Existence of Nondecreasing and Continuous Solutions for a Nonlinear Integral Equation with Supremum in the Kernel

J. Caballero, B. López and K. Sadarangani

Abstract. Using a technique associated with measures of noncompactness we prove the existence of nondecreasing solutions of an integral equation of Volterra type with supremum in the kernel, in the space $C[0, 1]$.

Keywords. Measure of noncompactness, fixed point theorem, nondecreasing solutions

Mathematics Subject Classification (2000). 45M99, 47H09

1. Introduction

Integral equations arise naturally in applications of real world problems [1, 3, 11, 13, 14, 15]. The theory of integral equations has been well developed with the help of various tools from functional analysis, topology and fixed-point theory.

The aim of this paper is to investigate the existence of nondecreasing solutions of an integral equation of Volterra type with supremum. Equations of such kind have been studied in other papers ([4, 16, 12], among others) and in the monograph [6]. These equations can be considered with connection to the following Cauchy problem:

$$x'(t) = u(t, \max_{[0,t]} |x(\tau)|), \quad x(0) = 0 \,.$$

The main tool used in our study is associated with the technique of measures of noncompactness. Especially, that technique is very useful in the existence theory for some functional, integral and differential equations [1, 9, 15]. Let us mention that in applications the most useful measures of noncompactness are those defined in an axiomatic way (c.f. [8, 10] and references therein). It is

Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain; jmena@ull.es, blopez@dma.ulpgc.es & ksadaran@dma.ulpgc.es

caused by the fact that using such measures of noncompactness and the fixed point theorem of Darbo type we are able to prove not only the existence of solutions of considered functional and integral equations but also we can obtain certain characterizations of those solutions.

In the present paper, we apply the measure of noncompactness defined in [10]. This measure is closely related to the monotonicity of real functions defined and continuous on bounded and closed interval.

2. Notation and auxiliary facts

Assume E is a real Banach space with norm $\| \cdot \|$ and zero element 0. Denote by B_r the closed ball centered at 0 and with radius r. If X is a nonempty subset of E we denote by $\overline{X}$, Conv X the closure and the closed convex closure of X, respectively. The symbols λX and $X + Y$ denote the usual algebraic operations on sets. Finally, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

Definition (see [9]). A function $\mu : \mathfrak{M}_E \longrightarrow [0, \infty)$ is said to be a *measure of noncompactness* in the space E if it satisfies the following conditions:

1. The family $\ker \mu = \{X \in \mathfrak{M}_E \ : \ \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(Conv X) = \mu(X)$.
4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5. If $\{X_n\}_n$ is a sequence of closed sets of $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$, for $n = 1, 2, \ldots$, and if $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described above is called *the kernel of the measure of noncompactness* μ. Further facts concerning measures of noncompactness and their properties may be found in [9].

Now, let us suppose that M is a nonempty subset of a Banach space E and the operator $T : M \mapsto E$ is continuous and transforms bounded sets onto bounded ones. We say that T satisfies the *Darbo condition* (with constant $k \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have $\mu(TX) \leq k\mu(X)$. If T satisfies the Darbo condition with $k < 1$, then it is called a contraction with respect to μ.

For our purpose we will only need the following fixed point theorem [9].

Theorem 2.1. *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and μ a measure of noncompactness in E. Let $F : Q \mapsto Q$ be a contraction with respect to μ. Then F has a fixed point in the set Q.*

Remark. Under the assumptions of the above theorem it can be shown that the set $\operatorname{Fix} F$ of fixed points of F belonging to Q is a member of $\ker \mu$. In fact, $\mu(\operatorname{Fix} F) = \mu(F(\operatorname{Fix} F)) \leq k\mu(\operatorname{Fix} F)$ and as $k < 1$, we deduce that $\mu(\operatorname{Fix} F) = 0$.

Let $C[0, 1]$ denote the space of all real functions defined and continuous on the interval $[0, 1]$. For convenience, we write $I = [0, 1]$ and $C(I) = C[0, 1]$. The space $C(I)$ is furnished with standard norm

$$\|x\| = \max\{|x(t)| : t \in I\}.$$

Next, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in Section 3. This measure was introduced and studied in [10].

