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# The New Maximal Measures for Stochastic Processes

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**Abstract.** In recent work the author proposed a reformed notion of stochastic processes, which in particular removes notorious problems with uncountable time domains. In case of a Polish state space the new stochastic processes are in one-to-one correspondence with the traditional ones. This implies for a stochastic process that the traditional *canonical* measure on the path space receives a certain distinguished *maximal* measure extension which has an immense domain. In the present paper we prove, under a certain local compactness condition on the Polish state space and for the time domain $[0, \infty[$, that the maximal domain in question has, for *all* stochastic processes, three distinguished members: the set of all continuous paths, the set of all paths with one-sided limits, and its subset of those paths which at each time are either left or right continuous. In all these cases the maximal measure of the set is equal to its outer canonical measure. However, the situation will be seen to be different for the set of the càdlàg paths, for example in the Poisson process.

**Keywords.** Stochastic processes, canonical measures, maximal measures, essential subsets, set of càdlàg paths, Poisson process, Choquet capacitability theorem, inner premeasures

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## 1. Introduction and basic measure theory

The present article continues the author's contributions to the fundamentals of stochastic processes [9]–[11]. These papers are based on his work in measure and integration [6, 8], the aim of which is to build adequate new structures. This work inspired a reformed concept of stochastic processes, which in particular removes notorious problems with uncountable time domains. The new stochastic processes are in one-to-one correspondence with the traditional ones whenever the state space is a Polish topological space. It is in this situation that the present article wants to add further evidence in favour of the reformed concept of stochastic processes.

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We start with two sections of introduction. The present Section 1 recalls from the earlier survey article [8] the fundamentals of the author's work in measure and integration. The subsequent Section 2 recalls from [10] the fundamentals on the two notions of stochastic processes, the traditional and the new one, and then formulates and discusses the new results of the present paper.

The heart of the new measure theory are parallel outer and inner extension procedures for certain set functions. The outer versions are similar to the famous extension procedure due to Carathéodory (1914), and therefore look more familiar than the inner ones. But in recent years the inner versions became more and more authoritative. In particular our treatment of stochastic processes will be based on the so-called inner $\tau$ version, and hence we shall restrict ourselves to the inner extension procedures.

Let $X$ be a nonvoid set. We start to recall the famous extension procedure of Carathéodory cited above. He defines on the one hand for a set function $\Theta : \mathfrak{P}(X) \to [0,\infty]$ with $\Theta(\varnothing) = 0$ the set system

$$\mathfrak{C}(\Theta) := \{A \subset X : \Theta(M) = \Theta(M \cap A) + \Theta(M \cap A')\ \forall\, M \subset X\},$$

the members of which are called *measurable* $\Theta$. It turns out that $\Theta|\mathfrak{C}(\Theta)$ is a content on an algebra in $X$. On the other hand he defines for a set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a set system $\mathfrak{S}$ in $X$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the so-called *outer measure* $\varphi^\circ : \mathfrak{P}(X) \to [0,\infty]$ to be

$$\varphi^\circ(A) = \inf\left\{ \sum_{l=0}^{\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } \bigcup_{l=0}^{\infty} S_l \supset A \right\}.$$

His main theorem then reads as follows. *If $\varphi : \mathfrak{S} \to [0,\infty]$ is a content on a ring and upward $\sigma$ continuous, then $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ is a measure on a $\sigma$ algebra in $X$ and an extension of $\varphi$.*

In the traditional theory this theorem is the most fundamental tool in order to produce nontrivial measures. However, it has been under quite some criticism. In the traditional frame the attacks are towards the formation $\mathfrak{C}(\cdot)$, as an unmotivated and artificial one, while as a rule no doubt falls upon the outer measure formation $\varphi \mapsto \varphi^\circ$. But the new structure to be described below will disclose that the opposite is true: There are in fact serious deficiencies around the Carathéodory theorem, but it is the particular form of his outer measure which must be blamed for them, whereas the formation $\mathfrak{C}(\cdot)$ remains the decisive methodical idea and even improves when put into the adequate context. The main defects of the theorem are as follows.

1) The measure extension it produces is of an obvious *outer regular* character, like $\varphi^\circ$ itself. It is mysterious how an *inner regular* counterpart could look – while inner regular aspects become more and more important.

2) The measure extension it produces is of an obvious *sequential* character. It is mysterious how a *nonsequential* counterpart could look – while nonsequential aspects become more and more important. Both times the *sum* in the definition of $\varphi^\circ$ is a crucial obstacle.

3) The proof of the theorem suffers a complete breakdown as soon as one attempts to pass from rings $\mathfrak{S}$ to less restrictive set systems like lattices – while lattices of subsets become more and more important.

All these defects will disappear under the new structure to which we proceed now, as said above in its *inner* version. Let as before $X$ be a nonvoid set. We adopt a kind of shorthand notation, in that $\bullet = \star\sigma\tau$ marks three parallel theories, where $\star$ stands for *finite*, $\sigma$ for *sequential* or countable, and $\tau$ for *nonsequential* or arbitrary. As an example, for a nonvoid set system $\mathfrak{S}$ in $X$ let $\mathfrak{S}_\bullet$ denote the system of the intersections and $\mathfrak{S}^\bullet$ the system of the unions of the nonvoid $\bullet$ subsystems of $\mathfrak{S}$.

In the sequel we assume that $\mathfrak{S}$ is a lattice in $X$ with $\varnothing \in \mathfrak{S}$ and that $\varphi : \mathfrak{S} \to [0,\infty[$ is an isotone set function with $\varphi(\varnothing) = 0$. Our basic definitions are as follows. We define an *inner $\bullet$ extension* of $\varphi$ to be an extension $\alpha : \mathfrak{A} \to [0,\infty]$ of $\varphi$ which is a content on a ring, and such that moreover $\mathfrak{S}_\bullet \subset \mathfrak{A}$ and

$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet < \infty$), and
$\alpha$ is inner regular $\mathfrak{S}_\bullet$.

We define $\varphi$ to be an *inner $\bullet$ premeasure* iff it admits inner $\bullet$ extensions. The subsequent *inner extension theorem* characterizes those $\varphi$ which are inner $\bullet$ premeasures, and then describes all inner $\bullet$ extensions of $\varphi$. The theorem is in terms of the *inner $\bullet$ envelopes* $\varphi_\bullet : \mathfrak{P}(X) \to [0,\infty]$ of $\varphi$, defined to be

$$\varphi_\bullet(A) = \sup\Big\{\inf_{M\in\mathfrak{M}}\varphi(M) : \mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with } \mathfrak{M} \downarrow\subset A\Big\},$$

where $\mathfrak{M} \downarrow\subset A$ means that $\mathfrak{M}$ is downward directed with intersection contained in $A$. We also need their *satellites* $\varphi_\bullet^B : \mathfrak{P}(X) \to [0,\infty]$ with $B \subset X$, defined to be

$$\varphi_\bullet^B(A) = \sup\left\{\inf_{M\in\mathfrak{M}}\varphi(M) : \begin{array}{c}\mathfrak{M} \subset \mathfrak{S} \text{ nonvoid } \bullet \text{ with}\\ \mathfrak{M} \downarrow\subset A \text{ and } M \subset B\ \forall M \in \mathfrak{M}\end{array}\right\}.$$

We note that $\varphi_\bullet$ is inner regular $\mathfrak{S}_\bullet$. Moreover $\varphi = \varphi_\bullet|\mathfrak{S}$ iff $\varphi$ is downward $\bullet$ continuous, and $\varphi_\bullet(\varnothing) = 0$ iff $\varphi$ is downward $\bullet$ continuous at $\varnothing$.

**Theorem 1.1** (Inner Extension Theorem). *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ is isotone with $\varphi(\varnothing) = 0$. Then $\varphi$ is an inner $\bullet$ premeasure iff*

*$\varphi$ is supermodular and downward $\bullet$ continuous, and*
*$\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

*Equivalently,*

> $\varphi$ *is supermodular and downward* $\bullet$ *continuous at* $\varnothing$*, and*
> $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$*.*

*In this case* $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is an inner* $\bullet$ *extension of* $\varphi$*, and a measure on a* $\sigma$ *algebra when* $\bullet = \sigma\tau$*; also* $\Phi$ *is complete. All inner* $\bullet$ *extensions of* $\varphi$ *are restrictions of* $\Phi$*. Moreover we have the* localization principle *which reads*

> *for* $A \subset X$*:* $S \cap A \in \mathfrak{C}(\varphi_\bullet)$ *for all* $S \in \mathfrak{S} \Longrightarrow A \in \mathfrak{C}(\varphi_\bullet)$*.*

Thus we have $\mathfrak{S} \subset \mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$. It is plain that the members of $\mathfrak{S}_\bullet$ are the most basic measurable subsets.

The prominent rôle of $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ as the *unique maximal inner* $\bullet$ *extension* of $\varphi$ emphasizes the fundamental nature of Carathéodory's formation $\mathfrak{C}(\cdot)$. There is no such fact in the traditional context: If $\varphi : \mathfrak{S} \to [0, \infty]$ is an upward $\sigma$ continuous content on a ring $\mathfrak{S}$ in $X$ then $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ need not be a *maximal measure extension* of $\varphi$ (for example for $\mathfrak{S} = \{\varnothing, X\}$ and $\varphi \neq 0$ one has $\varphi^\circ|\mathfrak{C}(\varphi^\circ) = \varphi$).

We also note a special case of particular importance: $\mathfrak{S}$ is called $\bullet$ *compact* (in the *set theoretical* sense in contrast to the *topological* one) iff each nonvoid $\bullet$ subsystem $\mathfrak{M} \subset \mathfrak{S}$ fulfils $\mathfrak{M} \downarrow \varnothing \Rightarrow \varnothing \in \mathfrak{M}$. It is obvious that in this case the above functions $\varphi$ are all downward $\bullet$ continuous at $\varnothing$. Thus the second equivalent condition in Theorem 1.1 becomes much simpler.

