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Uniqueness in Determining Polygonal Periodic Structures

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Abstract. We consider the inverse problem of recovering a two-dimensional perfectly reflecting diffraction grating from scattered waves measured above the structure. We establish the uniqueness within the class of general polygonal grating profiles by a minimal number of incoming plane waves, without excluding Rayleigh frequencies and further geometric constraints on the profile. This extends and improves the uniqueness results of Elschner, Schmidt and Yamamoto [Inverse Problems 19 (2003), 779–787].

Keywords. Diffraction grating, periodic Helmholtz equation, inverse Dirichlet and Neumann problems, polygonal grating profile

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1. Introduction

The problem of recovering a periodic structure from knowledge of the scattered field occurs in many applications, e.g., in diffractive optics; see [3, 18]. In this paper we consider the scattering of monochromatic plane waves by a perfectly reflecting diffraction grating in an isotropic lossless medium. Our goal is to prove global uniqueness in determining polygonal periodic grating profiles by near field observations with a minimal number of incident waves.

Let the profile of the diffraction grating be given by a Lipschitz curve $\Lambda \subset \mathbb{R}^2$ which is 2π-periodic with respect to x_1, where $(x_1, x_2) \in \mathbb{R}^2$. The unbounded domain above Λ is denoted by Ω_Λ. Suppose that a plane wave given by

$$u^i(x) := \exp(i\alpha x_1 - i\beta x_2), \quad (\alpha, \beta) = k(\sin\theta, \cos\theta)$$

is incident from the top, where $k > 0$ is the wave number and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the incident angle.

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We consider the scattering of u^i in the case of a perfectly reflecting grating profile Λ, which is modeled by the Dirichlet problem (TE polarization) or the Neumann problem (TM polarization). Then the total field $u = u(x_1, x_2)$, which is the sum of u^i and the scattered field u^s, satisfies

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_\Lambda, \qquad u = 0 \quad \text{or} \quad \partial_\nu u = 0 \quad \text{on } \Lambda, \tag{1}$$

and is assumed to be α-quasiperiodic in x_1:

$$u(x_1 + 2\pi, x_2) = \exp(2i\alpha\pi) u(x_1, x_2)\,. \tag{2}$$

Furthermore, u^s is required to satisfy the radiation condition

$$u^s(x) = \sum_{n\in\mathbb{Z}} A_n \exp\big(i(n+\alpha)x_1 + i\beta_n x_2\big) \quad \text{for } x_2 \text{ sufficiently large}, \tag{3}$$

with the Rayleigh coefficients $A_n \in \mathbb{C}$ and

$$\beta_n := \begin{cases} \big(k^2 - (n+\alpha)^2\big)^{\frac{1}{2}}, & \text{if } |n+\alpha| \le k \\ i\big((n+\alpha)^2 - k^2\big)^{\frac{1}{2}}, & \text{if } |n+\alpha| > k\,. \end{cases} \tag{4}$$

Note that β_n is real for at most a finite number of indices.

There always exists a solution $u \in H^1_{loc}(\Omega_\Lambda)$ of problem (1)–(3) which need not be unique in general. For the existence result see, e.g., [5, 7] where the more general TE and TM transmission problems have been studied, and nonuniqueness examples for both the direct Dirichlet and Neumann problems can be found in [13]. Moreover, it is known that the solution to the Dirichlet problem is always unique if the profile curve Λ is given by the graph of a function (see [16] for C^2 and [8] for Lipschitz functions), whereas this may be not true for the Neumann problem [15].

The inverse Dirichlet or Neumann problem can now be formulated as follows:

(IP): Determine the profile Λ from the knowledge of one wave number k, possibly several incident directions θ, and the total field $u|_{x_2=b}$ on a straight line $\{x \in \mathbb{R}^2 : x_2 = b\}$ lying in Ω_Λ.

Note that this problem also involves near field measurements since the evanescent modes cannot be measured far away from the grating profile.

In general, global uniqueness with one incident wave in problem (IP) is not true. This can be seen from the simple counterexample of the scattering of $u^i = \exp(-ikx_2)$ when one moves the flat grating in certain multiples of the wavelength. It was shown in [14] that a finite number of incident waves are sufficient to recover a C^2 grating profile from the total field above the structure.

In particular, one obtains the global uniqueness with one incident direction in the inverse Dirichlet problem if the wave number or the amplitude of the grating is sufficiently small.