Fix a nonempty and bounded subset X of $C(I)$. For $\varepsilon > 0$ and $x \in X$, denote by $w(x, \varepsilon)$ the modulus of continuity of x defined by

$$w(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\}.$$

Furthermore, put

$$w(X, \varepsilon) = \sup\{w(x, \varepsilon) : x \in X\}$$
$$w_0(X) = \lim_{\varepsilon \to 0} w(X, \varepsilon).$$

Next, let us define the following quantities:

$$i(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in I, t \leq s\}$$
$$i(X) = \sup\{i(x) : x \in X\}.$$

Observe that $i(X) = 0$ if and only if all functions belonging to X are nondecreasing on I. Finally, let

$$\mu(X) = w_0(X) + i(X). \tag{1}$$

It can be shown [10] that the function μ is a measure of noncompactness in the space $C(I)$. Moreover, the kernel $\ker \mu$ consists of all sets X belonging to $\mathfrak{M}_{C(I)}$ such that all functions from X are equicontinuous and nondecreasing on the interval I.

3. Main result

In this section we consider the following nonlinear integral equation of Volterra type:

$$x(t) = a(t) + (Tx)(t) \int_0^t u\Big(t, s, x(s), \max_{[0,s]} |x(\tau)|\Big) ds, \quad t \in I. \tag{2}$$

The functions a and u, as well as the operator T are given while $x = x(t)$ is an unknown function. We will study this equation under the following assumptions:

(i) $a \in C(I)$ and it is nondecreasing and nonnegative on the interval I.

(ii) The operator $T : C(I) \mapsto C(I)$ is continuous and satisfies the Darbo condition for the measure of noncompactness μ (defined in (1)) with a constant q. Moreover, T is a positive operator, i.e., $Tx \geq 0$ if $x \geq 0$.

(iii) There exist nonnegative constants c and d such that $\|Tx\| \leq c + d\|x\|$ for each $x \in C(I)$ and $t \in I$.

(iv) $u : I \times I \times \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ is continuous, $u : I \times I \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ and the function $t \mapsto u(t, s, x, y)$ is nondecreasing for each $(s, x, y) \in I \times \mathbb{R} \times \mathbb{R}_+$.

(v) There exist nonnegative constants α and β with $|u(t, s, x, y)| \leq \alpha + \beta|y|$.

(vi) There exists $L > 0$ such that

$$|u(t, s, x, y) - u(t, s, x', y')| \leq L(\max\{|x - x'|, |y - y'|\}).$$

(vii) There exists $r_0 > 0$ such that $\|a\| + (c + dr_0) \cdot (\alpha + \beta \cdot r_0) \leq r_0$ and $q(\alpha + \beta \cdot r_0) < 1$.

Before we formulate our main result we will prove the following lemmas which will be needed further on.

Lemma 3.1. *Suppose that $x \in C(I)$ and define*

$$(Gx)(t) = \max_{[0,t]} |x(\tau)| \quad for\ t \in I.$$

Then $Gx \in C(I)$.

Proof. Without loss of generality, we can assume that $x \geq 0$. We will prove that for $\varepsilon > 0$

$$w(Gx, \varepsilon) \leq w(x, \varepsilon).$$

Suppose contrary. This means that there exist $t_1, t_2 \in I$, $t_1 \leq t_2$, $t_2 - t_1 \leq \varepsilon$, such that

$$w(x, \varepsilon) < |(Gx)(t_2) - (Gx)(t_1)|. \tag{3}$$

As Gx is a nondecreasing function we have

$$0 < (Gx)(t_2) - (Gx)(t_1). \tag{4}$$

Further, let us find $0 \leq \tau_2 \leq t_2$ with the property $(Gx)(t_2) = x(\tau_2)$. Taking into account the inequality (4), we have $t_1 \leq \tau_2$. Thus,

$$(Gx)(t_2) - (Gx)(t_1) = x(\tau_2) - (Gx)(t_1) \leq x((\tau_2) - x(t_1)$$

and, as $\tau_2 - t_1 \leq t_2 - t_1 \leq \varepsilon$, we get

$$(Gx)(t_2) - (Gx)(t_1) \leq x(\tau_2) - x(t_1) \leq w(x, \varepsilon).$$

Thus, we arrive at a contradiction. Consequently, for $\varepsilon > 0$, $w(Gx, \varepsilon) \leq w(x, \varepsilon)$ and, as $x \in C(I)$, the proof is complete. $\square$