The most natural example is that $X$ is a Hausdorff topological space with $\mathfrak{S} = \mathrm{Comp}(X)$. For an isotone set function $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(\varnothing) = 0$ then the three conditions $\bullet = \star\sigma\tau$ in Theorem 1.1 turn out to be identical, and if fulfilled produce the same $\varphi_\bullet$ and hence the same $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. In this case $\varphi$ is called a *Radon premeasure* and $\Phi$ the *maximal Radon measure* which results from $\varphi$. The localization principle implies that $\mathfrak{C}(\varphi_\bullet) \supset \mathrm{Bor}(X)$.

## 2. Stochastic processes and new results

The present section assumes an infinite index set $T$ called the *time domain*, and a nonvoid set $Y$ called the *state space*. One forms the $T$-fold product set $X := Y^T$, called the *path space*, the members of which are the *paths* $x = (x_t)_{t \in T} : T \to Y$. For $t \in T$ let $H_t : X \to Y$ be the canonical projection $x \mapsto x_t$. Next let $I = I(T)$ consist of the nonvoid finite subsets $p, q, \ldots$ of $T$. For $p \in I$ one forms the product set $Y^p$, with $H_p : X \to Y^p$ the canonical projection $x \mapsto (x_t)_{t \in p}$, and for the pairs $p \subset q$ in $I$ the canonical projections $H_{pq} : Y^q \to Y^p$.

In the *traditional situation* one equips $Y$ with a $\sigma$ algebra $\mathfrak{B}$ of subsets. In $X = Y^T$ one forms the finite-based product set system

$$\mathfrak{B}^{[T]} := \Big\{ \prod_{t \in T} B_t : B_t \in \mathfrak{B} \ \forall\, t \in T \text{ with } B_t = Y \ \forall\!\forall\, t \in T \Big\},$$

where $\forall\forall$ means *for all except for finitely many*, and the generated $\sigma$ algebra $\mathfrak{A} := \mathrm{A}\sigma(\mathfrak{B}^{[T]})$, which is the smallest $\sigma$ algebra $\mathfrak{A}$ in $X$ such that the $H_t$, for all $t \in T$, are measurable $\mathfrak{A} \to \mathfrak{B}$. It is well known that for uncountable $T$ the formation $\mathfrak{A}$ is too narrow, because its members $A \in \mathfrak{A}$ are *countably determined* in the sense that $A = \{x \in X : (x_t)_{t \in D} \in E\}$ for some nonvoid countable $D \subset T$ and some $E \subset Y^D$. In this frame a *traditional stochastic process* with time domain $T$ and state space $(Y, \mathfrak{B})$, for short for $T$ and $(Y, \mathfrak{B})$, can be defined as a probability measure (*prob measure* for short) $\alpha : \mathfrak{A} \to [0, \infty[$ on $\mathfrak{A}$. In view of the size of the measurable space $(X, \mathfrak{A})$ it is a nontrivial problem how to produce such stochastic processes. The standard method is via *projective limits*.

For this purpose one forms in $Y^p$ for $p \in I$ the product set system $\mathfrak{B}^p := \mathfrak{B} \times \cdots \times \mathfrak{B}$ and the generated $\sigma$ algebra $\mathfrak{B}_p := \mathrm{A}\sigma(\mathfrak{B}^p)$. Then one considers the families $(\beta_p)_{p \in I}$ of prob measures $\beta_p : \mathfrak{B}_p \to [0, \infty[$ which are *projective* in the sense that $\beta_p = \beta_q(H_{pq}^{-1}(\cdot))|\mathfrak{B}_p$ for all pairs $p \subset q$ in $I$ (which makes sense because $H_{pq}$ is measurable $\mathfrak{B}_q \to \mathfrak{B}_p$). Each stochastic process $\alpha : \mathfrak{A} \to [0, \infty[$ produces such a projective family $(\beta_p)_{p \in I}$ via $\beta_p = \alpha(H_p^{-1}(\cdot))|\mathfrak{B}_p$ (which as before makes sense because $H_p$ is measurable $\mathfrak{A} \to \mathfrak{B}_p$). One notes that the correspondence $\alpha \mapsto (\beta_p)_{p \in I}$ is *injective*, but it need not be *surjective*. The projective family $(\beta_p)_{p \in I}$ is called *solvable* iff it comes from some and hence from a unique stochastic process $\alpha : \mathfrak{A} \to [0, \infty[$, called the *projective limit* of the family $(\beta_p)_{p \in I}$. Thus a stochastic process for $T$ and $(Y, \mathfrak{B})$ can also be defined as such a *solvable projective family* $(\beta_p)_{p \in I}$, called the family of *finite-dimensional distributions* of the process.

There is a famous particular situation $(Y, \mathfrak{B})$ in which *all* projective families $(\beta_p)_{p \in I}$ for all $T$ are solvable: this is the substance of the *projective limit theorem* due to Kolmogorov (1933). The fundamental fact behind the theorem is that in a *Polish* topological space $Y$ all finite (and all locally finite) measures on $\mathrm{Bor}(Y)$ are inner regular with respect to the lattice $\mathrm{Comp}(Y)$.

**Theorem 2.1.** *Assume that $Y$ is a* Polish *space and $\mathfrak{B} = \mathrm{Bor}(Y)$ its Borel $\sigma$ algebra. Then on $(Y, \mathfrak{B})$ all projective families $(\beta_p)_{p \in I}$ for all $T$ are solvable.*

However, the traditional theory remains burdened with the defect that for uncountable time domain $T$ the $\sigma$ algebra $\mathfrak{A}$ is much too small. For example, in case $T = [0, \infty[$ and $Y = \mathbb{R}$ the subset $A = \mathrm{C}(T, \mathbb{R}) \subset X = \mathbb{R}^T$ of continuous paths is not countably determined and hence not in $\mathfrak{A}$. One of the consequences is that in its more than fifty years the theory has not been able to produce for its stochastic processes an adequate notion of *essential subsets* in the path space. These problems will disappear under the reformed concept of stochastic processes based on the author's work in measure theory described in the previous section, to which we proceed now.

In the *new situation* one equips $Y$ with a *lattice* $\mathfrak{K}$ of subsets which contains the finite subsets and is $\bullet$ compact, for the moment with $\bullet = \star\sigma\tau$. In $X = Y^T$ one forms the finite-based product set system

$$(\mathfrak{K} \cup \{Y\})^{[T]} := \Big\{ \prod_{t \in T} S_t : S_t \in \mathfrak{K} \cup \{Y\} \; \forall\, t \in T \text{ with } S_t = Y \; \forall\forall\, t \in T \Big\},$$

and $\mathfrak{S} := ((\mathfrak{K} \cup \{Y\})^{[T]})^\star$. Thus $\mathfrak{S}$ is a lattice in $X$ with $\varnothing, X \in \mathfrak{S}$ and is $\bullet$ compact by [7, 2.6]. This formation is the basic step in the new enterprise. We also form in $Y^p$ for $p \in I$ the usual product set system $\mathfrak{K}^p := \mathfrak{K} \times \cdots \times \mathfrak{K}$ and the generated lattice $\mathfrak{K}_p = (\mathfrak{K}^p)^\star$.

We turn to the relevant set functions. These are on the one hand on $X = Y^T$ the inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ with $\varphi(X) = 1$, the *inner* $\bullet$ *prob premeasures* for short, with their maximal inner $\bullet$ extensions $\Phi = \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$ (thus with $\Phi(X) = 1$). On the other hand we consider the families $(\varphi_p)_{p \in I}$ of inner $\bullet$ *prob* premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$ with their $\Phi_p$ (thus with $\Phi_p(Y^p) = 1$), which are *projective* in the sense that $\varphi_p = (\varphi_q)_\bullet(H_{pq}^{-1}(\cdot))|\mathfrak{K}_p$ for all $p \subset q$ in $I$. These entities are connected via the subsequent comprehensive counterpart [10, Theorem 11] of the classical Kolmogorov projective limit theorem 2.1.

**Theorem 2.2.** *The family of the maps*

$$\varphi \mapsto \varphi_p := \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p \quad \textit{for } p \in I$$

*defines a one-to-one correspondence between the inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ and the projective families $(\varphi_p)_{p \in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0, \infty[$. For $B \subset Y^p$ and $p \in I$, we have*

$$(\varphi_p)_\bullet(B) = \varphi_\bullet(H_p^{-1}(B)) \quad \textit{and} \quad B \in \mathfrak{C}((\varphi_p)_\bullet) \Leftrightarrow H_p^{-1}(B) \in \mathfrak{C}(\varphi_\bullet).$$

*Moreover* $\Phi(A) = \inf_{p \in I} \Phi_p(H_p(A))$ *for* $A \in \mathfrak{S}_\bullet$.

The present result appears to be more favourable than the traditional one, because the relations between the families $(\varphi_p)_{p \in I}$ and their projective limits $\varphi$ look deeper than before. But the main benefit compared with the traditional situation is that in case $\bullet = \tau$ the resultant prob measure $\Phi = \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$ on $X$ has an immense domain: In fact, even the most prominent subclass $\mathfrak{S}_\tau \subset \mathfrak{C}(\varphi_\tau)$ contains for example all $A \subset X$ of the form $A = \prod_{t \in T} K_t$ with $K_t \in \mathfrak{K} \cup \{Y\}$ for all $t \in T$, and hence reaches far beyond the class of countably determined subsets. On the other side, it remains true that all inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ are rooted in the finite subsets of $T$.