Global uniqueness results for the inverse Dirichlet and Neumann problems with a minimal number of incident waves were first established within the class of profiles given by the graph of a piecewise linear function or a step function [10, 11]. The purpose of this paper is to extend these results to the practically important case of general polygonal grating profiles. Moreover, contrary to [10], we can allow Rayleigh frequencies and are able to remove the additional geometric assumption on the profile made in the Neumann case. Now we state our main result.

Theorem. *Let Λ_1 and Λ_2 be 2π-periodic polygonal profiles consisting of finitely many segments, and let us exclude the case where Λ_1 and Λ_2 are parallels to the x_1 axis. Let $u_j = u(\Lambda_j;\theta)$ satisfy the corresponding direct diffraction problem (1)–(3) in $\Omega_j = \Omega_{\Lambda_j}$, $j = 1, 2$, and choose b such that $\{x \in \mathbb{R}^2 : x_2 = b\} \subset \Omega_1 \cap \Omega_2$.*

(i) *In the Dirichlet case the relations*

$$u_1(x_1, b) = u_2(x_1, b) \quad \text{for all } x_1 \in (0, 2\pi) \tag{5}$$

for two different incident angles θ imply $\Lambda_1 = \Lambda_2$. If one excludes the Rayleigh frequencies by assuming

$$\beta_n \neq 0\,, \quad \text{i.e.,} \quad k^2 \neq (n+\alpha)^2 \quad \text{for all } n \in \mathbb{Z}\,, \tag{6}$$

then the relation (5) for one incident wave is sufficient.

(ii) *For the inverse Neumann problem, we have $\Lambda_1 = \Lambda_2$ if the relations (5) hold for four different incident directions, whereas three incoming waves are enough if the Rayleigh frequencies are excluded for each incident angle θ.*

It will be shown by appropriate counterexamples that a smaller number of incident waves is not sufficient to determine the grating profile uniquely, in general. We refer to the remarks at the end of the paper.

The proof of the theorem is carried out in the next section and relies on a refinement of the arguments in [10] in combination with those developed in [1, 6, 9] for inverse scattering by polygonal sound-soft and sound-hard obstacles.

2. Proof of the theorem

2.1. Finding an "exit direction". Arguing by contradiction, let $\Lambda_1 \neq \Lambda_2$ be two periodic polygonal curves, with the case $\Lambda_j = \{x \in \mathbb{R}^2 : x_2 = c_j\}$, $c_1 < c_2$, excluded. Consider the solutions $u_j = u(\Lambda_j;\theta) \in H^1_{loc}(\Omega_j)$ of the corresponding

direct diffraction problems, and let Ω be the unbounded connected component of $\Omega_1 \cap \Omega_2$. Note that by elliptic regularity, each function u_j is infinitely smooth up to the boundary, with the exception of the corner points of Λ_j. Moreover, since u_j satisfies the Helmholtz equation, u_j is real-analytic in Ω_j. Since $u_1 = u_2$ and then also $\partial_2 u_1 = \partial_2 u_2$ on $\{x_2 = b\}$, we have

$$u(x) = u(\theta)(x) := u_1(x) = u_2(x) \quad \text{in } \Omega\,; \tag{7}$$

see, e.g., [2]. Henceforth by a ray we mean a straight line starting from one point and extended to infinity.

The following auxiliary result extends Lemma 2 in [10] to general polygonal grating profiles. Note that the proof is more straightforward in the case where Λ_1 and Λ_2 are given by the graphs of piecewise linear functions; see [10, Sec. 2.2].

Proposition 1. *There exists a ray $S \subset \Omega$ such that $u|_S = 0$ in the Dirichlet case and $\partial_\nu u|_S = 0$ in the Neumann case.*

Proof. Henceforth A_1A_2 stands for the open segment in $\mathbb{R}^2$ with end points A_1 and A_2. Since $\Lambda_1 \neq \Lambda_2$, there exists a segment $A_1A_2 \subset \partial\Omega \cap \Omega_1$ without loss of generality (see [9, Sec. 3] for the details), and by (7) we have $u_1 = 0$ resp. $\partial_\nu u_1 = 0$ on A_1A_2. Thus the set $\mathcal{G}$ defined by

$$\mathcal{G} = \left\{ S : \begin{array}{l} S \text{ is a (finite or infinite) open segment extended to maximal length} \\ \text{in } \Omega_1 \text{ such that } u_1 = 0 \text{ resp. } \partial_\nu u_1 = 0 \text{ on } S \end{array} \right\}$$

is not empty. If $\mathcal{G}$ contains an infinite segment, then we already have the desired ray. So we may assume that the subset of finite segments is not empty, and by periodicity we can restrict ourselves to the set

$$\mathcal{G}_0 = \left\{ S : \begin{array}{l} S \text{ is a finite open segment with both ends on } \Lambda_1, \text{ starting in a fixed} \\ \text{period, say } 0 \le x_1 \le 2\pi, \text{ and ending in the same, in the preceding} \\ \text{or subsequent period, such that } u_1 = 0 \text{ resp. } \partial_\nu u_1 = 0 \text{ on } S \end{array} \right\}.$$