Lemma 3.2. *Let $(x_n), x \in C(I)$. Suppose that $x_n \to x$ in $C(I)$. Then $Gx_n \to Gx$ uniformly on I.*

Proof. Note that, for $t \in I$ and $y \in C(I)$, $(Gy)(t) = \|y_{|[0,t]}\|$, where $y_{|[0,t]}$ denotes the restriction of the function y on the interval $[0,t]$ and the norm is considered in the space $C([0,t])$. In view of this fact we can deduce

$$\|Gx_n - Gx\| = \sup_{t \in I} |(Gx_n)(t) - (Gx)(t)| = \sup_{t \in I} \left| \|x_{n|[0,t]}\| - \|x_{|[0,t]}\| \right| \leq \|x_n - x\|.$$

As $x_n \to x$ in $C(I)$ we obtain the desired result. $\qquad\square$

Now we present our main result.

Theorem 3.3. *Under assumptions* (i)–(vii) *equation* (2) *has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on the interval I.*

Proof. Let us consider the operator A defined on the space $C(I)$ by

$$(Ax)(t) = a(t) + (Tx)(t) \int_0^t u(t,s,x(s),G(x)(s))ds$$

where G is defined in Lemma 3.1

Firstly, if we consider the previous assumptions, we can see clearly that if $x \in C(I)$ then $Ax \in C(I)$. Consequently, the operator A transforms the space $C(I)$ into itself. Moreover, for each $t \in I$ we have

$$|(Ax)(t)| = \left| a(t) + (Tx)(t) \int_0^t u(t,s,x(s),G(x)(s))ds \right|$$
$$\leq \|a\| + (c + d\|x\|) \int_0^t (\alpha + \beta G(x)(s))ds$$
$$\leq \|a\| + (c + d\|x\|) \cdot (\alpha + \beta\|x\|).$$

Hence, $\|Ax\| \leq \|a\| + (c + d\|x\|) \cdot (\alpha + \beta\|x\|)$. From assumption (vii), we can get that the operator A transforms the ball $B_{r_0} = B(0, r_0)$ into itself.

In the sequel, we consider the operator A on the subset $B_{r_0}^+$ of the ball B_{r_0} defined by

$$B_{r_0}^+ = \{x \in B_{r_0} : x(t) \geq 0 \text{ for } t \in I\}.$$

Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. On the other hand, in view of our assumptions (i), (ii) and (iv), if $x \in B_{r_0}^+$, then $Ax \in B_{r_0}^+$.

Next, we prove that A is continuous on $B_{r_0}^+$. To do this , let $\{x_n\}$ be a sequence in $B_{r_0}^+$ such that $x_n \to x$ and we will prove that $Ax_n \to Ax$. In fact,

for each $t \in I$ we have

$$
\begin{aligned}
&\left|(Ax_n)(t) - (Ax)(t)\right| \\
&= \left|(Tx_n)(t)\int_0^t u(t,s,x_n(s),G(x_n)(s))ds - (Tx)(t)\int_0^t u(t,s,x(s),G(x)(s))ds\right| \\
&\leq \left|(Tx_n)(t)\int_0^t u(t,s,x_n(s),G(x_n)(s))ds - (Tx)(t)\int_0^t u(t,s,x_n(s),G(x_n)(s))ds\right| \\
&\quad + \left|(Tx)(t)\int_0^t u(t,s,x_n(s),G(x_n)(s))ds - (Tx)(t)\int_0^t u(t,s,x(s),G(x)(s))ds\right| \\
&\leq \left|(Tx_n)(t) - (Tx)(t)\right| \int_0^t \left|u(t,s,x_n(s),G(x_n)(s))\right|ds \\
&\quad + \left|(Tx)(t)\right| \int_0^t \left|u(t,s,x_n(s),G(x_n)(s)) - u(t,s,x(s),G(x)(s))\right|ds \,.
\end{aligned}
$$