Thus we are entitled to define a *stochastic process* with time domain $T$ and *state space* $(Y,\mathfrak{K})$, for $T$ and $(Y,\mathfrak{K})$ for short, to be an inner $\tau$ prob premeasure $\varphi:\mathfrak{S}\rightarrow[0,\infty[$. The maximal inner $\tau$ extension $\Phi=\varphi_\tau|\mathfrak{C}(\varphi_\tau)$ of $\varphi$ will be called its *maximal measure*.

We proceed to the comparison with the traditional situation in the most fundamental particular case. The result is [10, Theorem 13]. Its proof combines the above Theorems 1.1 and 2.2 with the basic properties of Polish spaces.

**Theorem 2.3.** *Assume that $Y$ is a Polish space with $\mathfrak{B}=\mathrm{Bor}(Y)$ and $\mathfrak{K}=\mathrm{Comp}(Y)$. There is a one-to-one correspondence between*

*the traditional stochastic processes $\alpha:\mathfrak{A}\rightarrow[0,\infty[$ for $T$ and $(Y,\mathfrak{B})$, and*
*the new stochastic processes $\varphi:\mathfrak{S}\rightarrow[0,\infty[$ for $T$ and $(Y,\mathfrak{K})$.*

*The correspondence rests upon $\mathfrak{S}\subset\mathfrak{A}\subset\mathfrak{C}(\varphi_\tau)$ and reads $\varphi=\alpha|\mathfrak{S}$ and $\alpha=\Phi|\mathfrak{A}$. Moreover $\varphi_\tau=(\alpha^\star|\mathfrak{S}_\tau)_\star\leqq\alpha^\star$.*

In the present particular case the decisive point is that the *canonical measure* $\alpha$ with its inadequate domain $\mathfrak{A}$ receives the *maximal measure* $\Phi$ as a well-defined and highly distinguished measure extension with the immense domain $\mathfrak{C}(\varphi_\tau)$. In the traditional context there is no such extension of $\alpha$ like $\Phi$, at least beyond the case of compact $Y$ to which we shall come back at once. There were attempts to use the *outer* canonical measure $\alpha^\star$ instead, in particular via the idea of Doob [2] that the *essential* subsets $A\subset X$ for a stochastic process $\alpha$ be those with $\alpha^\star(A)=1$. But this idea is bound to fail except in particular instances; there is a drastic illustration in [10, Theorem 4]. In the new context the natural notion of an *essential* subset $A\subset X$ for a stochastic process $\varphi$ is that the maximal measure $\Phi$ lives on $A$. The sequel will reveal a certain remarkable and pleasant *partial coincidence* between the two notions.

At this point we turn to the present new results. For the first result we introduce for Hausdorff topological spaces $Y$ the condition

(COMP) There exists an isotone sequence $(K(n))_n$ of compact subsets $K(n)\subset Y$ such that each compact $K\subset Y$ satisfies $K\subset K(n)$ for some $n\in\mathbb{N}$.

Note that (COMP) is fulfilled when $Y$ is locally compact and second countable.

Now assume that $T=[0,\infty[$ and let $Y$ be a Polish space. We then define the subsets $C\subset D\subset E\subset F\subset X$ as follows: $C$ consists of the *continuous* paths $x:T\rightarrow Y$, and $F$ of the paths $x:T\rightarrow Y$ which possess all one-sided limits $x_t^\pm\in Y$ for $t\in T$, with the convention $x_0^-:=x_0$. Then $E$ consists of the paths $x\in F$ which at each $t\in T$ are either left or right continuous, and $D$ of the paths $x\in F$ which are right continuous at all $t\in T$, the so-called *càdlàg* ones. Note that none of these subsets is countably determined, and hence not a member of $\mathfrak{A}$.

**Theorem 2.4.** *Assume that* $T = [0, \infty[$ *and that the Polish space* $Y$ *fulfils* (COMP)*. For each couple* $\alpha$ *and* $\varphi$ *of stochastic processes then* $C, E, F$ *are members of* $\mathfrak{C}(\varphi_\tau)$ *and fulfil* $\alpha^\star(\cdot) = \Phi(\cdot)$.

Theorem 2.4 had certain partial predecessors in the 1959 paper of Nelson [13] and in the 1972 and 1980 books of Tjur [14, 15]. Its proof owes basic methodical ideas to [13, Theorem 3.4] in case $F$ and to [15, Proposition 10.5.1] in case $E$. The former one is close to the approach via *numbers of upcrossings* due to Doob [3, Chapter VII, Section 3] , which reappears in [14, 15]. The result [13, 3.4] has been reproduced in the 1989 textbook of Dudley [4, Theorem E. 6, p. 426].

In all these sources the adequate treatment was restricted to the case of *compact* Polish spaces $Y$ (and *compact* intervals $T$), and in place of $\Phi$ to the Radon measure on the compact product space $X = Y^T$ which results from an appropriate Radon measure version of the Kolmogorov theorem 2.1, or from the so-called regularity extension of Baire probability measures for which we refer to [4, Theorem 7.3.1] and [6, 8.14]. The step beyond compact Polish state spaces could not be done in natural manner before the new measure-theoretical foundations in [6, 8] had been laid down. The reason is that the proper kind of *compactness* required in the procedure is *not topological* compactness but the more flexible *set-theoretical* $\tau$ compactness, manifested in the formation of the lattice $\mathfrak{S}$ in [9, 10]. Before this achievement, a typical severe consequence was that the case of the state space $\mathbb{R}$ could not be treated right away, but required the problematic detour via $\overline{\mathbb{R}}$. For this point see for example Bourbaki [1, p. 120].

On the other side the new maximal measure $\Phi = \varphi_\tau | \mathfrak{C}(\varphi_\tau)$ is able to illuminate the rôle of the set $D$ of the càdlàg paths. Our subsequent second main result is for the most relevant Poisson process, in the sense of [10, Section 5]. The result is somewhat weaker than the full truth which is still in the dark, but it suffices to raise the suspicion that the actual importance of the class $D$ could be inferior to the one attributed to it in the traditional view of our days. In this connection the author is indebted to [15, Section 10.1.2] and to the 1990 paper of Dudley [5].

**Theorem 2.5.** *Assume that* $Y = \mathbb{R}$ *and* $T = [0, \infty[$*. Then for the Poisson process* $\alpha$ *and* $\varphi$ *the set* $D \subset X$ *has* $\varphi_\tau(D) = 0$*, and hence is either nonmeasurable* $\mathfrak{C}(\varphi_\tau)$ *or is in* $\mathfrak{C}(\varphi_\tau)$ *with* $\Phi(D) = 0$.

Note that $\alpha^\star(D) = 1$, for example from [10, Remark 8].

The subsequent Sections 3 and 4 will be devoted to the proof of Theorem 2.4. In Section 3 we prove a certain consequence of the Choquet capacitability theorem – or rather of its mirror assertion with co-Suslin in place of Suslin – on which our theorem is based. We also include the respective consequence of the actual Choquet theorem in view of its proper interest. The final Theorem 3.3 in Section 3 can be expected to form the basis for future relatives of the present Theorem 2.4. Then Sections 5 and 6 will present the proof of Theorem 2.5, and in fact of a much more comprehensive result.

## 3. Consequences of the Choquet capacitability theorems

Our treatement of the Choquet theorem and its mirror theorem will be based on [6, Section 10], where these theorems found substantial extensions. For the Theorems 3.1 and 3.2 below we assume a nonvoid set $X$ and a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ in $X$. Let $\mathfrak{S}^{\#}$ and $\mathfrak{S}_{\#}$ denote the Suslin and co-Suslin set systems for such an $\mathfrak{S}$ as defined in [6, Section 10].

**Theorem 3.1.** *Let $\varphi : \mathfrak{S} \to [0, \infty]$ be an outer $\bullet$ premeasure with $\Phi = \varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ ($\bullet = \sigma\tau$). Assume that $A \in (\mathfrak{S}^\bullet)^{\#}$, and either $\Phi|\mathfrak{S}^\bullet < \infty$ or $(\Phi|\mathfrak{S}^\bullet)_\sigma(A) < \infty$. Then $A \in \mathfrak{C}(\varphi^\bullet)$ and $\Phi(A) = (\Phi|\mathfrak{S}_\sigma)_\star(A)$.*

We note that the final assertion looks like in the Choquet theorem [8, 2.4] itself. However, in the present situation that theorem, applied to $\varphi^\bullet$ and $\mathfrak{S}^\bullet$ under the assumption $\varphi^\bullet|\mathfrak{S}^\bullet = \Phi|\mathfrak{S}^\bullet < \infty$, furnishes but the weaker assertion $\varphi^\bullet(A) = \big(\varphi^\bullet|(\mathfrak{S}^\bullet)_\sigma\big)_\star(A)$. Thus it is clear that some more work is required.

*Proof.* 1) From [6, Theorem 10.12 with 10.14] applied to $\Phi = \varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ and $\mathfrak{S}^\bullet \subset \mathfrak{C}(\varphi^\bullet)$ we obtain $A \in \mathfrak{C}(\varphi^\bullet)$ and $\Phi(A) = (\Phi|\mathfrak{S}^\bullet)_\sigma(A)$. To be shown is

$$\Phi(A) = \sup\{\Phi(D) : D \in \mathfrak{S}_\sigma \text{ with } D \subset A\},$$

where $\geqq$ is obvious. Fix a real $c < \Phi(A)$, and then an $\varepsilon > 0$ with $c + \varepsilon < \Phi(A)$. Then take a sequence $(A_n)_n$ in $\mathfrak{S}^\bullet$ with

$$A_n \downarrow\subset A \quad \text{and} \quad c + \varepsilon < \lim_{n\to\infty} \Phi(A_n) \leqq (\Phi|\mathfrak{S}^\bullet)_\sigma(A).$$

From both assumptions it follows that $c + \varepsilon < \Phi(A_n) < \infty$ for $n \in \mathbb{N}$.

