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On the Infinite Dimensionality of the Middle L^2 Cohomology of Complex Domains

By

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Introduction

In 1983, H. Donnelly and C. Fefferman [3] discovered a strikingly new phenomenon in complex analysis by establishing a vanishing theorem for the invariant L^2 cohomology. According to their result, for any strictly pseudoconvex bounded domain D in a Stein manifold of dimension n, the L^2 cohomology groups of D vanish except for that of the middle degree n. Their proof is based on a rather original estimate of H. Donnelly and F. Xavier [4] which may well be called the Hardy's inequality on manifolds. As for this new estimate, K. Takegoshi and the author [10] noticed that it is a direct consequence of Jacobi's identity, and applied it later to show an extension theorem for L^2 holomorphic functions (cf. [9]). It has applications to the Hodge theory on singular complex spaces, too. (cf. [7] and [8]). As for the L^2 cohomology in the middle degree, it was also shown in [3] that their (p, q) components are all infinite dimensional. Compared to the vanishing theorem, the basic reason for such infinite dimensionality seems to remain less transparent, although it is discussed under several different geometric situations (cf. [1], [6]). Therefore it might make sense to ask for a general geometric situation under which the infinite dimensionality is valid. The present paper is meant for that purpose.

Let D be a domain in a connected complex manifold of dimension n. By definition, a regular boundary point of D is a point $p \in \partial D$ which admits a real-valued C^∞ function φ defined on a neighbourhood $U \ni p$ such that $d\varphi(p) \neq 0$ and $U \cap D = \{x \in U;\ \varphi(x) < 0\}$. We call such φ a defining function of ∂D around p. A regular boundary point $p \in \partial D$ shall be called non-degenerate if rank $\partial\bar{\partial}e^{\varphi} = n$ at p, for any defining function φ around p. Then our result is stated as follows.

Theorem. *Let D be a domain in a connected complex manifold of dimension*

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n. Suppose that there exists a non-degenerate regular boundary point $p \in \partial D$. Let φ be any defining function around p and let ds^2 be any Hermitian metric on D satisfying

$$C^{-1}ds^2 < (-\varphi)^{-a}ds_U^2 + (-\varphi)^{-b}\partial\varphi\bar{\partial}\varphi < Cds^2$$

on $U \cap D$ for some Hermitian metric ds_U^2 defined on a neighbourhood $U \ni p$. Here a, b and C are positive numbers with $1 \leq a \leq b < a+3$. Then, for any positive integers *p and q with $p+q=n$,*

$$\dim H_{(2)}^{p,q}(D) = \infty.$$

Here $H_{(2)}^{p,q}(D)$ denote the L^2 $\bar{\partial}$-cohomology group of type (p, q) with respect to ds^2.

Note that

$$\dim H_{(2)}^{n,0}(D) = \dim H_{(2)}^{0,n}(D) = \infty$$

for any Hermitian metric on D, if D is a bounded domain in a Stein manifold of dimension n. Thus we rediscover Donnely-Fefferman's infinite-dimensionality theorem recalling that $a=1$ and $b=2$ if D is strictly pseudoconvex and ds^2 is the Bergman metric (cf. [2] or [5]).

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The Infinite Dimensionality Theorem

Let (D, ds^2) be given as in Theorem, and let $p \in \partial D$, $U \ni p$ and φ: $U \to \mathbf{R}$ be fixed likewise. Before going into the proof we prepare the notations. Let $C^{p,q}(D)$ (resp. $L^{p,q}(D)$) be the set of C^∞ (resp. square integrable) (p, q) forms on D and let $\bar{\partial}$ be the complex exterior derivative of type (0, 1). We put

$$\mathscr{D}_1^{p,q} := C^{p,q}(D)$$

$$\mathscr{D}_2^{p,q} := \{f \in \mathscr{D}_1^{p,q};\ \bar{\partial}f \in L^{p,q+1}(D)\}$$

$$\mathscr{D}_3^{p,q} := \mathscr{D}_2^{p,q} \cap L^{p,q}(D)$$

$$Z^{p,q} := \{f \in \mathscr{D}_1^{p,q};\ \bar{\partial}f = 0\}$$

$$Z_{(2)}^{p,q} := \{f \in L^{p,q}(D);\ \bar{\partial}f = 0\}$$

$$B_{(2)}^{p,q} := \{f \in L^{p,q}(D);\ \exists g \in L^{p,q-1}(D)\ s.t.\ \bar{\partial}g = f\}$$

$$H_{(2)}^{p,q}(= H_{(2)}^{p,q}(D)) := Z_{(2)}^{p,q}/B_{(2)}^{p,q}.$$

Then the operator $\bar{\partial}$: $\mathscr{D}_2^{p,q} \to Z_{(2)}^{p,q+1}$ induces a linear map, say $[\bar{\partial}]$, from $\mathscr{D}_2^{p,q}/(\mathscr{D}_3^{p,q} + Z^{p,q})$ to $H_{(2)}^{p,q+1}$. By the regularity theorem for elliptic operators $[\bar{\partial}]$ turns out to be injective. Therefore, in order to prove Theorem it suffices to show the following.

Proposition. *Under the assumption of Theorem, one has*

$$\dim \mathscr{D}_2^{p,q-1}/(\mathscr{D}_3^{p,q-1} + Z^{p,q-1}) = \infty$$

for any positive integers p and q satisfying $p+q=n$.