In virtue of Lemma 3.2,

$$
\begin{aligned}
&\|Ax_n - Ax\| \\
&\quad \leq \|Tx_n - Tx\|(\alpha + \beta\|x_n\|) \\
&\qquad + (c+d\|x\|)L\int_0^t \max\left\{|x_n(s)-x(s)|, \left[\max_{[0,s]}|x_n(\tau)| - \max_{[0,s]}|x(\tau)|\right]\right\}ds \\
&\quad \leq \|Tx_n - Tx\|(\alpha + \beta \cdot r_0) + (c + d\cdot r_0)L\,\|x_n - x\|.
\end{aligned} \tag{5}
$$

As T is a continuous operator, there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$, we have

$$
\|Tx_n - Tx\| \leq \frac{\varepsilon}{2(\alpha + \beta r_0)}\,.
$$

Moreover, we can find $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have that $\|x_n - x\| \leq \frac{\varepsilon}{2L(c+dr_0)}$. Finally, if we take $n \geq \max\{n_1, n_2\}$, from (5) we get $\|Ax_n - Ax\| \leq \varepsilon$. This fact proves that A is continuous in $B_{r_0}^+$.

In the sequel, we prove that the operator A satisfies the Darbo condition with respect to the measure of noncompactness introduced in Section 2. Let X be a nonempty subset of $B_{r_0}^+$. Fix $\varepsilon > 0$ and $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \varepsilon$. Without loss of generality, we may assume that $t_1 \leq t_2$. Then we obtain

$$
\begin{aligned}
|(Ax)(t_2) - (Ax)(t_1)| = \bigg|&a(t_2) + (Tx)(t_2)\int_0^{t_2} u(t_2,s,x(s),G(x)(s))ds \\
&-a(t_1) - (Tx)(t_1)\int_0^{t_1} u(t_1,s,x(s),G(x)(s))ds\bigg|
\end{aligned}
$$

$$\leq |a(t_2) - a(t_1)|$$

$$+ \left|(Tx)(t_2)\int_0^{t_2} u(t_2, s, x(s), G(x)(s))ds - (Tx)(t_1)\int_0^{t_2} u(t_2, s, x(s), G(x)(s))ds\right|$$

$$+ \left|(Tx)(t_1)\int_0^{t_2} u(t_2, s, x(s), G(x)(s))ds - (Tx)(t_1)\int_0^{t_2} u(t_1, s, x(s), G(x)(s))ds\right|$$

$$+ \left|(Tx)(t_1)\int_0^{t_2} u(t_1, s, x(s), G(x)(s))ds - (Tx)(t_1)\int_0^{t_1} u(t_1, s, x(s), G(x)(s))ds\right|$$

$$\leq w(a, \varepsilon) + |(Tx)(t_2) - (Tx)(t_1)|\int_0^{t_2} |u(t_2, s, x(s), G(x)(s))|ds$$

$$+ |(Tx)(t_1)|\int_0^{t_2} |u(t_2, s, x(s), G(x)(s)) - u(t_1, s, x(s), G(x)(s))|ds$$

$$+ |(Tx)(t_1)|\int_{t_1}^{t_2} |u(t_1, s, x(s), G(x)(s))|ds$$

$$\leq w(a, \varepsilon) + w(Tx, \varepsilon) \cdot (\alpha + \beta r_0) + (c + dr_0) \cdot w_u(\varepsilon, \cdot) + \varepsilon \cdot (c + dr_0)(\alpha + \beta r_0).$$

Hence,

$$w(Ax, \varepsilon) \leq w(a, \varepsilon) + w(Tx, \varepsilon) \cdot (\alpha + \beta r_0) + (c + dr_0) \cdot w_u(\varepsilon, \cdot) + \varepsilon \cdot (c + dr_0)(\alpha + \beta r_0).$$

Consequently,

$$w(AX, \varepsilon) \leq w(a, \varepsilon) + w(TX, \varepsilon) \cdot (\alpha + \beta r_0) + (c + dr_0) \cdot w_u(\varepsilon, \cdot) + \varepsilon \cdot (c + dr_0)(\alpha + \beta r_0).$$