Proceeding similarly to the case of scattering by polygonal bounded obstacles (cf. [1, 6, 9]), we can prove that $\mathcal{G}_0$ is a finite set, which then implies the existence of a ray S with the desired properties or a contradiction to the assumption that $\Lambda_1 \neq \Lambda_2$. Let us give a sketch of the finiteness proof, first for the inverse Dirichlet problem and then for the Neumann case, and we refer to [6, 9] for the details.

If $\mathcal{G}_0$ contains infinitely many segments, we can choose sequences of points $\{P_j\}$, $\{Q_j\}$ such that $P_j \neq P_{j'}$ if $j \neq j'$, $P_j, Q_j \in \Lambda_1$, $P_jQ_j \subset \Omega_1$ and $u_1 = 0$ on P_jQ_j for all j. Since the length of Λ_1 restricted to $\{x \in \mathbb{R}^2 : -2\pi \le x_1 \le 4\pi\}$ is finite, we find subsequences $P_j \to P_\infty$, $Q_j \to Q_\infty$, lying at one side of P_∞, Q_∞, respectively. Moreover, P_j are not vertices of Λ_1 and P_jP_{j+1}, $Q_jQ_{j+1} \subset \Lambda_1$

for all j. It may happen that the stationary sequence $Q_j = Q_\infty$ occurs or that $P_\infty = Q_\infty$. Consider the polygonal open set D_j bounded by the segments P_jQ_j, P_jP_{j+1}, $P_{j+1}Q_{j+1}$, Q_jQ_{j+1}, which is a triangle or quadrangle, or consists of two triangles with a common vertex.

For any j, u_1 satisfies the homogeneous Dirichlet problem for the Helmholtz operator $\Delta+k^2$ on D_j, and the area of D_j tends to zero as $j \to \infty$. However, this contradicts the Poincaré inequality for H^1 functions with vanishing boundary values (see, e.g., [12, Ch. 7.8]) and thus proves the finiteness of the set $\mathcal{G}_0$.

In the case of the Neumann problem, more sophisticated arguments are needed to show that the corresponding set $\mathcal{G}_0$ is finite. As in [6] or [9, Sec. 3], we can construct a sequence of triangles D_j, whose diameters converge to zero as $j \to \infty$, and such that u_1 satisfies the homogeneous Neumann problem for the Helmholtz operator on D_j for all j. Then we obtain a contradiction to the optimal Poincaré inequality of [17] for planar convex domains. In the n-dimensional case when $n \geq 3$, a gap in the original proof of this inequality was indicated and fixed in [4].

We now complete the proof of Proposition1 following the proof of Lemma 3.7 in [1]. Consider the open set

$$G^* := \Big(\Omega_1 \setminus \bigcup_{S\in\mathcal{G}_0} S\Big) \cap \big\{x \in \mathbb{R}^2 : 0 < x_1 < 2\pi\big\},$$

which has one unbounded connected component G_∞ and a finite number of bounded connected components. We note that there exists only one unbounded connected component, because the boundaries of any components of $\Omega_1 \setminus \mathcal{G}_0$ consist of finite segments.

Now we choose a point $P \in \partial G_\infty$ lying on exactly one open segment S of $\mathcal{G}_0$, and select points $P^+ \in G_\infty$ and $P^- \in G^* \setminus \bar{G}_\infty$ sufficiently close to P. Moreover, we can choose a continuous curve $\gamma(t)$, $t \in [t_1, \infty)$, which intersects the set $\mathcal{G}_0$ only at the point $P \in S$ and satisfies, for some $t_2 > \bar{t} > t_1$,

$$\begin{aligned} \gamma(t_1) = P^-; &\quad \gamma(t) \in G^* \setminus \bar{G}_\infty, \ t \in (t_1, \bar{t}) \\ \gamma(\bar{t}) = P; &\quad \gamma(t) \in G_\infty, \qquad t > \bar{t} \end{aligned}$$

$$\text{with } \gamma(t_2) = P^+ \text{ and } \gamma(t) \text{ leading to infinity as } t \to \infty\,.$$

We will find a segment $S^* \neq S$ with $u_1 = 0$ on S^*, intersecting γ at some $t > \bar{t}$. Then, either S^* can be extended to a ray in Ω on which u_1 vanishes (an exit direction), or we have a finite segment belonging to $\mathcal{G}_0$ which is a contradiction.