2) Since $\Phi|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous, there are $S_n \in \mathfrak{S}$ with $S_n \subset A_n$ and $\Phi(S_n) > \Phi(A_n) - \frac{\varepsilon}{2^n}$ for $n \in \mathbb{N}$. We form $D_n := S_1 \cap \cdots \cap S_n \in \mathfrak{S}$, so that $D_n \subset S_n \subset A_n$ and $D_n \downarrow D \in \mathfrak{S}_\sigma$ with $D \subset A$. We claim that $\Phi(D_n) > \Phi(A_n) - \varepsilon\big(1 - \frac{1}{2^n}\big)$ for $n \in \mathbb{N}$; from this we obtain $\Phi(D) \geqq c$ and hence the assertion.

3) The case $n = 1$ is clear. For the induction step $1 \leqq n \Rightarrow n + 1$ we note that $D_{n+1} = D_n \cap S_{n+1}$ and $A_n \supset D_n \cup S_{n+1}$, and hence

$$\begin{aligned}\Phi(D_{n+1}) + \Phi(A_n) &\geqq \Phi(D_n) + \Phi(S_{n+1}) \\ &> \Phi(A_n) - \varepsilon\big(1 - \tfrac{1}{2^n}\big) + \Phi(A_{n+1}) - \tfrac{\varepsilon}{2^{n+1}}.\end{aligned}$$

It follows that $\Phi(D_{n+1}) > \Phi(A_{n+1}) - \varepsilon\big(1 - \frac{1}{2^{n+1}}\big)$. ☐

**Theorem 3.2.** *Let* $\varphi : \mathfrak{S} \to [0, \infty[$ *be an inner* $\bullet$ *premeasure with* $\Phi = \varphi_\bullet | \mathfrak{C}(\varphi_\bullet)$ $(\bullet = \sigma\tau)$ *and* $\Phi(X) = \varphi_\bullet(X) < \infty$. *For* $A \in (\mathfrak{S}_\bullet)_\#$ *then* $A \in \mathfrak{C}(\varphi_\bullet)$ *and* $\Phi(A) = (\Phi|\mathfrak{S}^\sigma)^\star(A)$.

*Proof.* 1) From the counterpart [6, 10.13 with 10.16] applied to $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ and $\mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$ we obtain $A \in \mathfrak{C}(\varphi_\bullet)$ and $\Phi(A) = (\Phi|\mathfrak{S}_\bullet)^\sigma(A)$. To be shown is

$$\Phi(A) = \inf\{\Phi(V) : V \in \mathfrak{S}^\sigma \text{ with } V \supset A\},$$

where $\leqq$ is obvious. Fix a real $c > \Phi(A)$, and then an $\varepsilon > 0$ with $c > \Phi(A) + \varepsilon$. Then take a sequence $(A_n)_n$ in $\mathfrak{S}_\bullet$ with $A_n \uparrow\supset A$ and $c - \varepsilon > \lim_{n\to\infty} \Phi(A_n)$. Thus $\Phi(A_n) < c - \varepsilon$ for $n \in \mathbb{N}$.

2) Since $\Phi|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous, there are $S_n \in \mathfrak{S}$ with $S_n \supset A_n$ and $\Phi(S_n) < \Phi(A_n) + \frac{\varepsilon}{2^n}$ for $n \in \mathbb{N}$. We form $V_n := S_1 \cup \cdots \cup S_n \in \mathfrak{S}$, so that $V_n \supset S_n \supset A_n$ and $V_n \uparrow V \in \mathfrak{S}^\sigma$ with $V \supset A$. We claim that $\Phi(V_n) < \Phi(A_n) + \varepsilon\big(1 - \frac{1}{2^n}\big)$ for $n \in \mathbb{N}$; from this we obtain $\Phi(V) \leqq c$ and hence the assertion.

3) The case $n = 1$ is clear. For the induction step $1 \leqq n \Rightarrow n+1$ we note that $V_{n+1} = V_n \cup S_{n+1}$ and $A_n \subset V_n \cap S_{n+1}$, and hence

$$\begin{aligned}\Phi(V_{n+1}) + \Phi(A_n) &\leqq \Phi(V_n) + \Phi(S_{n+1}) \\ &< \Phi(A_n) + \varepsilon\big(1 - \tfrac{1}{2^n}\big) + \Phi(A_{n+1}) + \tfrac{\varepsilon}{2^{n+1}}.\end{aligned}$$

It follows that $\Phi(V_{n+1}) < \Phi(A_{n+1}) + \varepsilon\big(1 - \frac{1}{2^{n+1}}\big)$. □

In this connection we recall the basic fact that a $\sigma$ algebra $\mathfrak{A}$ in $X$ which carries a *complete* finite measure satisfies $\mathfrak{A}^\# = \mathfrak{A}_\# = \mathfrak{A}$. This is a well-known consequence of the Choquet capacitability theorems, for example contained in [6, 10.12 and 10.13].

The basis for the sequel will be the consequence of Theorem 3.2 which follows. We adopt an assumption from the previous section: an infinite time domain $T$ and a Polish state space $Y$ with $\mathfrak{B}$ and $\mathfrak{K}$, and the resultant path space $X = Y^T$ with $\mathfrak{A}$ and $\mathfrak{S}$.

**Theorem 3.3.** *Let* $\alpha : \mathfrak{A} \to [0, \infty[$ *and* $\varphi : \mathfrak{S} \to [0, \infty[$ *be a couple of stochastic processes as above. Then* $(\mathfrak{S}_\tau)_\# \subset \mathfrak{C}(\varphi_\tau)$ *and* $\alpha^\star = \Phi$ *on* $(\mathfrak{S}_\tau)_\#$.

This is a wide extension of the basic fact that $\mathfrak{S}_\tau \subset \mathfrak{C}(\varphi_\tau)$ and of the special case that $\alpha^\star = \Phi$ on $\mathfrak{S}_\tau$, which is contained in the previous formula $\varphi_\tau = (\alpha^\star|\mathfrak{S}_\tau)_\star$.

*Proof.* The assertion $(\mathfrak{S}_\tau)_\# \subset \mathfrak{C}(\varphi_\tau)$ is in Theorem 3.2. For $A \in (\mathfrak{S}_\tau)_\#$ then $\Phi(A) = (\Phi|\mathfrak{S}^\sigma)^\star(A) = (\alpha|\mathfrak{S}^\sigma)^\star(A)$, because $\mathfrak{S}^\sigma \subset \mathfrak{A} \subset \mathfrak{C}(\varphi_\tau)$. We combine this with the obvious relations $(\alpha|\mathfrak{S}^\sigma)^\star(A) \geqq \alpha^\star(A) \geqq \Phi^\star(A) = \Phi(A)$ to obtain the final assertion. □

## 4. Proof of Theorem 2.4

We assume $T = [0, \infty[$ as before, and $Y$ equipped with a fixed metric $d$ and with $\mathfrak{B} = \mathrm{Bor}(Y)$ and $\mathfrak{K} = \mathrm{Comp}(Y)$ in the metric topology. As before let $X = Y^T$ consist of the paths $x = (x_t)_{t \in T} : T \to Y$.

For a nondegenerate interval $I \subset T$ we define $C(I) \subset Y^I$ to consist of the continuous paths $x \in Y^I$, and $F(I) \subset Y^I$ to consist of the paths $x \in Y^I$ which possess all relevant one-sided limits, that is $x_t^- \in Y$ in the $t \in I$ with $t > \inf I$ and $x_t^+ \in Y$ in the $t \in I$ with $t < \sup I$. Then $E(I) \subset F(I)$ consists of the paths $x \in F(I)$ which at each $\inf I < t < \sup I$ are either left or right continuous.

**Lemma 4.1.** *Let $I \subset T$ be a nondegenerate compact interval $I = [a, b]$. For $x \in F(I)$ define the* value set *$M \subset Y$ to be $M = M^\circ \cup M^- \cup M^+$ with $M^\circ = \{x_t : a \leqq t \leqq b\}$ and $M^- = \{x_t^- : a < t \leqq b\}$ and $M^+ = \{x_t^+ : a \leqq t < b\}$. Then $M \in \mathfrak{K} = \mathrm{Comp}(Y)$.*

*Proof.* To be shown is that each sequence in $M$ has a subsequence which converges to some member of $M$. A sequence in $M$ either has

($\circ$) a subsequence $\big(x_{s(n)}\big)_n$ in $M^\circ$, or

($-$) a subsequence $\big(x_{t(n)}^-\big)_n$ in $M^-$, or

($+$) a subsequence $\big(x_{t(n)}^+\big)_n$ in $M^+$.