Proof. Let $z=(z_1,\dots,z_n)$ be a holomorphic local coordinate around p defined on a coordinate neighbourhood V. By choosing (V, z) appropriately, we may assume that z is a coordinate on $\bar{V}$, $(-1,0,\dots,0)\in D\cap V\subset U$, $\varphi_{z_1}\neq 0$ and $(\partial\bar{\partial}\varphi)^{n-1}\neq 0$ on V. We put $\zeta_i:=z_i(z_1+1)^{-1}$ for $i\geq 2$, $\zeta:=(\zeta_2,\dots,\zeta_n)$ and $\|\zeta\|^2:=\sum_{i=2}^{n}|\zeta_i|^2$. Then we take a C^∞ function $\chi:\mathbf{R}\to[0,1]$ in such a way that $\chi(\|\zeta\|)(1-\chi(|z_1+1|))$ extends to a C^∞ function on $\bar{D}$, say ρ, which is zero outside V and $=1$ on a neighbourhood of p. Note that $\rho=0$ on a neighbourhood of $\{z\in V\cap\bar{D};\ z_1=-1\}$. Thus for any C^∞ function g in ζ, we can regard ρg as a C^∞ function on $\bar{D}$. From the assumptions on ds^2 and ζ, we find that there exists a constant C_0 such that

$$C_0^{-1}ds^2<(-\varphi)^{-a}\sum_{i=2}^{n}d\zeta_i d\bar{\zeta}_i+(-\varphi)^{-b}\partial\varphi\bar{\partial}\varphi<C_0 ds^2$$

on supp $\rho\cap D$. For positive integers p and q with $p+q=n$ we put

$$u:=\bar{\zeta}_{q+1}\rho g\,d\bar{\zeta}_2\wedge\cdots\wedge d\bar{\zeta}_q\wedge d\zeta_2\wedge\cdots\wedge d\zeta_{p+1},$$

where g is a C^∞ function of ζ which shall be specified later. As above, we regard u as a C^∞ $(p, q-1)$ form on $\bar{D}$. Since $\partial\bar{\partial}\varphi$ is non-degenerate on the complex tangent spaces of $\partial D\cap V$, we can split $\bar{\partial}u$ as

$$\bar{\partial}u=\partial\bar{\partial}\varphi\wedge v_1+\partial\varphi\wedge v_2+\bar{\partial}\varphi\wedge v_3+\varphi v_4,$$

for some C^∞ forms v_i on $\bar{D}$, where v_1 is chosen in particular in such a way that

$$v_1|_{\partial D\cap V}=\sum_{I,J}v_{1IJ}d\zeta_I\wedge d\bar{\zeta}_J|_{\partial D\cap V}$$

for some (uniquely determined) C^∞ functions v_{1IJ} on $\partial D\cap V$. We put $w:=u+\partial\varphi\wedge v_1$. Then $\bar{\partial}w\in L^{p,q}(D)$, since $\partial\varphi\wedge\bar{\partial}v_1$, $\partial\varphi\wedge v_2$, $\bar{\partial}\varphi\wedge v_3\in L^{p,q}(D)$ and $\varphi v_4\in L^{p,q}(D)$ by the condition $a+3>b$. Therefore $w\in\mathscr{D}_2^{p,q-1}$. Suppose that $w\in\mathscr{D}_3^{p,q-1}+Z^{p,q-1}$. Then $\bar{\partial}w=\bar{\partial}w_0$ for some $w_0\in\mathscr{D}_3^{p,q-1}$. We express w_0 on $V\cap D$ as

$$w_0=\sum_{I,J}w'_{0IJ}d\zeta_I d\bar{\zeta}_J+\partial\varphi\wedge\sum_{K,L}w''_{0KL}d\zeta_K d\bar{\zeta}_L\bar{\partial}\varphi\wedge w'''_0,$$

where I, J, K, L run through the multi-indices with $|I|+|J|=n-1$ and $|K|+|L|=n-2$. Then the L^2 condition implies

$$(1)\qquad \left|\int_{D\cap V}\varphi^{-a}|w''_{0KL}|^2dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n\right|<\infty$$

for any K and L.

Since $a \geq 1$, (1) shows that there exists a sequence $\{\varepsilon_\mu\}_{\mu=1}^{\infty} \subset (-\infty, 0)$ such that $\lim_{\mu \to \infty} \varepsilon_\mu = 0$ and

$$\lim_{\mu \to \infty} \int_{\varphi^{-1}(\varepsilon_\mu) \cap V} |w''_{0KL}|^2 ds_{(\varepsilon_\mu)} = 0. \tag{2}$$

Here $ds_{(\varepsilon_\mu)}$ denote the volume form of the hypersurface $\varphi^{-1}(\varepsilon_\mu) \cap V$, which is induced from the metric $\sum_{i=1}^{n} dz_i d\bar{z}_i$.

Since w is C^∞ on $\bar{D}$, (2) immediately implies that

$$\bar{\partial} v_1 \wedge \bar{\partial}\varphi = 0 \text{ on } \partial D \cap V \cap (\text{supp } v_2)^c. \tag{3}$$

Clearly (3) does not hold if we choose g in advance to be a nonzero polynomial in ζ_2 without the constant term. Thus we have finished the proof of Proposition. Q.E.D.

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