Let $G^\pm$ be the connected components of $(\Omega_1 \setminus S) \cap \{x \in \mathbb{R}^2 : 0 < x_1 < 2\pi\}$ containing $P^\pm$, and consider the connected components $E^\pm$ of $G^\pm \cap \Pi(G^\mp)$ containing $P^\pm$, where Π denotes the symmetric transform with respect to the

extended straight line of S. Then $E^- = \Pi(E^+)$. Let $E = E^- \cup E^+ \cup S$. Note that ∂E consists of segments of Λ_1 and $\Pi(\Lambda_1)$.

Since u_1 is odd symmetric with respect to S, we obtain $u_1 = 0$ on ∂E. Note that E is bounded since G^- is bounded. Therefore, for some $t > \bar{t}$, $\gamma(t)$ intersects $\partial E \cap G_\infty$ and thus another segment $S^* \neq S$ with $u_1 = 0$ on S^*. This finishes the proof of Proposition 1 in the Dirichlet case.

The proof in the Neumann case is analogous; we only have to use the fact that u_1 is even symmetric with respect to S in the above argument. □

2.2. Reduction to a finite sum of propagating waves. The function u defined in (7) has the Rayleigh expansion

$$\begin{aligned} u(x) &= u(\theta)(x) \\ &= \Big(A\exp(i\alpha x_1 - i\beta x_2) + \sum_{n\in P} A_n \exp\big(i(\alpha+n)x_1 + i\beta_n x_2\big)\Big) \\ &\quad + \Big(\sum_{n\in\mathbb{Z}\setminus P} A_n \exp\big(i(n+\alpha)x_1 + i\beta_n x_2\big)\Big) := v + w \qquad (x_2 > b)\,, \end{aligned} \tag{8}$$

where $A = 1$ and P denotes the finite set $\{n \in \mathbb{Z} : \beta_n \in \mathbb{R}\}$. Note that $-i\beta_n \geq C > 0$ for all $n \in \mathbb{Z}\setminus P$ and $\beta_n \sim |n|i$ as $|n| \to \infty$. Of course, α, β, β_n and A_n depend on the incident direction θ.

Proposition 2. *The existence of an exit direction (cf. Proposition* 1*) implies that* $w = 0$ *in* (8)*, i.e.,* $A_n = 0$ *for all* $n \in \mathbb{Z}\setminus P$.

Proof. Since a translation of the x coordinates only amounts to different coefficients in the expansion (8) with $A \neq 0$, we can assume that the ray S of Proposition 1 starts at $x = 0$. We have

$$\partial_\nu u = -\sin\phi\, \partial_1 u + \cos\phi\, \partial_2 u \quad \text{on } S, \tag{9}$$

if the ray S is given by $S = \{(t\cos\phi, t\sin\phi) : t > 0\}$. We have to consider the following two cases:

(i) `Dirichlet case`: As in [10], $u|_S = 0$ implies $A_n = 0$ for all $n \in \mathbb{Z}\setminus P$ if $\phi \neq 0$. For $\phi = 0$, we obviously even have $A_n = 0$ for $n \neq 0$.

(ii) `Neumann case`: From (8) and (9) we have

$$\begin{aligned} \partial_\nu u(x) &= B\exp(i\alpha x_1 - i\beta x_2) + \sum_{n\in\mathbb{Z}} B_n \exp\big(i(\alpha+n)x_1 + i\beta_n x_2\big) \quad \text{on } S \\ B &= (-i\alpha\sin\phi - i\beta\cos\phi)A \\ B_n &= \big(-i(n+\alpha)\sin\phi + i\beta_n\cos\phi\big)A_n\,, \quad n \in \mathbb{Z}\,. \end{aligned} \tag{10}$$

Then, for $n \in \mathbb{Z}\backslash P$, we have $B_n \neq 0$ if and only if $A_n \neq 0$. Note that $\beta_n \notin \mathbb{R}$ for these n, implying the assertion for $\phi \neq \frac{\pi}{2}$, and that by (4) $|n+\alpha| > k > 0$ for $n \in \mathbb{Z}\backslash P$ if $\phi = \frac{\pi}{2}$. Then Proposition 2 follows as in the Dirichlet case; see [10]. □