From the uniform continuity of the function u on the set $I \times I \times \mathbb{R}_+ \times \mathbb{R}_+$ and the continuity of the function a on I, we have that $w_u(\varepsilon, \cdot) \to 0$ and $w(a, \varepsilon) \to 0$ as $\varepsilon \to 0$. So, applying limit when $\varepsilon \to 0$, we obtain

$$w_0(AX) \leq (\alpha + \beta r_0)w_0(TX). \tag{6}$$

Now, we study the term related to the monotonicity. Fix $x \in X$ and $t_1, t_2 \in I$ with $t_1 < t_2$. Then, taking into account our assumptions, we have

$$|(Ax)(t_2) - (Ax)(t_1)| - ((Ax)(t_2) - (Ax)(t_1))$$

$$= \left|a(t_2) + (Tx)(t_2)\int_0^{t_2} u(t_2, s, x(s), G(x)(s))ds - a(t_1) - \right.$$

$$\left. - (Tx)(t_1)\int_0^{t_1} u(t_1, s, x(s), G(x)(s))ds\right|$$

$$- \left((a(t_2) + (Tx)(t_2)\int_0^{t_2} u(t_2, s, x(s), G(x)(s))ds - a(t_1) - \right.$$

$$\left. - (Tx)(t_1)\int_0^{t_1} u(t_1, s, x(s), G(x)(s))ds\right)$$

$$\leq [|a(t_2) - a(t_1)| - (a(t_2) - a(t_1))]$$

$$+ \left| (Tx)(t_2) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - (Tx)(t_1) \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds \right|$$

$$- \left((Tx)(t_2) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - (Tx)(t_1) \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds \right)$$

$$\leq \left| (Tx)(t_2) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - (Tx)(t_1) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds \right|$$

$$+ \left| (Tx)(t_1) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - (Tx)(t_1) \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds \right|$$

$$- \left((Tx)(t_2) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - (Tx)(t_1) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds \right)$$

$$- \left((Tx)(t_1) \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - (Tx)(t_1) \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds \right),$$

and hence

$$|(Ax)(t_2) - (Ax)(t_1)| - ((Ax)(t_2) - (Ax)(t_1))$$

$$\leq [|(Tx)(t_2) - (Tx)(t_1)| - ((Tx)(t_2) - (Tx)(t_1))] \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds$$

$$+ (Tx)(t_1) \left[\left| \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds \right| \right. \quad (7)$$

$$\left. - \left(\int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds \right) \right].$$

Now, we will prove that

$$\int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds \geq 0.$$

In fact, notice that

$$\int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds$$

$$= \int_0^{t_2} u(t_2, s, x(s), G(x)(s)) ds - \int_0^{t_2} u(t_1, s, x(s), G(x)(s)) ds$$

$$+ \int_0^{t_2} u(t_1, s, x(s), G(x)(s)) ds - \int_0^{t_1} u(t_1, s, x(s), G(x)(s)) ds$$

$$= \int_0^{t_2} \big(u(t_2, s, x(s), G(x)(s)) - u(t_1, s, x(s), G(x)(s)) \big) ds$$

$$+ \int_{t_1}^{t_2} u(t_1, s, x(s), G(x)(s)) ds \, .$$

Since $t \to u(t, s, x, y)$ is nondecreasing, we have that

$$u(t_2, s, x(s), G(x)(s)) \geq u(t_1, s, x(s), G(x)(s))$$

(assumption (iv)), and, consequently,

$$\int_0^{t_2} (u(t_2, s, x(s), G(x)(s)) - u(t_1, s, x(s), G(x)(s)))ds \geq 0. \qquad (8)$$

On the other hand, as $u : I \times I \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ then

$$\int_{t_1}^{t_2} u(t_1, s, x(s), G(x)(s))ds \geq 0. \qquad (9)$$

Finally, (8) and (9) imply

$$\int_0^{t_2} u(t_2, s, x(s), G(x)(s))ds - \int_0^{t_1} u(t_1, s, x(s), G(x)(s))ds \geq 0.$$