In the latter cases there is a sequence $\big(x_{s(n)}\big)_n$ in $M^\circ$ such that $d\big(x_{t(n)}^\pm, x_{s(n)}\big) < \frac{1}{n}$ for $n \in \mathbb{N}$. This reduces the task to the situation ($\circ$). In this situation we pass to another subsequence to achieve that the sequence $\big(s(n)\big)_n$ is monotone, and once more to achieve that $\big(s(n)\big)_n$ is either constant or strictly monotone. But then the sequence $\big(x_{s(n)}\big)_n$ converges either to some member of $M^\circ$ or to some member of $M^-$ or $M^+$. □

**Proposition 4.2.** *Let $I \subset T$ be a nondegenerate compact interval $I = [a, b]$.*

1) *We have*

$$C(I) = \bigcap_{k \in \mathbb{N}} \bigcup_{r \in \mathbb{N}} \bigcap_{\mathfrak{s} \in \mathfrak{M}(I,r)} M(\mathfrak{s}, k),$$

*where $\mathfrak{M}(I, r)$ consists of the pairs $\mathfrak{s} = (u, v)$ of points $a \leqq u < v \leqq b$ with $v - u \leqq \frac{1}{r}$, and $M(\mathfrak{s}, k) := \big\{x \in Y^I : d(x_u, x_v) \leqq \frac{1}{k}\big\}$.*

2) *If $(Y, d)$ is complete, then we have*

$$E(I) = \bigcap_{k \in \mathbb{N}} \bigcup_{r \in \mathbb{N}} \bigcap_{\mathfrak{s} \in \mathfrak{M}(I,r)} M(\mathfrak{s}, k),$$

*where $\mathfrak{M}(I, r)$ consists of the triples $\mathfrak{s} = (u, v, w)$ of points $a \leqq u < v < w \leqq b$ with $w - u \leqq \frac{1}{r}$, and*

$$M(\mathfrak{s}, k) := \big\{x \in Y^I : d(x_u, x_v) \wedge d(x_v, x_w) \leqq \tfrac{1}{k}\big\}.$$

3) *If* $(Y,d)$ *is complete, then we have*

$$F(I)=\bigcap_{k\in\mathbb{N}}\bigcup_{r\in\mathbb{N}}\bigcap_{\mathfrak{s}\in\mathfrak{M}(I,r)}M(\mathfrak{s},k),$$

*where* $\mathfrak{M}(I,r)$ *consists of the sequences* $\mathfrak{s}=(u(1),v(1),\dots,u(r),v(r))$ *of points* $a\leqq u(1)<v(1)<\dots<u(r)<v(r)\leqq b$, *and*

$$M(\mathfrak{s},k):=\bigcup_{l=1}^{r}\big\{x\in Y^I: d(x_{u(l)},x_{v(l)})\leqq\tfrac{1}{k}\big\}.$$

*Proof.* 1) is clear, because it says that the $x\in C(I)$ are uniformly continuous.

2) Let $R\subset Y^T$ denote the second member.

2.i) We prove $E(I)\subset R$. Fix $x\in E(I)$ and $k\in\mathbb{N}$. For each $s\in I$ there exists $\delta(s)>0$ such that

$$\begin{aligned}&u,v\in I\cap]s-\delta(s),s[\;\Rightarrow d(x_u,x_v)<\tfrac{1}{k}\\&u,v\in I\cap]s,s+\delta(s)[\;\Rightarrow d(x_u,x_v)<\tfrac{1}{k}.\end{aligned}$$

This implies that

$$u<v<w \text{ in } I\cap]s-\delta(s),s+\delta(s)[\;\Rightarrow\; d(x_u,x_v)\wedge d(x_v,x_w)\leqq\tfrac{1}{k}.$$

In fact, the assertion is clear for $v<s$ and $v>s$, while in case $v=s$ this point must be an interior one of $I$, so that $x_v=x_v^-$ or $x_v=x_v^+$, and thus the assertion is clear as well. Now since $I$ is compact there exist $s_1,\dots,s_m\in I$ with

$$I\subset\bigcup_{j=1}^{m}\big]s_j-\tfrac{1}{2}\delta(s_j),s_j+\tfrac{1}{2}\delta(s_j)\big[.$$

We take an $r\in\mathbb{N}$ with $\frac{1}{r}\leqq\min_{1\leqq j\leqq m}\frac{1}{2}\delta(s_j)>0$. For each $\mathfrak{s}=(u,v,w)\in\mathfrak{M}(I,r)$ we have $u\in\big]s_j-\frac{1}{2}\delta(s_j),s_j+\frac{1}{2}\delta(s_j)\big[$ and hence $u,v,w\in]s_j-\delta(s_j),s_j+\delta(s_j)[$ for some $1\leqq j\leqq m$. From the above it follows that $d(x_u,x_v)\wedge d(x_v,x_w)\leqq\frac{1}{k}$, that is $x\in M(\mathfrak{s},k)$. Therefore $x\in R$.

2.ii) We prove $R\subset E(I)$. Fix $x\in R$.

2.ii.1) We start with an intermediate assertion: If $a<s\leqq b$ and $k\in\mathbb{N}$ then there exists $t\in[a,s[$ such that $t<u,v<s\;\Rightarrow\; d(x_u,x_v)\leqq\frac{4}{k}$. In fact, take an $r\in\mathbb{N}$ with

$$x\in\bigcap_{\mathfrak{s}\in\mathfrak{M}(I,r)}M(\mathfrak{s},k),$$

that is, for all $a\leqq u<v<w\leqq b$ with $w-u\leqq\frac{1}{r}$, one has $d(x_u,x_v)\wedge d(x_v,x_w)\leqq\frac{1}{k}$. Now we distinguish two cases. First assume that there exists $t\in I\cap[s-\frac{1}{r},s[$

with $d(x_t, x_s) \leqq \frac{1}{k}$. For $t < u < s$ we have $d(x_t, x_u) \wedge d(x_u, x_s) \leqq \frac{1}{k}$, and hence $d(x_u, x_s) \leqq \frac{2}{k}$. Therefore $d(x_u, x_v) \leqq \frac{4}{k}$ for all $t < u, v < s$. The opposite case is $d(x_t, x_s) > \frac{1}{k}$ for all $t \in I \cap [s - \frac{1}{r}, s[$. For $t < u < s$ we have $d(x_t, x_u) \wedge d(x_u, x_s) \leqq \frac{1}{k}$, which combined with $d(x_u, x_s) > \frac{1}{k}$ implies that $d(x_t, x_u) \leqq \frac{1}{k}$. Thus for each fixed $t \in I \cap [s - \frac{1}{r}, s[$ we have $d(x_u, x_v) \leqq \frac{2}{k}$ for all $t < u, v < s$. This proves the intermediate assertion.

2.ii.2) Since $(Y, d)$ is complete we see from 2.ii.1) that for $a < s \leqq b$ the limit $x_s^- \in Y$ exists. Likewise for $a < s \leqq b$ the limit $x_s^+ \in Y$ exists. Now fix $a < s < b$. For each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ with

$$x \in \bigcap_{\mathfrak{s} \in \mathfrak{M}(I,r)} M(\mathfrak{s}, k),$$

so that for $a \leqq u < s < v \leqq b$ with $v - u \leqq \frac{1}{r}$ one has $d(x_u, x_s) \wedge d(x_s, x_v) \leqq \frac{1}{k}$. It follows that $d(x_s^-, x_s) \wedge d(x_s, x_s^+) \leqq \frac{1}{k}$ for all $k \in \mathbb{N}$. Therefore $x_s = x_s^-$ or $x_s = s_s^+$. Thus we have $x \in E(I)$.

3) Let $R \subset Y^I$ denote the second member. For $x \in Y^I$ thus $x \in R'$ means that there exists $k \in \mathbb{N}$ such that for each $r \in \mathbb{N}$ there is an $\mathfrak{s} \in \mathfrak{M}(I, r)$ with $d(x_{u(l)}, x_{v(l)}) > \frac{1}{k}$ for all $1 \leqq l \leqq r$.

3.i) We prove $F(I) \subset R$. Assume not, and fix $x \in F(I)$ such that $x \in R'$ with some $k \in \mathbb{N}$ as above. For each $s \in I$ there exists $\delta(s) > 0$ such that

$$u, v \in I \cap ]s - \delta(s), s[ \Rightarrow d(x_u, x_v) < \tfrac{1}{k}$$
$$u, v \in I \cap ]s, s + \delta(s)[ \Rightarrow d(x_u, x_v) < \tfrac{1}{k}.$$

Since $I$ is compact there exist $s_1, \ldots, s_m \in I$ with

$$I \subset \bigcup_{j=1}^{m} ]s_j - \delta(s_j), s_j + \delta(s_j)[ = \bigcup_{j=1}^{m} ]s_j - \delta(s_j), s_j[ \cup \{s_j\} \cup ]s_j, s_j + \delta(s_j)[. \quad (\star)$$

Thus each of the $3m$ intervals $A$ in $(\star)$ fulfils $u, v \in I \cap A \Rightarrow d(x_u, x_v) < \frac{1}{k}$. Now let $r \in \mathbb{N}$, and take some $\mathfrak{s} \in \mathfrak{M}(I, r)$ with $d(x_{u(l)}, x_{v(l)}) > \frac{1}{k}$ for all $1 \leqq l \leqq r$. Then it cannot happen that two $u(l)$ for different $1 \leqq l \leqq r$ are in $I \cap A$ for the same interval $A$ in $(\star)$. Thus $r \leqq 3m$, which is a contradiction.

3.ii) We prove $R \subset F(I)$. Fix $x \in (F(I))'$, and assume for example that in some $s \in [a, b[$ the limit $x_s^+ \in Y$ does not exist. Since $(Y, d)$ is complete this means that there exists $k \in \mathbb{N}$ such that for each $\delta > 0$ there are $u, v \in I \cap ]s, s + \delta[$ with $d(x_u, x_v) > \frac{1}{k}$. Thus we obtain two infinite sequences $s < \cdots < u(r) < v(r) < \cdots < u(1) < v(1) \leqq b$ with $u(r), v(r) \downarrow s$ and $d(x_{u(l)}, x_{v(l)}) > \frac{1}{k}$ for all $l \in \mathbb{N}$. Hence the $\mathfrak{s} = (u(r), v(r), \ldots, u(1), v(1)) \in \mathfrak{M}(I, r)$ for all $r \in \mathbb{N}$ show that $x \in R'$. $\square$

**Proposition 4.3.** *Assume that the metric space* $(Y, d)$ *is complete and fulfils condition* (COMP). Note that this implies that $(Y, d)$ is separable and hence Polish. *Then the subsets* $C(T) = C, E(T) = E, F(T) = F$ *of* $Y^T = X$ *are members of* $(\mathfrak{S}_\tau)_\#$.

*Proof.* The three cases admit a common proof. For the nondegenerate intervals $I \subset T$ we write $P(I) \subset Y^I$ for each fixed one of the three $C(I), E(I), F(I)$.