2.3. End of proof in the Dirichlet case. We now have from (8) and Proposition 2 that

$$u = v = A\exp(i\alpha x_1 - i\beta x_2) + \sum_{n\in P} A_n \exp\big(i(\alpha+n)x_1 + i\beta_n x_2\big),\ A \neq 0. \quad (11)$$

Moreover, v is analytic in $\mathbb{R}^2$ and satisfies $v|_L = 0$ on each straight line L extending a segment of $\Lambda_1 \cup \Lambda_2$. So, by assumption, there exist at least two lines L_1, L_2 intersecting at $x = 0$, where v vanishes. Let ψ denote the polar angle between L_1 and L_2. Using the reflection argument with odd extension, we end up with the following two cases (cf. [10]):

(i): $\psi = \lambda\pi,\ \lambda \in (0,1)$ irrational, and $v|_{L_{j\psi}} = 0$ for all $j \in \mathbb{N}$.

Here $L_{j\psi}$ denotes the line of polar angle $j\psi$ with respect to L_1. Since the directions of $L_{j\psi}$ are dense in $[0, 2\pi)$, we obtain $v = 0$ in $\mathbb{R}^2$ which is a contradiction to $A \neq 0$. Therefore, case (i) cannot occur.

(ii): $\psi = \lambda\pi,\ \lambda$ rational, and $v|_{L_{j\pi/N}} = 0,\ j = 0, \ldots, N-1$, for some integer $N \geq 2$.

Since v is odd symmetric with respect to the lines $L_{j\pi/N}$, we obtain $v(x) = (-1)^N v(-x)\,, \quad x \in \mathbb{R}^2\,$, or equivalently,

$$\begin{aligned} &A\exp(i\alpha x_1 - i\beta x_2) + \sum_{n\in P} A_n \exp\big(i(n+\alpha)x_1 + i\beta_n x_2\big) \\ &= (-1)^N A\exp(-i\alpha x_1 + i\beta x_2) + (-1)^N \sum_{n\in P} A_n \exp\big(-i(n+\alpha)x_1 - i\beta_n x_2\big); \end{aligned} \quad (12)$$

see [10]. This implies, for some $n_0 \in P$ and all $x \in \mathbb{R}^2$,

$$(-1)^N A\exp(-i\alpha x_1 + i\beta x_2) = A_{n_0}\exp\big(i(n_0+\alpha)x_1 + i\beta_{n_0}x_2\big)\,,$$

and because of $A \neq 0$,

$$n_0 = -2\alpha, \quad \beta_{n_0} = \beta, \quad A(-1)^N = A_{n_0}\,. \quad (13)$$

Moreover, (12) gives for all $x \in \mathbb{R}^2$

$$\begin{aligned} &\sum_{n\in P\backslash\{n_0\}} A_n \exp\big(i(n+\alpha)x_1 + i\beta_n x_2\big) \\ &\qquad = (-1)^N \sum_{n\in P\backslash\{n_0\}} A_n \exp\big(-i(n+\alpha)x_1 - i\beta_n x_2\big)\,. \end{aligned} \quad (14)$$

Therefore, for any $n \in P\backslash\{n_0\}$, we must have either $A_n = 0$, or $\beta_n = 0$ in which case a Rayleigh frequency occurs. If there is no Rayleigh frequency, we have $A_n = 0$ for all $n \in P\backslash\{n_0\}$. Otherwise there exists $n_1 \in \mathbb{Z}$ with $\beta_{n_1} = 0$ and $n_1 + \alpha = k$, and by equality (14) there is another index $n_2 \in \mathbb{Z}$ such that

$$\beta_{n_1} = \beta_{n_2} = 0, \quad n_1 + \alpha = -(n_2 + \alpha) = k, \quad A_{n_2} = (-1)^N A_{n_1}\,. \tag{15}$$

Furthermore, we have $A_n = 0$ if $n \neq n_0$, n_1, n_2. Hence, in terms of (11), (13) and (15), for even N, v takes the form

$$v(x) = A\cos(\alpha x_1 - \beta x_2) + B\cos(kx_1), \quad A \neq 0, \tag{16}$$

where $B = 0$ if there is no Rayleigh frequency. Similarly, for odd N, cos has to be replaced by sin:

$$v(x) = A\sin(\alpha x_1 - \beta x_2) + B\sin(kx_1), \quad A \neq 0. \tag{17}$$

To complete the proof of the theorem in the Dirichlet case, we now investigate the relation $v|_L = 0$, where v is given by (16) or (17), and L is the straight line given by $L = L(\phi) = \{(t\cos\phi, t\sin\phi) : t \in \mathbb{R}\}$, where $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Since $(\alpha, \beta) = k(\sin\theta, \cos\theta)$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\alpha x_1 - \beta x_2 = tk\sin(\theta - \phi)$ on $L(\phi)$, and (16) can be written as

$$A\cos(tk\,\sin(\theta - \phi)) + B\cos(tk\,\cos\phi) = 0, \quad t \in \mathbb{R}. \tag{18}$$

