsilon \xrightarrow{2} e \quad \text{in } L^p(\mathbb{R}^3 \times \mathcal{Y})^9 \quad (p \in]1, +\infty[), \\ \sigma_\varepsilon &\xrightarrow{2} \sigma \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^9, \quad \nabla \cdot \sigma_\varepsilon \xrightarrow{2} \nabla \cdot \bar{\sigma} + \nabla_y \cdot \sigma_1 \quad \text{in } L^2(\mathbb{R}^3 \times \mathcal{Y})^3. \end{aligned} \quad (7.9)$$

Notice that u is independent of the fine-scale variable y , cf. Theorem 2.2. Defining \bar{e} , \tilde{e} and \bar{f} , \tilde{f} as in (1.7), we then get the coarse- and fine-scale laws:

$$\begin{aligned} \nabla^s u &= \bar{e}, & \nabla \cdot \bar{\sigma} &= \bar{f} & \text{a.e. in } \mathbb{R}^3 \\ \nabla_y^s u_1 &= \tilde{e}, & \nabla_y \cdot \sigma &= 0, & \nabla_y \cdot \sigma_1 &= \tilde{f} & \text{a.e. in } \mathbb{R}^3 \times \mathcal{Y}. \end{aligned} \quad (7.10)$$

8. On the homogenization of some elliptic equations

So far we confined ourselves to first-order linear differential operators with constant coefficients. It is clear that several results take over to operators with x -dependent coefficients. One may also include coefficients that also depend on the unknown function u_ε , provided that the latter converges strongly two-scale.

The results that we derived so far are based on the L^2 -boundedness of some first-order operators, and may be applied to the homogenization of a large number of partial differential equations. Here we illustrate a simple example.

A quasilinear elliptic equation. We deal with the system

$$\begin{aligned} \nabla \times [A(u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot (\nabla \times u_\varepsilon + f)] &= 0 \\ \nabla \cdot u_\varepsilon &= 0 \end{aligned} \quad \text{in } \mathbb{R}^3. \quad (8.1)$$

This equation may account for equilibrium in a heterogeneous and anisotropic electric conductor that occupies a domain Ω of \mathbb{R}^3 and is surrounded by an insulating environment, e.g. air. In this case u_ε represents the magnetic field ($u_\varepsilon = H_\varepsilon$), $-f$ is a prescribed electric current field supported outside Ω , and A is the electric resistance (that vanishes outside Ω). (8.1)₁ then follows from the stationary Ampère and Faraday equations coupled with Ohm's laws:

$$\nabla \times H_\varepsilon = J_\varepsilon, \quad \nabla \times E_\varepsilon = 0, \quad E_\varepsilon = A \cdot (J_\varepsilon + f) \quad \text{a.e. in } \mathbb{R}^3.$$

The resistance is here assumed to depend on the magnetic-field, as it is prescribed by the classical *Hall effect*, cf. e.g. [20]. The magnetic field H_ε must also be related by a constitutive law to the magnetic induction B_ε , which is divergence free. If the magnetic permeability is scalar and uniform, i.e., B_ε is proportional to H_ε , then we retrieve (8.1)₂.

Let Ω be a bounded Lipschitz domain of \mathbb{R}^3 and $A (= \{a_{ij} : i, j = 1, 2, 3\})$ a function $\mathbb{R}^3 \times \Omega \times \mathcal{Y} \rightarrow \mathbb{R}^9$ such that

$$\begin{aligned} A(v, \cdot, \cdot) &\text{ is measurable w.r.t. the } \sigma\text{-algebra} \\ &\text{generated by } \mathcal{B}(\Omega) \times \mathcal{L}(\mathcal{Y}), \quad \forall v \in \mathbb{R}^3 \end{aligned} \quad (8.2)$$

(our discussion would be unchanged if $\mathcal{B}(\Omega) \times \mathcal{L}(\mathcal{Y})$ were replaced by $\mathcal{L}(\Omega) \times \mathcal{B}(\mathcal{Y})$),

$$v \mapsto A(v, x, y) \text{ is continuous, for any } x \text{ and a.a. } y, \tag{8.3}$$

$$\begin{aligned} \exists c_1, c_2 \in \mathbb{R} \ (0 < c_1 < c_2) \text{ such that for any } (v, x) \text{ and a.a. } y \\ c_1 \xi_i \xi_i \leq a_{ij}(v, x, y) \xi_i \xi_j, \quad |\{a_{ij}(v, x, y) \xi_i\}| \leq c_2 |\xi| \quad \forall \xi \in \mathbb{R}^3. \end{aligned} \tag{8.4}$$

The function $(v, x) \mapsto A(v, x, \frac{x}{\varepsilon})$ is thus of Caratheodory class, and the mapping $x \mapsto A(v(x), x, \frac{x}{\varepsilon})$ is measurable in Ω whenever so is the function v .