This together with (7) yields

$$|(Ax)(t_2) - (Ax)(t_1)| - ((Ax)(t_2) - (Ax)(t_1))$$

$$\leq [|(Tx)(t_2) - (Tx)(t_1)| - ((Tx)(t_2) - (Tx)(t_1))] \int_0^{t_2} u(t_2, s, x(s), G(x)(s))ds$$

$$\leq (\alpha + \beta \cdot r_0) \cdot i(Tx).$$

Therefore, $i(Ax) \leq (\alpha + \beta \cdot r_0) \cdot i(Tx)$ and, consequently,

$$i(AX) \leq (\alpha + \beta \cdot r_0) \cdot i(TX). \qquad (10)$$

Finally, combining (6) and (10), we get

$$\mu(AX) = w_0(AX) + i(AX) \leq (\alpha + \beta \cdot r_0) \cdot \mu(TX) \leq (\alpha + \beta \cdot r_0) \cdot q \cdot \mu(X).$$

Since $(\alpha + \beta \cdot r_0) \cdot q < 1$ (assumption (vii)), Theorem 1 guarantees the existence of a nondecreasing solution of (2). $\qquad \square$

4. Examples

In this section we show that assumptions of our existence result are rather easy to verify. We illustrate this assertion with the help of some examples.

Example 4.1. Consider the integral equation

$$x(t) = 1 + \frac{1}{8}x(t) \int_0^t \left(t + \sin s + |\cos(x(s))| + \left| \cos\left(\max_{[0,s]} |x(\tau)| \right) \right| \right)ds. \qquad (11)$$

In this case $a(t) = 1$ and this function verifies assumption (i). Moreover,

$(Tx)(t) = \frac{1}{8}x(t)$ and satisfies assumptions (ii) and (iii) with $q = \frac{1}{8}$, $c = 0$ and $d = \frac{1}{8}$. In our case the function u is given by $u(t, s, x, y) = t + \sin s + |\cos x| + |\cos y|$ and satisfies assumptions (iv), (v) and (vi) with $\alpha = 4$, $\beta = 0$ and $L = 2$. In this case the inequality $\|a\| + (c + dr_0)(\alpha + \beta r_0) \leq r_0$ appearing in assumption (vii) has the form $1 + (\frac{1}{8}r_0) \cdot 4 \leq r_0$ which is satisfied for $r_0 = 2$ and, moreover, $q(\alpha + \beta r_0) = \frac{1}{8} \cdot 4 = \frac{1}{2} < 1$. Theorem 3.3 guarantees that the integral equation (11) has a nondecreasing solution in the ball B_2 of the space $C(I)$.

Example 4.2. Consider the integral equation

$$x(t) = e^t + \frac{1}{6} \max_{[0,t]} |x(\tau)| \int_0^t \left(t + \sin s + e^{-\max_{[0,s]} |x(\tau)|} \right) ds. \qquad (12)$$

In this case $a(t) = e^t$ and it is obvious that such function verifies assumption (i) with $\|a\| = e$. The operator T has the form $(Tx)(t) = \frac{1}{6} \max_{[0,t]} |x(\tau)|$. In virtue of Lemma 3.1, such operator satisfies assumptions (ii) and (iii) with $q = \frac{1}{6}$, $c = 0$ and $d = \frac{1}{6}$. In our case, the function u has the form $u(t, s, x, y) = t + \sin s + e^{-y}$ and satisfies assumptions (iv), (v) and (vi) with $\alpha = 3$, $\beta = 0$ and $L = 1$. The inequality $\|a\| + (c + dr_0)(\alpha + \beta r_0) \leq r_0$ appearing in assumption (vii) has the form $e + (\frac{1}{6}r_0) \cdot 3 \leq r_0$ and it is satisfied for $r_0 = 2e$ and, as, $q(\alpha + \beta r_0) = \frac{1}{6} \cdot 3 = \frac{1}{2} < 1$, Theorem 3.3 guarantees that our integral equation has a nondecreasing solution in the ball B_{2e} of the space $C(I)$.