This is impossible for $B = 0$. To explore the case $B \neq 0$, we note that for a, $b \in \mathbb{R}$ the functions

$$\cos(at),\ \cos(bt) \text{ are linearly independent on } \mathbb{R} \quad \text{if and only if} \quad a = \pm b. \tag{19}$$

Hence (18) implies that

$$\sin(\theta - \phi) = \pm\cos\phi = \pm\sin\left(\frac{\pi}{2} - \phi\right) \tag{20}$$

giving the relations $\theta - \phi = \frac{\pi}{2} + \phi$ or $\theta - \phi = \phi - \frac{\pi}{2}$ under the above constraints on θ and ϕ.

Therefore, the functions (16) can only vanish on the line $L(\phi)$ if, for the given incident direction θ,

$$\phi = \frac{\theta}{2} - \frac{\pi}{4} \qquad \text{or} \qquad \phi = \frac{\theta}{2} + \frac{\pi}{4}\,. \tag{21}$$

On the other hand, we know that v vanishes on two lines $L_j = L(\phi_j)$, $j = 1, 2$, where $\phi_1 < \phi_2$ without loss of generality. Hence, by (21) we have

$$\theta = 2\phi_1 + \frac{\pi}{2} = 2\phi_2 - \frac{\pi}{2}\,. \tag{22}$$

Now we observe that the relations (22) are not possible for another incident direction $\theta_1 \neq \theta$. This completes the proof if v has the representation (16).

Finally we consider the case $v|_L = 0$ with $L = L(\phi)$ and v given by (17). Then we have

$$A \sin(tk \sin(\theta - \phi)) + B \sin(tk \cos\phi) = 0, \quad t \in \mathbb{R}. \tag{23}$$

For $B = 0$, (23) implies that $\phi = \theta$ which is not possible for $\phi = \phi_1,\ \phi_2,\ \phi_1 < \phi_2$. Thus we have proved the uniqueness with one incident wave if the Rayleigh frequencies are excluded.

We are left with the case $B \neq 0$, i.e., there exist Rayleigh frequencies. Since (19) also holds with cos replaced by sin, we obtain the relations (20)–(22) again. However, taking a different incident direction $\theta_1 \neq \theta$, we obtain a contradiction as above. This finishes the proof of assertion (i) of the theorem. □

2.4. End of proof in the Neumann case. This time the finite expansion (11) satisfies $\partial_\nu u|_L = 0$ on each straight line extending a segment of $\Lambda_1 \cup \Lambda_2$. By assumption, there are at least two lines $L_j = L(\phi_j)$, $j = 1, 2,\ \phi_1 < \phi_2$, with this property. Let again $\psi = \phi_2 - \phi_1$ be the angle between L_1 and L_2. Using now the reflection argument with even extension, we arrive at the following two cases (compare Subsection 2.3):

(i): $\psi = \lambda\pi,\ \lambda \in (0, 1)$ irrational, and $\partial_\nu v|_{L_{j\psi}} = 0$ for all $j \in \mathbb{N}$.
Hence we have $\partial_\nu v|_L = 0$ for all directions $L = L(\phi),\ \phi \in [0, 2\pi)$. Passing to polar coordinates $x = r(\cos\phi, \sin\phi)$ and multiplying by r, from (10) we obtain the relation

$$\begin{aligned} &-(i\alpha x_2 + i\beta x_1)\, A \exp(i\alpha x_1 - i\beta x_2) \\ &\qquad + \sum_{n\in P} \big(- i(n+\alpha)x_2 + i\beta_n x_1 \big)\, A_n \exp\big(i(n+\alpha)x_1 + i\beta_n x_2\big) = 0 \end{aligned} \tag{24}$$

for all $x \in \mathbb{R}^2$. However, (24) is not possible since $A \neq 0$ and the functions on the left hand side of this equality are linearly independent on $\mathbb{R}^2$. Therefore the case (i) cannot happen.