Let $f \in L^2(\Omega)^3$ and for any $\varepsilon > 0$ consider the problem of finding $u_\varepsilon \in H_*^1(\Omega)^3$ such that (8.1) is fulfilled in $\mathcal{D}'(\Omega)$. By the compactness of the injection $H_*^1(\Omega)^3 \rightarrow L^2(\Omega)^3$, it is easy to see that this equation has a solution.

Theorem 8.1. *Assume that the conditions (8.2)–(8.4) are fulfilled. For any $\varepsilon > 0$ let $u_\varepsilon \in H_*^1(\Omega)^3$ be a solution of (8.1), and let $u_{1\varepsilon}^* \in \ell^2(H_*^1(\mathcal{Y})^3)$ be such that (extending u_ε with vanishing value outside Ω and omitting restrictions)*

$$\begin{aligned} \nabla \cdot u_{1\varepsilon}^*(m, y) = 0 \quad \text{for a.a. } y \in \mathcal{Y} \\ \int_{\mathcal{Y}} [\nabla \times u_{1\varepsilon}^*(m, y) - (\nabla \times u_\varepsilon)(\varepsilon(m + y))] \cdot \nabla \times \zeta \, dy = 0 \quad \forall \zeta \in H^1(\mathcal{Y})^3. \end{aligned} \tag{8.5}$$

(By Lemma 4.1 this problem has a unique solution.) Let us set

$$\begin{aligned} u_{1\varepsilon}(\varepsilon(m + y)) &:= u_{1\varepsilon}^*(m, y) \\ z_\varepsilon(\varepsilon(m + y)) &:= \nabla \times u_{1\varepsilon}^*(m, y) \end{aligned} \quad \text{for a.a. } y \in Y, \forall m \in \mathbb{Z}^3. \tag{8.6}$$

Then there exist $u \in H_*^1(\Omega)^3$ and $u_1 \in L^2(\Omega; H_*^1(\mathcal{Y})^3)$ such that, as $\varepsilon \rightarrow 0$ along a suitable subsequence,

$$u_\varepsilon \rightharpoonup u \quad \text{in } H_*^1(\Omega)^3, \tag{8.7}$$

$$u_{1\varepsilon} \xrightarrow{2} u_1, \quad z_\varepsilon \xrightarrow{2} \nabla_y \times u_1 \quad \text{in } L^2(\Omega \times \mathcal{Y})^3, \tag{8.8}$$

$$\nabla \times u_\varepsilon \xrightarrow{2} \nabla \times u + \nabla_y \times u_1 \quad \text{in } L^2(\Omega \times \mathcal{Y})^3. \tag{8.9}$$

This entails that

$$\begin{aligned} \iint_{\Omega \times \mathcal{Y}} \{A(u(x), x, y) \cdot [\nabla \times u(x) + \nabla_y \times u_1(x, y) + f(x)]\} \\ [\nabla \times v(x) + \nabla_y \times v_1(x, y)] \, dx \, dy = 0 \end{aligned} \tag{8.10}$$

$$\forall v \in H_*^1(\Omega)^3, \forall v_1 \in L^2(\Omega; H^1(\mathcal{Y})^3)$$

$$\nabla \cdot u = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad \nabla_y \cdot u_1 = 0 \quad \text{in } \mathcal{D}'(\Omega \times \mathcal{Y}). \tag{8.11}$$

By the arbitrariness of v and v_1 , the equation (8.10) is equivalent to the system

$$\begin{aligned} \nabla \times \int_{\mathcal{Y}} \{A(u(x), x, y) \cdot [\nabla \times u(x) + \nabla_y \times u_1(x, y) + f(x)]\} \, dy = 0 \quad \text{in } \mathcal{D}'(\Omega)^3 \\ \nabla_y \times \{A(u(x), x, y) \cdot [\nabla \times u(x) + \nabla_y \times u_1(x, y) + f(x)]\} = 0 \quad \text{in } \mathcal{D}'(\Omega \times \mathcal{Y})^3. \end{aligned} \tag{8.12}$$

Proof. In this argument all fields defined in Ω are extended to \mathbb{R}^3 with vanishing value. Multiplying (8.1) by u_ε it is easy to see that the family $\{u_\varepsilon\}$ is uniformly bounded in $H_*^1(\Omega)^3$. Theorem 3.2 then yields (8.7), (8.8) and

$$\nabla \times u_\varepsilon \xrightarrow{2} \nabla \times u + \nabla_y \times u_1 \quad \text{in } L^2(\Omega \times \mathcal{Y})^3,$$

as ε vanishes along a suitable sequence. Thus $u_\varepsilon \rightarrow u$ in $L^2(\Omega)^3$, whence

$$A(u_\varepsilon(x), x, \frac{x}{\varepsilon}) \xrightarrow{2} A(u(x), x, y) \quad \text{in } L^2(\Omega \times \mathcal{Y})^9. \quad (8.13)$$