Example 4.3. Consider the integral equation

$$x(t) = 1 + \frac{1}{2} \int_0^t \frac{t}{s + \text{arctg}\, x(s) + \text{arctg}(\max_{[0,s]} |x(\tau)|) + 2\pi} ds. \qquad (13)$$

In this case $a(t) = 1$ and $\|a\| = 1$. This function satisfies assumption (i). The operator is given by $(Tx)(t) = \frac{1}{2}$ which satisfies assumption (ii) and (iii) with $q = 0$, $c = \frac{1}{2}$ and $d = 0$. In our case the function u is given by $u(t, s, x, y) = t(s + \text{arctg}\, x + \text{arctg}\, y + 2\pi)^{-1}$ and such function verifies assumptions (iv), (v) and (vi) with $\alpha = 1$, $\beta = 0$ and $L = 2$. The inequality $\|a\| + (c + dr_0)(\alpha + \beta r_0) \leq r_0$ in assumption (vii) has the form $1 + \frac{1}{2} \leq r_0$ which it is satisfied for $r_0 = \frac{3}{2}$. Moreover, as, $q(\alpha + \beta r_0) = 0 \cdot 1 = 0 < 1$, Theorem 3.3 guarantees that our integral equation has a nondecreasing solution in the ball $B_{\frac{3}{2}}$ of the space $C(I)$.

References

[1] Agarwal, R. P., O'Regan, D. and Wong, P. J. Y., *Positive Solutions of Differential, Difference and Integral Equations*. Dordrecht: Kluwer 1999.

[2] Angelov, V. G. and Bainov, D. D., On the functional differential equations with "maximums". *Appl. Anal.* 16 (1983), 187 – 194.

[3] Argyros, I. K., Quadratic equations and applications to Chandrasekhar's and related equations. *Bull. Austral. Math. Soc.* 32 (1985), 275 – 292.

[4] Bainov, D. D. and Hristova, S. G., Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential equations with "supremum". *J. Math. Anal. Appl.* 172 (1993), 339 – 352.

[5] Bainov, D. D., Milusheva, S. D. and Nieto, J. J., Partially multiplicative averaging for impulsive differential equations with supremum. *Proc. A. Razdmadze Math. Inst.* 110 (1994), 1 – 18.

[6] Bainov, D. D. and Mishev, D. P., *Oscillation Theory for Neutral Differential Equations with Delay.* Bristol: Adam Hilger 1991.

[7] Bainov, D. D. and Zahariev, A. I., Oscillating and asymptotic properties of a class of functional differential equations with maxima. *Czechoslovak Math. J.* 34 (1984), 247 – 251.

[8] Banaś, J., *Applications of Measures of Noncompactness to Various Problems* (in Russian). Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz. 34. Rzeszów 1987.

[9] Banaś, J. and Goebel, K., *Measures of Noncompactness in Banach Spaces.* Lecture Notes Pure Appl. Math. 60. New York: Marcel Dekker 1980.

[10] Banaś, J. and Olszowy, L., Measures of noncompactness related to monotonicity. *Comment. Math.* 41 (2001), 13 – 23.

[11] Banaś, J. and Sadarangani, K., Solvability of Volterra–Stieltjes operator-integral equations and their applications. *Comput. Math. Appl.* 41 (2001), 1535 – 1544.

[12] Caballero, J., López, B. and Sadarangani, K., On monotonic solutions of an integral equation of Volterra type with supremum. *J. Math. Anal. Appl.* 305 (2005), 304 – 315.

[13] Deimling, K., *Nonlinear Functional Analysis.* Berlin: Springer 1985.

[14] Hu, S., Khavanin, M. and Zhuang, W., Integral equations arising in the kinetic theory of gases. *Appl. Anal.* 34 (1989), 261 – 266.

[15] Meehan, M. M. and O'Regan, D., *Existence Theory for Nonlinear Integral and Integrodifferential Equations.* Dordrecht: Kluwer 1998.

[16] Myshkis, A. D., On some problems of the theory of differential equations with deviating argument (in Russian). *Uspehi Mat. Nauk* 32 (1977), 173 – 202.