(ii): λ rational, and $\partial_\nu v|_{L_{j\pi/N}} = 0,\ j = 0, \ldots, N-1$, for some integer $N \geq 2$. Since v is symmetric with respect to the lines $L_{j\pi/N}$, we obtain this time $v(x) = v(-x),\ x \in \mathbb{R}^2$, implying the relations (12)–(15) without the factor $(-1)^N$. Therefore, v takes the form (16), and we have to investigate the relation $\partial_\nu v|_L = 0$ with $L = L(\phi)$, which can be written as

$$\partial_\nu v|_L = (\alpha \sin\phi + \beta \cos\phi)\, A \sin(tk \sin(\theta - \phi)) + Bk \sin\phi \sin(tk \cos\phi) = 0$$

for $t \in \mathbb{R}$, or equivalently,

$$A \cos(\theta - \phi) \sin(tk \sin(\theta - \phi)) + B \sin\phi \sin(tk \cos\phi) = 0, \quad t \in \mathbb{R}. \tag{25}$$

The last relation holds for $\phi = \phi_j$, $j = 1, 2$, $\phi_1 < \phi_2$. First we consider the case that $\phi_1 = 0$ or $\phi_2 = \frac{\pi}{2}$. Then (25) and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ imply that $\theta = 0$, so that two incident directions are enough to determine the profile in this case.

From now on, we can assume that ϕ_1, $\phi_2 \notin \{0, \frac{\pi}{2}\}$. If $B = 0$ (e.g., no Rayleigh frequency occurs), then (25) gives

$$\theta = \phi, \qquad \text{or} \qquad \theta = \phi - \frac{\pi}{2} \quad \text{if } \phi > 0 \quad \text{and} \quad \theta = \phi + \frac{\pi}{2} \quad \text{if } \phi < 0\,.$$

Hence, by ϕ_1, $\phi_2 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$, we have $\phi_1 < 0 < \phi_2$, so that

$$\phi_1 = \theta \qquad \text{or} \qquad \phi_2 = \theta, \quad \text{with } \phi_2 - \phi_1 = \frac{\pi}{2},\ -\frac{\pi}{2} < \phi_1 < 0 < \phi_2 < \frac{\pi}{2}. \quad (26)$$

From (26) we observe that three different incident directions θ with $B = B(\theta) = 0$ in (25) and $\partial_\nu v(\theta)|_{L_j} = 0$, $j = 1, 2$, are impossible. This proves assertion (ii) of the theorem if Rayleigh frequencies are excluded for each incoming wave.

Finally, let θ be an incident direction such that (15) holds and (25) is fulfilled with $B \neq 0$ and $\phi = \phi_j$, $j = 1, 2$, $\phi_1 < \phi_2$. Since $B \sin\phi \, \sin(tk\cos\phi)$ does not vanish identically in $\mathbb{R}$ by $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$, we have $\cos(\theta - \phi) \neq 0$ in (25), and as in Subsection 2.3 we further obtain relations (22). Note that these relations cannot be true for another incident direction $\theta_1 \neq \theta$ unless the corresponding coefficient $B = B(\theta_1)$ in (25) is zero. Since the equality $B(\theta) = 0$ can only hold for at most two different incident angles θ, we have thus proved that measurements with four incoming waves are always sufficient to ensure the uniqueness in the inverse Neumann problem. This finishes the proof of the theorem. □

Remark 1. The counterexamples of [10] show that in the case of Rayleigh frequencies one incident wave is not enough to ensure the uniqueness in the inverse Dirichlet problem in general.

We now present an example for non-uniqueness in the inverse Neumann problem with two incident waves if no Rayleigh frequencies occur:

Consider the quadratic grid generated by the 2π-periodic extensions of the lines $L(\phi)$, $\phi = \pm\frac{\pi}{4}$, and the incident waves with $\theta_1 = -\frac{\pi}{4}$, $\theta_2 = \frac{\pi}{4}$, and let $k = \frac{1}{\sqrt{2}}$. Then we have

$$k\sin\theta_j = \frac{(-1)^j}{2}\,, \quad k\cos\theta_j = \frac{1}{2}\,,$$
$$k^2 = \frac{1}{2} \neq (n + k\sin\theta_j)^2 = n^2 + (-1)^j n + \frac{1}{4} \quad \text{for all } n \in \mathbb{Z}\,,$$

i.e., no Rayleigh frequencies for θ_1 and θ_2, and the functions

$$v_j(x) = v(\theta_j)(x) = 2\cos\Big((-1)^j\,\frac{x_1}{2} - \frac{x_2}{2}\Big)$$

satisfy the Helmholtz equation in the whole plane and the corresponding quasi-periodicity and radiation conditions. Note that (11) takes the form (16) with $A = 2$ and $B = 0$ for $\theta = \theta_1, \theta_2$. Moreover, the normal derivatives $\partial_\nu v_j$, $j = 1, 2$, vanish on the lines $\{x_2 = \pm x_1\}$ since

$$\partial_\nu v_j|_{L(\pm\frac{\pi}{4})} = 2\cos\left(\theta_j \pm \frac{\pi}{4}\right) \sin\left(tk \sin\left(\theta_j \pm \frac{\pi}{4}\right)\right) = 0$$

for all $t \in \mathbb{R}$ and $j = 1, 2$; compare the left hand side of (25).