1) For a nondegenerate compact interval $I \subset T$ and $k, r \in \mathbb{N}$ we form

$$P(k, r, I) := \bigcap_{\mathfrak{s} \in \mathfrak{M}(I,r)} M(\mathfrak{s}, k) \subset Y^I,$$

with the entities defined in Proposition 4.2. Thus we have

$$P(I) = \bigcap_{k \in \mathbb{N}} \bigcup_{r \in \mathbb{N}} P(k, r, I).$$

The definitions show that $P(k, r, I)$ is closed in the product topology of $Y^I$. Thus for each $K \in \mathfrak{K}$ the intersection $K^I \cap P(k, r, I)$ is compact in that product topology, and hence in view of [7, 2.4.2)] is a member of $((\mathfrak{K}^I)^\star)_\tau$. It follows that the product set $(K^I \cap P(k, r, I)) \times Y^{T \setminus I}$ is a member of

$$\big((\mathfrak{K}^I)^\star\big)_\tau \times Y^{T \setminus I} = \big((\mathfrak{K}^I \times Y^{T \setminus I})^\star\big)_\tau \subset \mathfrak{S}_\tau.$$

Therefore the product set $(K^I \cap P(I)) \times Y^{T \setminus I}$ is a member of $((\mathfrak{S}_\tau)^\sigma)_\sigma \subset (\mathfrak{S}_\tau)_\#$, with the last inclusion because $(\mathfrak{S}_\tau)_\#$ is stable under countable unions and intersections, for example by [6, 10.3].

2) From Lemma 4.1 we have

$$P(I) = \bigcup_{K \in \mathfrak{K}} K^I \cap P(I).$$

Moreover (COMP) implies that $K^I \subset \bigcup_{n \in \mathbb{N}} (K(n))^I$. Thus

$$P(I) = \bigcup_{n \in \mathbb{N}} (K(n))^I \cap P(I).$$

Combined with 1) it follows that the product set $P(I) \times Y^{T \setminus I}$ is a member of $((\mathfrak{S}_\tau)_\#)^\sigma = (\mathfrak{S}_\tau)_\#$.

3) At last we conclude from

$$P(T) = \bigcap_{m \in \mathbb{N}} P([0, m]) \times Y^{T \setminus [0,m]},$$

combined with 2) that $P(T)$ is a member of $((\mathfrak{S}_\tau)_\#)_\sigma = (\mathfrak{S}_\tau)_\#$. This is the assertion. □

Proposition 4.3 and Theorem 3.3 combine at once to furnish the first main Theorem 2.4.

## 5. Preparations for the proof of Theorem 2.5

The present final part of the paper assumes $T = [0, \infty[$ and $Y = \mathbb{R}$ with the path space $X = \mathbb{R}^T$. We start with some preparations which involve the product topology of $X$.

We define $X^\circ \subset X$ to consist of the paths $x = (x_t)_{t \in T} \in X$ with the properties

i) $x$ has values in $\mathbb{N}0 := \mathbb{N} \cup \{0\}$ with $x_0 = 0$ and is monotone increasing, and hence has one-sided limits $x_t^\pm \in \mathbb{N}0$ for $t \in T$, with the convention $x_0^- := x_0 = 0$;

ii) $x_t^+ - x_t^- \leqq 1$ for all $t \in T$;

iii) $x$ is unbounded, that is $x_t \uparrow \infty$ for $t \uparrow \infty$.

For $x \in X^\circ$ and $r \in \mathbb{N}$ we define

$$t(x, r) := \sup\{t \in T : x_t \leqq r - 1\} = \inf\{t \in T : x_t \geqq r\} \in T,$$

which is the point in $T$ where the r-th jump of $x$ takes place. We list a few immediate properties.

**Remark 5.1.** Let $x \in X^\circ$ and $r \in \mathbb{N}$.

i) For $t \in T$ we have

$$\begin{aligned} t < t(x,r) \;&\Rightarrow\; x_t \leqq r - 1 \Rightarrow\; t \leqq t(x,r), \\ t > t(x,r) \;&\Rightarrow\; x_t \geqq r \qquad\;\; \Rightarrow\; t \geqq t(x,r). \end{aligned}$$

ii) We have $0 \leqq t(x, 1) < \cdots < t(x, r) < \cdots$ and $t(x, r) \uparrow \infty$ for $r \uparrow \infty$.

iii) We have $x_t = r$ for $t(x, r) < t < t(x, r+1)$, and $x_t = 0$ for $0 \leqq t < t(x, 1)$ when $t(x, 1) > 0$. Hence $x^-_{t(x,r)} = r - 1$ and $x^+_{t(x,r)} = r$, and $x_{t(x,r)}$ is either $= r - 1$ or $= r$.

**Remark 5.2.** For $r \in \mathbb{N}$ the function $t(\cdot, r) : X^\circ \to [0, \infty[$ is continuous in the product topology of $X$ restricted to $X^\circ$.

*Proof.* We fix $a \in X^\circ$ with $t(a, r) > 0$; the case $t(a, r) = 0$ is a simpler variant. Let $0 < \varepsilon < t(a, r)$ and put $v := t(a, r) + \varepsilon$ and $u := t(a, r) - \varepsilon$. The set

$$A := \{x \in X^\circ : x_v \geqq r \text{ and } x_u \leqq r - 1\} = \{x \in X^\circ : x_v > r - 1 \text{ and } x_u < r\}$$

is open, and from 5.1.i) we see that $a \in A$ and that the $x \in A$ fulfil $u \leqq t(x, r) \leqq v$ or $|t(x, r) - t(a, r)| \leqq \varepsilon$. $\square$

Next we form in $X^\circ$ for $r \in \mathbb{N}$ the subsets

$$\begin{aligned} L(r) &:= \left\{x \in X^\circ : x_{t(x,r)} = r - 1 = x^-_{t(x,r)}\right\} \\ R(r) &:= \left\{x \in X^\circ : x_{t(x,r)} = \quad r \quad = x^+_{t(x,r)}\right\}, \end{aligned}$$

thus the sets of the paths $x \in X^\circ$ which are left/right continuous at $t(x,r)$. Note that $t(x,r) > 0$ for $x \in R(r)$. We have $X^\circ = L(r) \cup R(r)$ and $L(r) \cap R(r) = \varnothing$. The main result of the second part will be in terms of $L(r)$ and $R(r)$. The basis are the remarkable properties of the compact subsets of $L(r)$ and $R(r)$ which follow.

**Remark 5.3.** For fixed $r \in \mathbb{N}$ let $K \subset L(r)$ be compact $\neq \varnothing$. Define $t(K,r) := \sup\{t(x,r) : x \in K\}$. Thus $t(K,r) \in T$, and

$$K^r := \{x \in K : t(x,r) = t(K,r)\} \subset K$$

is compact $\neq \varnothing$ by 5.2. Also define the compact subsets

$$K_t^r := \{x \in K : x_t \geqq r\} \subset K \quad \text{for } t \in T.$$

Then

i) $K_t^r$ is monotone increasing in $t \in T$ with $K_0^r = \varnothing$ and $K_t^r = K$ for $t > t(K,r)$.

ii) In case $K_t^r \neq \varnothing$ we have $t(K_t^r, r) < t$, and $K_s^r = K_t^r$ for $t(K_t^r, r) < s < t$.

iii) We have $K^r = \{x \in K : x_{t(K,r)} = r - 1\}$, and

$$K \setminus K^r = \{x \in K : t(x,r) < t(K,r)\}$$

is $= \{x \in K : x_{t(K,r)} \geqq r\} = K^r_{t(K,r)}$, and hence is compact as well.

*Proof.* i) is clear from 5.1.i).

ii) We have $t > 0$ from i). For $x \in K_t^r$ we have $x_t \geqq r$ by definition and hence $t \geqq t(x,r)$ from 5.1.i). But $t = t(x,r)$ cannot happen since $x_{t(x,r)} = r - 1$. Thus $t > t(x,r)$ for all $x \in K_t^r$ and hence $t > t(K_t^r, r)$. Next let $t(K_t^r, r) < s < t$ For $x \in K_t^r$ then $s > t(x,r)$ and hence $x_s \geqq r$ from 5.1.i), that is $x \in K_s^r$. Thus $K_t^r \subset K_s^r$, while $K_t^r \supset K_s^r$ from i).

iii) For $x \in K$ we have

$$\begin{aligned} x \in K^r &\Rightarrow x_{t(K,r)} = x_{t(x,r)} = r - 1 && \text{because } x \in L(r) \\ x \in K \setminus K^r &\Rightarrow x_{t(K,r)} \geqq r && \text{from 5.1.i);} \end{aligned}$$

thus we have $\Leftrightarrow$ both times. □

**Remark 5.4.** For fixed $r \in \mathbb{N}$ let $K \subset R(r)$ be compact $\neq \varnothing$. Define $t(K,r) := \inf\{t(x,r) : x \in K\} \in T$. Thus $t(K,r) > 0$, and

$$K^r := \{x \in K : t(x,r) = t(K,r)\} \subset K$$

is compact $\neq \varnothing$ by 5.2. Also define the compact subsets

$$K_t^r := \{x \in K : x_t \leqq r - 1\} \subset K \quad \text{for } t \in T.$$

Then:

i) $K_t^r$ is monotone decreasing in $t \in T$ with $K_t^r = K$ for $0 \leqq t < t(K,r)$ and $\bigcap_{t\in T} K_t^r = \varnothing$, so that $K_t^r = \varnothing$ for sufficiently large $t \in T$.

ii) In case $K_t^r \neq \varnothing$ we have $t(K_t^r, r) > t$, and $K_s^r = K_t^r$ for $t < s < t(K_t^r, r)$.

iii) We have $K^r = \{x \in K : x_{t(K,r)} = r\}$, and

$$K \setminus K^r = \{x \in K : t(x,r) > t(K,r)\}$$

is $= \{x \in K : x_{t(K,r)} \leqq r-1\} = K_{t(K,r)}^r$, and hence is compact as well.