For any $v \in \mathcal{D}(\Omega)^3$ and any $v_1 \in \mathcal{D}(\Omega \times \mathcal{Y})^3$, multiplying (8.1) by $v(x) + \varepsilon v_1(x, \frac{x}{\varepsilon})$ and integrating in Ω we get

$$\begin{aligned} & \iint_{\Omega \times \mathcal{Y}} \{A(u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot [\nabla \times u_\varepsilon(x) + f(x)]\} \\ & \cdot [\nabla \times v(x) + \varepsilon \nabla_x \times v_1(x, \frac{x}{\varepsilon}) + \nabla_y \times v_1(x, \frac{x}{\varepsilon})] dx dy = 0. \end{aligned} \quad (8.14)$$

Passing to the limit in (8.1) we then get (8.10). The equations (8.11) are direct consequences of (8.1)₂ and (8.5)₁.

Let us now come to (8.9). By (8.7), denoting by ε' a suitable subsequence, $A(u_{\varepsilon'}(x), x, \frac{x}{\varepsilon'}) \xrightarrow{2} A(u(x), x, y)$ a.e. in $\Omega \times \mathcal{Y}$. Let us set $g_{\varepsilon'} := \nabla \times u_{\varepsilon'}$ and $g := \nabla \times u + \nabla_y \times u_1$. Recalling the upper estimate of (8.4), the two-scale Vitali theorem (cf. Lemma 2.3) then yields

$$\int_{\Omega} \{[A(u_{\varepsilon'}(x), x, \frac{x}{\varepsilon'}) \cdot g_{\varepsilon'}(x)] \cdot g_{\varepsilon'}(x) - [A(u(x), x, \frac{x}{\varepsilon'}) \cdot g_{\varepsilon'}(x)] \cdot g_{\varepsilon'}(x)\} dx \rightarrow 0.$$

By (8.10) and (8.14) (here with $v = g_{\varepsilon'}$ and $v_1 \equiv 0$) we then get

$$\int_{\Omega} [A(u(x), x, \frac{x}{\varepsilon'}) \cdot g_{\varepsilon'}(x)] \cdot g_{\varepsilon'}(x) dx \rightarrow \iint_{\Omega \times \mathcal{Y}} [A(u(x), x, y) \cdot g(x, y)] \cdot g(x, y) dx dy.$$

Notice that

$$\begin{aligned} v & \mapsto \int_{\Omega} [A(u(x), x, \frac{x}{\varepsilon'}) \cdot v(x)] \cdot v(x) dx, \\ w & \mapsto \iint_{\Omega \times \mathcal{Y}} [A(u(x), x, y) \cdot w(x, y)] \cdot w(x, y) dx dy \end{aligned}$$

are the square of uniformly convex norms over $L^2(\Omega)^3$ and $L^2(\Omega \times \mathcal{Y})^3$, respectively. By a two-scale extension of the property of *compactness by strict convexity* [40], (8.12) then yields (8.9) for the subsequence $\{g_{\varepsilon'}\}$. This actually holds for the whole sequence $\{g_\varepsilon\}$, since the same argument applies to any subsequence. \square

Other quasilinear elliptic equations. If the magnetic induction $w_\varepsilon = B_\varepsilon$ is a nonlinear function of the magnetic field u_ε , then (8.1) must be coupled with the condition $\nabla \cdot w_\varepsilon \equiv 0$. If A is independent of u_ε , for any ε existence of a solution of the corresponding system can be proved via a procedure based on Murat's and Tartar's *compensated compactness* and on *compactness by strict convexity*. As ε vanishes, convergence to a homogenized problem may then be derived via two-scale extensions of those compactness techniques, cf. [39], using the spaces $L^2_{\text{rot}}(\mathbb{R}^3)^3$ and $L^2_{\text{div}}(\mathbb{R}^3)^3$, and applying of the results of Sections 4, 5.

The analogous of the system (8.1) for the gradient operator reads

$$-\nabla \cdot [A(u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot (\nabla u_\varepsilon + f)] = 0 \quad \text{in } \Omega; \quad (8.15)$$

here u_ε is a scalar function, and no further condition like (8.1)₂ is needed. If u_ε is interpreted as a temperature field, this may represent thermal equilibrium in a heterogeneous and anisotropic material, with a temperature-dependent heat conductivity tensor A . It is easy to see that this equation has at least one solution. This problem has been studied in a large literature, see the monographs quoted in the Introduction. The two-scale limit behaviour as ε vanishes may be treated along the lines of the procedure that we used for (8.1).