Remark 2. To construct a non-uniqueness example for the inverse Neumann problem with three incident angles θ_j, $j = 1, 2, 3$, we consider a rectangular grid generated by the 2π-periodic extensions of two lines $L_i = L(\phi_i)$, $i = 1, 2$, $\phi_1 < \phi_2$. We have to show that, for each j, there is a solution $v_j = v(\theta_j)$ of the Helmholtz equation in the whole plane satisfying homogeneous Neumann conditions on both lines L_i as well as the corresponding quasiperiodicity and radiation conditions. Then, by the considerations at the end of the above proof, we are left with the following case:

$$\phi_1 = \theta_2 = \frac{\theta_1}{2} - \frac{\pi}{4} < 0, \quad \phi_2 = \theta_3 = \frac{\theta_1}{2} + \frac{\pi}{4} > 0\,; \tag{27}$$

compare (22) and (26). Moreover, each $v(\theta_j)$ takes the form (16) with $B(\theta_1) \neq 0$, $B(\theta_2) = B(\theta_3) = 0$, and a Rayleigh frequency occurs at least for $\theta = \theta_1$.

Let $k = \frac{25}{2}$, and choose the incident angles θ_j such that

$$\sin\theta_1 = \frac{7}{25}, \quad \sin\theta_2 = -\frac{3}{5}, \quad \sin\theta_3 = \frac{4}{5}\,, \tag{28}$$

implying the relations

$$\begin{aligned} &\cos\theta_2 = \sin\theta_3 = \frac{4}{5}\,, \quad \cos\theta_3 = -\sin\theta_2 = \frac{3}{5}\,, \\ &\cos\theta_1 = \frac{24}{25} = \cos\left(2\theta_2 + \frac{\pi}{2}\right) = -\sin 2\theta_2\,. \end{aligned} \tag{29}$$

Therefore, the incident angles satisfy the constraints (27), and from (28) and (29) we further have that

$$\begin{aligned} &k(\sin\theta_1, \cos\theta_1) = \Big(\frac{7}{2}, 12\Big), \quad k(\sin\theta_2, \cos\theta_2) = \Big(-\frac{15}{2}, 10\Big), \\ &k(\sin\theta_3, \cos\theta_3) = \Big(10, \frac{15}{2}\Big) \end{aligned} \tag{30}$$

and the straight lines L_i are given by

$$L_1 = \left\{\Big(\frac{4}{5}t, -\frac{3}{5}t\Big) : t \in \mathbb{R}\right\}, \quad L_2 = \left\{\Big(\frac{3}{5}t, \frac{4}{5}t\Big) : t \in \mathbb{R}\right\}. \tag{31}$$

Using the ansatz (16) (with $A = 2$ for each incident direction), together with (30) and (31), it can easily be checked that the functions v_j defined by

$$\begin{aligned} v_1(x) &= 2\cos\Big(\frac{7}{2}x_1 - 12x_2\Big) + 2\cos\Big(\frac{25}{2}x_1\Big), \\ v_2(x) &= 2\cos\Big(\frac{15}{2}x_1 + 10x_2\Big), \quad v_3(x) = 2\cos\Big(10x_1 - \frac{15}{2}x_2\Big) \end{aligned} \tag{32}$$

are the required solutions to the Helmholtz equation. In particular, the functions (32) satisfy homogeneous Neumann conditions on $L_1 \cup L_2$ and thus on the grid generated by the 2π-periodic extensions of these lines. Moreover, for $\theta = \theta_1$, we note that (15) holds with $n_1 = 9$ and $n_2 = -16$, so that indeed a Rayleigh frequency occurs.

From (27) we also obtain the uniqueness in the inverse Neumann problem with three different incident directions if all incident angles are either non-negative or non-positive.

Finally, we remark that we excluded in our theorem the standard non-uniqueness examples where the two profiles are parallels to the x_1 axis. In that case it is easy to find counterexamples to the uniqueness in the inverse Dirichlet and Neumann problems for an arbitrarily large number of incident waves.

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