The proof is parallel to that of 5.3 in all parts. So far the preparations on the path space $X$ and its subspace $X^\circ$.

We come to the point where the measures and hence the inner premeasures enter the scene. We start to recall two basic facts: a special case of [6, Theorem 21.17 ] on product formation and [9, Theorem 3.10 ] on direct images. We adopt the former notations.

**Theorem 5.5** (Recollection). *Let $X$ and $Y$ be nonvoid sets, and $\mathfrak{S}$ in $X$ and $\mathfrak{T}$ in $Y$ be lattices with $\varnothing$. Let $\varphi : \mathfrak{S} \to [0,\infty[$ and $\psi : \mathfrak{T} \to [0,\infty[$ be inner $\bullet$ premeasures with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ and $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$ $(\bullet = \sigma\tau)$, and $\vartheta = \varphi \times \psi : (\mathfrak{S} \times \mathfrak{T})^\star \to [0,\infty[$ be their product inner $\bullet$ premeasure by* [6, *Theorem* 21.9]*; thus $\Theta = \vartheta_\bullet|\mathfrak{C}(\vartheta_\bullet)$ is an extension of $\Phi \times \Psi$. Assume that $\Psi(Y) < \infty$. Then for $E \in \mathfrak{C}(\vartheta_\bullet)$ the sections $E(x) := \{y \in Y : (x,y) \in E\} \subset Y$ for $x \in X$ fulfil*

$$\psi_\bullet(E(\cdot)) : X \to [0,\infty[ \text{ is measurable } \mathfrak{C}(\varphi_\bullet),$$

*and $\Theta(E) = \int \psi_\bullet(E(x))d\Phi(x)$.*

**Theorem 5.6** (Recollection). *Let $X$ and $Y$ be nonvoid sets and $H : X \to Y$. Let $\mathfrak{S}$ in $X$ and $\mathfrak{T}$ in $Y$ be lattices with $\varnothing$ such that*

$$(\Rightarrow)\ H(\mathfrak{S}_\bullet) \subset \mathfrak{T}_\bullet \quad \text{and} \quad (\Leftarrow)\ H^{-1}(\mathfrak{T}_\bullet) \subset \mathfrak{S}\top\mathfrak{S}_\bullet \quad (\bullet = \star\sigma\tau).$$

*Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ is an inner $\bullet$ premeasure with $\Phi = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ such that $\psi := \varphi_\bullet(H^{-1}(\cdot))|\mathfrak{T} < \infty$. Then $\psi : \mathfrak{T} \to [0,\infty[$ is an inner $\bullet$ premeasure with $\Psi = \psi_\bullet|\mathfrak{C}(\psi_\bullet)$ which fulfils $\psi_\bullet = \varphi_\bullet(H^{-1}(\cdot))$ on $\mathfrak{P}(Y)$ and $\Psi = \overrightarrow{H}\Phi$.*

After this we return to the situation $T = [0,\infty[$ and $Y = \mathbb{R}$ with $X = \mathbb{R}^T$. As before let $\mathfrak{B} = \mathrm{Bor}(\mathbb{R})$ and $\mathfrak{K} = \mathrm{Comp}(\mathbb{R})$ with the resultant $\mathfrak{A}$ and $\mathfrak{S}$ in $X$. We recall from [10, Corollary 14] that $\mathrm{Comp}(X) \subset \mathfrak{S}_\tau \subset \mathrm{Cl}(X)$. For $p = \{t(1),\ldots,t(n)\} \in I$ with $0 \leqq t(1) < \cdots < t(n)$ one forms $\mathfrak{K}_p := (\mathfrak{K}^p)^\star$ on $Y^p = \mathbb{R}^p := \mathbb{R}^n$. We recall from 2.2 the fundamental connection between the inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0,\infty[$ and the projective families $(\varphi_p)_{p\in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ $(\bullet = \star\sigma\tau)$.

As in [9, Section 6] and [10, Section 5] we fix a family $(\gamma_t)_{t\in T}$ of Radon prob premeasures $\gamma_t : \mathfrak{K} \to [0,\infty[$ with $\gamma_0 = \delta_0|\mathfrak{K}$ which under convolution fulfils $\gamma_s \star \gamma_t = \gamma_{s+t}$ for $s,t \in T$, and consider the resultant projective family $(\varphi_p)_{p\in I}$ of inner $\tau$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$. We want to deduce the inductive version of their definition, which had been written down without proof in [11, 1.1]. Note that the $\Gamma_t = (\gamma_t)_\bullet|\mathfrak{C}((\gamma_t)_\bullet)$ are independent of $\bullet = \star\sigma\tau$. After this $(\gamma_t)_{t\in T}$ will then be specialized to the Poisson semigroup.

**Proposition 5.7.** *Let* $q = \{t(0), t(1), \dots, t(n)\}$ *with* $0 \leqq t(0) < t(1) < \cdots < t(n)$ *and* $p = \{t(1), \dots, t(n)\}$. *Then for* $E \in \mathfrak{C}((\varphi_q)_\tau)$ *the function* $s \mapsto (\varphi_{p-t(0)})_\tau\big(E(s) - (s,\dots,s)\big) : \mathbb{R} \to [0,\infty[$ *is measurable* $\mathfrak{C}((\gamma_{t(0)})_\tau)$, *and*

$$(\varphi_q)_\tau(E) = \int (\varphi_{p-t(0)})_\tau\big(E(s) - (s,\dots,s)\big) d\Gamma_{t(0)}(s).$$

*Proof.* As in [9, Section 6] we use the family $(\gamma_p)_{p\in I}$ of the product inner $\tau$ prob premeasures $\gamma_p : \mathfrak{K}_p \to [0,\infty[$ formed after [7] section 1, of which the inductive definition reads $\gamma_q := \gamma_{t(0)} \times \gamma_{p-t(0)}$. Then the above 5.6 will be applied to the homeomorphisms $G_p : \mathbb{R}^p \to \mathbb{R}^p$ defined to be

$$G_p : (s_1,\dots,s_n) \mapsto (t_1,\dots,t_n) \quad \text{with } t_l = \sum_{k=1}^{l} s_k \quad \text{for } 1 \leqq l \leqq n,$$

and to the lattice $\mathfrak{K}_p$ on both sides. By [9, Proposition 6.5] this furnishes the desired projective family $(\varphi_p)_{p\in I}$ of the $\varphi_p := (\gamma_p)_\tau(G_p^{-1}(\cdot))|\mathfrak{K}_p$.

Now let $E \in \mathfrak{C}((\varphi_q)_\tau)$, which by 5.6 is equivalent to $G_q^{-1}(E) \in \mathfrak{C}((\gamma_q)_\tau)$. Thus 5.5 asserts that the function $s \mapsto (\gamma_{p-t(0)})_\tau\big((G_q^{-1}(E))(s)\big) : \mathbb{R} \to [0,\infty[$ is measurable $\mathfrak{C}((\gamma_{t(0)})_\tau)$, and that

$$(\varphi_q)_\tau(E) = (\gamma_q)_\tau(G_q^{-1}(E)) = \int (\gamma_{p-t(0)})_\tau\big((G_q^{-1}(E))(s)\big) d\Gamma_{t(0)}(s).$$

It remains to rewrite the integrand. For $s \in \mathbb{R}$ the section $(G_q^{-1}(E))(s)$ consists of the $(s_1,\dots,s_n) \in \mathbb{R}^n$ such that $(s, s_1,\dots,s_n) \in G_q^{-1}(E)$, which means

$$G_q(s, s_1,\dots,s_n) = \big(s, G_{p-t(0)}(s_1,\dots,s_n) + (s,\dots,s)\big) \in E$$

or $G_{p-t(0)}(s_1,\dots,s_n) \in E(s) - (s,\dots,s)$. Thus

$$(G_q^{-1}(E))(s) = G_{p-t(0)}^{-1}\big(E(s) - (s,\dots,s)\big),$$

so that 5.6 furnishes

$$\begin{aligned}(\gamma_{p-t(0)})_\tau\big((G_q^{-1}(E))(s)\big) &= (\gamma_{p-t(0)})_\tau\Big(G_{p-t(0)}^{-1}\big(E(s) - (s,\dots,s)\big)\Big)\\ &= (\varphi_{p-t(0)})_\tau\big(E(s) - (s,\dots,s)\big).\end{aligned}$$

The assertion follows. □

## 6. Proof of Theorem 2.5

We continue under the notions and notations of Section 5. In the sequel we specialize $(\gamma_t)_{t\in T}$ to be the Poisson semigroup, defined for $t>0$ to be

$$\gamma_t = e^{-t}\sum_{l=0}^{\infty}\frac{t^l}{l!}\delta_l|\mathfrak{K} \quad \text{and hence} \quad (\gamma_t)_\tau = e^{-t}\sum_{l=o}^{\infty}\frac{t^l}{l!}\delta_l.$$

Thus the family $(\varphi_p)_{p\in I}$ obtained above furnishes via 2.2 the Poisson process $\varphi:\mathfrak{S}\to[0,\infty[$ and its traditional version $\alpha:\mathfrak{A}\to[0,\infty[$. We recall from [10, Corollary 1 combined with Theorem 6] that $\varphi_\tau$ is inner regular $\mathrm{Comp}(X)\subset\mathfrak{S}_\tau$.