Finally let us come to equations of the form

$$\nabla \times [A(\nabla \times u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot \nabla \times u_\varepsilon] = f \quad \text{in } \Omega, \quad (8.16)$$

$$-\nabla \cdot [A(\nabla u_\varepsilon(x), x, \frac{x}{\varepsilon}) \cdot \nabla u_\varepsilon] = f \quad \text{in } \Omega. \quad (8.17)$$

For each of them existence of a solution may be proved, provided that the corresponding operator is monotone; if it is even cyclically monotone, then this is the Euler equation of a convex functional, and variational techniques may be used, cf. [40]. More general quasi-linear equations will be studied apart.

9. Conclusions

Let a sequence $\{u_\varepsilon\}$ be such that $u_\varepsilon \xrightarrow[2]{} u$ in $L^2(\mathbb{R}^N \times \mathcal{Y})^M$, and $P : \text{Dom}(P) \rightarrow L^2(\mathbb{R}^N)^{\tilde{M}}$ be a first-order differential operator (with constant coefficients), for suitable integers M, \tilde{M} . In each of the five cases that we dealt with in Sections 2–6, we constructed a sequence $\{u_{1\varepsilon}\}$ such that, as ε vanishes along a suitable subsequence, defining \bar{u} as in (1.7),

$$u_{1\varepsilon} \xrightarrow[2]{} u_1, \quad \varepsilon P u_{1\varepsilon} \xrightarrow[2]{} P_y u_1, \quad P u_\varepsilon \xrightarrow[2]{} P \bar{u} + P_y u_1 \quad \text{in } L^2(\mathbb{R}^N \times \mathcal{Y})^{\tilde{M}}; \quad (9.1)$$

here \bar{u} is the weak one-scale limit of $\{u_\varepsilon\}$, P_y is the *version* of P acting on the variable y . For instance, in Section 2, $M = 1$, $\tilde{M} = N$, $P = \nabla$, $\text{Dom}(P) = H^1(\mathbb{R}^N)$; in Section 4, $M = \tilde{M} = 3$, $P = \nabla \times$, $\text{Dom}(P) = L^2_{\text{rot}}(\mathbb{R}^3)^3$.

We encountered two basic situations. In Sections 2, 3 the domain of P has compact injection in $L^2(\mathbb{R}^N)^M$, so that $u_\varepsilon \rightarrow u$ in this space; the two-scale limit function then coincides with u , which does not depend on the fine-scale variable y . On the other hand in Sections 4, 5, 6 the sequence does not strongly one-scale converge in $L^2(\mathbb{R}^N)^M$, so that the two-scale limit function u may depend on y ; however of course $u_\varepsilon \rightarrow \bar{u}$ in $L^2(\mathbb{R}^N)^M$. We then represented the weak two-scale limit of Pu_ε in the form $P\bar{u} + P_y u_1$; this entails that the fine-scale information are confined to the term $\nabla_y \times u_1$. Notice that $P\bar{u}$ and $P_y u_1$ are orthogonal in $L^2(\mathbb{R}^N \times \mathcal{Y})^M$.

An alternative representation of the form $Pu + P_y \tilde{u}_1$ for the curl and divergence operators was studied by Wellander and Kristensson [41], who used a technique that is reminiscent of that of Theorem 3 of [27] and Proposition 1.14 of [1] for the gradient.

Remark added in proofs. Here we briefly compare the latter alternative representation of the two-scale limit of the curl with that of Section 4. By Theorem 4.2 $\nabla_y \times u = 0$, hence there exists $\varphi \in L^2(\mathbb{R}^3; H_*^1(\mathcal{Y}))$ such that $u(x, y) = \bar{u}(x) + \nabla_y \varphi(x, y)$ for a.a. $(x, y) \in \mathbb{R}^3 \times \mathcal{Y}$. Thus

$$\nabla \times \bar{u} = \nabla_x \times u - \nabla_x \times \nabla_y \varphi = \nabla_x \times u + \nabla_y \times \nabla_x \varphi \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathcal{Y})^3.$$

Setting $u_1^* := u_1 + \nabla_x \varphi \in H^{-1}(\mathbb{R}^3; H_*^1(\mathcal{Y})^3)$ we then infer that

$$\nabla \times \bar{u} + \nabla_y \times u_1 = \nabla_x \times u + \nabla_y \times (\nabla_x \varphi + u_1) = \nabla_x \times u + \nabla_y \times u_1^*. \quad (9.2)$$

However it is not obvious that $\nabla_x \times u \in L^2(\mathbb{R}^3 \times \mathcal{Y})^3$, and thus the latter equality a priori just holds in $H^{-1}(\mathbb{R}^3; L^2(\mathcal{Y})^3)$; in other terms, both $\nabla_x \times u$ and $\nabla_y \times u_1^*$ might be elements of the latter space.

An analogous remark concerning the representation of the two-scale limit of the divergence applies to Theorems 5.2 and 6.1.

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