**Lemma 6.1.** *For $r\in\mathbb{N}$ and $c\in T$ we have $\varphi_\tau(\{x\in X^\circ : t(x,r)=c\})=0$.*

*Proof.* To be shown is $\varphi_\tau(K)=0$ for $K\subset X^\circ$ compact $\neq\varnothing$ with $t(x,r)=c$ for all $x\in K$. From [10, Theorem 11] we know that

$$\varphi_\tau(K)\leqq(\varphi_q)_\tau(H_q(K))=(\varphi_q)_\tau(\{(x_t)_{t\in q}:x\in K\}) \quad \text{for all } q\in I.$$

We take $q=\{u,v\}$ with $0=c=u<v$ in case $c=0$ (note that in this case $r=1$) and $0<u<c<v$ in case $c>0$. In view of $c=t(x,r)<t(x,r+1)$ for $x\in K$ and hence $c<\inf\{t(x,r+1):x\in K\}$ by 5.1.ii) and 5.2 we can take $c<v<\inf\{t(x,r+1):x\in K\}$, so that $x_v=r$ for $x\in K$ by 5.1.iii). Likewise in case $r\geqq 2$ we can take $\sup\{t(x,r-1):x\in K\}<u<c$, so that $x_u=r-1$ for $x\in K$; this relation is also true in case $r=1$ for both $c>0$ and $c=0$. It follows that $\varphi_\tau(K)\leqq(\varphi_{\{u,v\}})_\tau(\{r-1\}\times\{r\})$. Now 5.7 furnishes

$$(\varphi_{\{u,v\}})_\tau(\{r-1\}\times\{r\})=(\varphi_{\{v-u\}})_\tau(\{1\})\Gamma_u(\{r-1\})\leqq(\gamma_{v-u})_\tau(\{1\})\leqq v-u,$$

so that $\varphi_\tau(K)\leqq v-u$. The assertion follows. □

We come to the main result of the second part. In combination with [10, Section 5] it will furnish much more than the desired Theorem 2.5.

**Theorem 6.2.** *For the Poisson process we have $\varphi_\tau(L(r))=\varphi_\tau(R(r))=0$ for all $r\in\mathbb{N}$.*

*Proof.* The proofs of the two assertions are parallel in all essentials. But this is not clear a priori, because there are certain differences between left and right. Therefore we shall present both proofs.

`Proof for` $L(r)$. To be shown is $\varphi_\tau(K)=0$ for $K\subset L(r)$ compact $\neq\varnothing$. Assume that $\varphi_\tau(K)>0$. We define

$$c:=\inf\big\{t\in T:\varphi_\tau(K_t^r)=\varphi_\tau(K)\big\},$$

and obtain the properties which follow.

1) $0 \leqq c \leqq t(K,r)$ from 5.3.i).

2) $\varphi_\tau(K_c^r) < \varphi_\tau(K)$. In fact, assume that $\varphi_\tau(K_c^r) = \varphi_\tau(K) > 0$. Then $K_c^r \neq \varnothing$, so that in particular $c > 0$ from 5.3.i). From 5.3.ii) it follows that $K_s^r = K_c^r$ and hence $\varphi_\tau(K_s^r) = \varphi_\tau(K_c^r) = \varphi_\tau(K)$ for certain $0 < s < c$, in contradiction to the definition of $c$.

3) Define $M := \bigcap_{c<t<\infty} K_t^r \subset K \subset L(r)$, so that $M$ is compact with $\varphi_\tau(M) = \varphi_\tau(K) > 0$ and hence $\neq \varnothing$. We claim that $c = t(M,r)$. In fact, on the one hand we have for $t > c$ by definition $M \subset K_t^r$, for $x \in M$ therefore $x_t \geqq r$, and combined with $x_{t(x,r)} = r-1$ from $x \in L(r)$ hence $t > t(x,r)$. Therefore $c \geqq t(M,r)$. On the other hand $c > t(M,r)$ is impossible, because for $x \in M$ then $c > t(x,r)$ and hence $x_c \geqq r$ or $x \in K_c^r$, so that $M \subset K_c^r$ and hence $\varphi_\tau(M) \leqq \varphi_\tau(K_c^r) < \varphi_\tau(K)$ from 2) which is wrong. Thus $c = t(M,r)$ as claimed.

Now 5.3.iii) shows that $M = M^r \cup M^r_{t(M,r)} = M^r \cup M_c^r$ with $M^r = \{x \in M : t(x,r) = t(M,r) = c\}$ and $M_c^r$ compact and disjoint. From 6.1 we have $\varphi_\tau(M^r) = 0$. It follows that $\varphi_\tau(K) = \varphi_\tau(M) = \varphi_\tau(M_c^r) \leqq \varphi_\tau(K_c^r) < \varphi_\tau(K)$ from $M \subset K$ and 2), which is a contradiction. Thus we obtain $\varphi_\tau(K) = 0$.

`Proof for` $R(r)$. To be shown is $\varphi_\tau(K) = 0$ for $K \subset R(r)$ compact $\neq \varnothing$. Assume that $\varphi_\tau(K) > 0$. We define

$$c := \sup\{t \in T : \varphi_\tau(K_t^r) = \varphi_\tau(K)\},$$

and obtain the properties which follow.

1) $0 < t(K,r) \leqq c < \infty$ from 5.4.i).

2) $\varphi_\tau(K_c^r) < \varphi_\tau(K)$. In fact, assume that $\varphi_\tau(K_c^r) = \varphi_\tau(K) > 0$. Then $K_c^r \neq \varnothing$. From 5.4.ii) it follows that $K_s^r = K_c^r$ and hence $\varphi_\tau(K_s^r) = \varphi_\tau(K_c^r) = \varphi_\tau(K)$ for certain $s > c$, in contradiction to the definition of $c$.

3) Define $M := \bigcap_{0\leqq t<c} K_t^r \subset K \subset R(r)$, so that $M$ is compact with $\varphi_\tau(M) = \varphi_\tau(K) > 0$ and hence $\neq \varnothing$. We claim that $c = t(M,r)$. In fact, on the one hand we have for $t < c$ by definition $M \subset K_t^r$, for $x \in M$ therefore $x_t \leqq r-1$, and combined with $x_{t(x,r)} = r$ from $x \in R(r)$ hence $t < t(x,r)$. Therefore $c \leqq t(M,r)$. On the other hand $c < t(M,r)$ is impossible, because for $x \in M$ then $c < t(x,r)$ and hence $x_c \leqq r-1$ or $x \in K_c^r$, so that $M \subset K_c^r$ and hence $\varphi_\tau(M) \leqq \varphi_\tau(K_c^r) < \varphi_\tau(K)$ from 2) which is wrong. Thus $c = t(M,r)$ as claimed.

Now 5.4.iii) shows that $M = M^r \cup M^r_{t(M,r)} = M^r \cup M_c^r$ with $M^r = \{x \in M : t(x,r) = t(M,r) = c\}$ and $M_c^r$ compact and disjoint. From 6.1 we have $\varphi_\tau(M^r) = 0$. It follows that $\varphi_\tau(K) = \varphi_\tau(M) = \varphi_\tau(M_c^r) \leqq \varphi_\tau(K_c^r) < \varphi_\tau(K)$ from $M \subset K$ and 2), which is a contradiction. Thus we obtain $\varphi_\tau(K) = 0$. □

To conclude the second part we combine the new main result 6.2 with those for the Poisson process in [10, Section 5]. In the path space $X = \mathbb{R}^T$ we have the chain of subsets $C \subset D \subset E \subset F \subset X$. Beside them the present consideration was for the subset $X^\circ \subset X$ and the $L(r), R(r) \subset X^\circ$ for $r \in \mathbb{N}$. One notes that

$$X^\circ \subset E \quad \text{and} \quad X^\circ \cap D = \bigcap_{r\in\mathbb{N}} R(r).$$

The principal actor in [10, Section 5] was the set $E(T) \subset X$. We summarize the former results and add the present consequences.

1) By [10, Theorem 27] we have $E(T) \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(E(T)) = 1$.

2) The subsets $E(T), X^\circ \subset X$ are not far from each other, but none of them is contained in the other. However, by [10, Proposition 30.2]) there exists an $A \in \mathfrak{A}$ with $\alpha(A) = 1$ such that $E(T) \cap A \subset X^\circ$. From 1) of course $E(T) \cap A \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(E(T) \cap A) = 1$, and since $\Phi$ is complete it follows that $X^\circ \in \mathfrak{C}(\varphi_\tau)$ with $\Phi(X^\circ) = 1$.

3) Much weaker than 6.2 is the assertion that $\varphi_\tau(X^\circ \cap D) = 0$. It combines with 2) to furnish

$$\varphi_\tau(D) = \varphi_\tau(D \cap X^\circ) + \varphi_\tau(D \cap (X^\circ)') = 0.$$

Thus we have proved Theorem 1.2.

4) We combine 2) with 6.2 and with $X^\circ = L(r) \cup R(r)$ and $L(r) \cap R(r) = \varnothing$, to conclude that $L(r)$ and $R(r)$ are not in $\mathfrak{C}(\varphi_\tau)$. The substance of this assertion is already in Tjur [15, 10.1.2 and 10.9.4] (but not the quantitative 6.2!). However, it remains open whether the intersection $\bigcap_{r\in\mathbb{N}} R(r) = X^\circ \cap D$ and hence $D$ are in $\mathfrak{C}(\varphi_\tau)$ or not.

5) The traditional assertion on the rôle of $D \subset X$ for the Poisson process is $\alpha^\star(D) = 1$. It appears in [10, remark 29] in the sharper form $\alpha^\star(E(T) \cap D) = 1$. We conclude for $A \in \mathfrak{A}$ with $\alpha(A) = 1$ that

$$1 = \alpha^\star(E(T) \cap D) = \alpha^\star(E(T) \cap D \cap A) + \alpha^\star(E(T) \cap D \cap A') = \alpha^\star((E(T) \cap A) \cap D).$$

Thus [10, Proposition 30.2]) cited above implies that $\alpha^\star(X^\circ \cap D) = 1$.

All this manifests an essential difference between the traditional treatment, this time of the Poisson process, and the present one on the basis of the new maximal measure $\Phi = \varphi_\tau | \mathfrak{C}(\varphi_\tau)$. We shall come back to this point in a subsequent paper [12].

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