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Quasi-Periodic Solutions in Nonlinear Asymmetric Oscillations

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Abstract. The existence of Aubry–Mather sets and infinitely many subharmonic solutions to the following p-Laplacian like nonlinear equation

$$(p-1)^{-1}(\phi_p(x'))' + [\alpha\phi_p(x^+) - \beta\phi_p(x^-)] + g(x) = h(t)$$

is discussed, where $\phi_p(u) = |u|^{p-2}u$, $p > 1$, α, β are positive constants satisfying $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = \frac{2}{n}$ with $n \in \mathbb{N}$, h is piece-wise two times differentiable and $2\pi_p$-periodic, $g \in C^1(R)$ is bounded, $x^{\pm} = \max\{\pm x, 0\}$, $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$.

Keywords. Aubry–Mather sets, p-Laplacian, resonance, quasi-periodic solutions

Mathematics Subject Classification (2000). 34C25

1. Introduction

In this paper, we consider the existence of Aubry–Mather sets and quasi-periodic solutions to the following p-Laplacian like nonlinear differential equation

$$(p-1)^{-1}(\phi_p(x'))' + [\alpha\phi_p(x^+) - \beta\phi_p(x^-)] + g(x) = h(t) \quad ('= \tfrac{d}{dt}), \tag{1}$$

where $\phi_p(u) = |u|^{p-2}u, p > 1$ is a constant, $x^{\pm} = \max\{\pm x, 0\}, \alpha, \beta$ are positive constants satisfying

$$\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = \frac{2}{n}\,, \tag{2}$$

h is piece-wise two times differentiable and $2\pi_p$-periodic, $g \in C^1(R)$ is bounded and $\pi_p = \frac{2\pi}{p\sin(\pi/p)}$. If $p = 2$, then $\pi_2 = \pi$ and (1) reduces to second order differential equation

$$x'' + \alpha x^+ - \beta x^- + g(x) = h(t). \tag{3}$$

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The existence of Aubry–Mather sets and quasi-periodic solutions of (3) was discussed recently in [3] if $g \in C^2$ and the limits

$$\lim_{x\to+\infty} g(x) = g(+\infty), \qquad \lim_{x\to-\infty} g(x) = g(-\infty)$$

exist and g satisfies some further approximate properties at infinity. Capietto and Liu [3], by applying a version of Aubry–Mather theory due to Pei [11], proved the existence of quasi-periodic solutions in generalized sense and multiplicity of subharmonic solutions to equation (3) under the so-called "resonance" situation, i.e., (2) holds for $p = 2$ and some $n \in \mathbb{N}$.

Let C be the solution of the initial value problem

$$x'' + \alpha x^+ - \beta x^- = 0, \qquad x(0) = 1, \;\; x'(0) = 0.$$

Assume $\alpha \neq \beta$ and $\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2m}{n}$, $m, n \in \mathbb{N}$. Alonso and Ortega [2] proved that if the function

$$\phi_f(\theta) = \int_0^{2\pi} C\left(\tfrac{m\theta}{n} + t\right) h(t)\, dt$$

has only simple zeros, then any solution x of the linear equation

$$x'' + \alpha x^+ - \beta x^- = h \tag{4}$$

with large initial values, that is, if $|x(t_0)| + |x'(t_0)| \gg 1$ for some $t_0 \in \mathbb{R}$, goes to infinity in the future or in the past. Moreover, they showed the existence of h such that unbounded solutions and 2π-periodic solutions of (4) can coexist.

Fabry and Fonda [4] and Fabry and Mawhin [5] obtained, by degree methods, sufficient conditions for the existence of 2π-periodic solutions and of unbounded solutions as well as subharmonic solutions for (5) below, respectively. More precisely, in [5] the following equation is considered:

$$x'' + \alpha x^+ - \beta x^- = g(x) + f(x) + h(t), \tag{5}$$

and it is proved that if the function

$$\Phi_h(\theta) = \frac{n}{\pi}\left[\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\right] + \frac{1}{2\pi\sqrt{\alpha}} \int_0^{2\pi} C(t+\theta) h(t)\, dt$$

has zeros and all of them are simple, then all solutions of (5) with large initial values are unbounded if the following resonance condition is satisfied:

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n}, \quad n \in \mathbb{N},$$

and f has a sublinear primitive, that is, $\lim_{|x|\to\infty}\frac{1}{x}\int_0^x f(s)ds = 0$. Later, the author of this paper [12] discussed the more general equation (1) and considered the following function:

$$\phi(\theta) = D_p - \frac{p}{\alpha^{\frac{1}{p}}}\int_0^{2\pi_p} h(mt)C_p(mt+\theta)\,dt,$$

where

$$D_p = \frac{2n}{m}B\left(\frac{2}{p},1-\frac{1}{p}\right)\left[\frac{g(+\infty)}{\alpha^{\frac{2}{p}}} - \frac{g(-\infty)}{\beta^{\frac{2}{p}}}\right],$$

$B(r,s)$ is the β-function $B(r,s) = \int_0^1(1-t)^{r-1}t^{s-1}\,dt$ for $r>0$, $s>0$, and $C_p(t)$ is the $\frac{2\pi_p m}{n}$ -periodic solution of the following initial value problem:

$$(p-1)^{-1}(\phi_p(u'))' + [\alpha\phi_p(u^+) - \beta\phi_p(u^-)] = 0, \quad u(0)=1,\ u'(0)=0,$$

if α and β satisfy $\alpha^{-\frac{1}{p}} + \beta^{-\frac{1}{p}} = \frac{2m}{n}$, $m,\,n\in\mathbb{N}$.

It was shown in [12] that if the function $\phi(\theta)$ has no zero for all $\theta\in\mathbb{R}$, then all solutions of (1) are bounded. For more recent results on boundedness and existence 2π-periodic solutions of (1) and (3), we refer [1], [6]–[9], [11]–[15] and the references therein.

In the rest of this paper, we denote by S the unique solution of the initial value problem

$$(p-1)^{-1}(\phi_p(x'))' + [\alpha\phi_p(x^+) - \beta\phi_p(x^-)] = 0, \quad x(0)=0, \quad x'(0)=1. \tag{6}$$

Definition 1.

(A) A solution of $(x_\omega(t),\,x'_\omega(t))$ of (1) is called of *Mather type with rotation number* ω if $\omega = \frac{k}{m}$ is rational, the solutions $(x_\omega(t+2i\pi),\,x'_\omega(t+2i\pi))$, $1\le i\le m-1$, are mutually unlinked periodic solutions of periodic $2m\pi$ and, in this case,

$$\lim_{\omega\to n}\min_{t\in\mathbb{R}}(|x_\omega(t)| + |x'_\omega(t)|) = +\infty.$$

(B) If ω is irrational, the solution $(x_\omega(t),\,x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one, that is, the closed set

$$\overline{\{(x_\omega(2i\pi),\,x'_\omega(2i\pi)),\,i\in\mathbb{Z}\}}$$

is a *Denjoy's minimal set*.

The main results of this paper are formulated in the following theorems:

Theorem 1. *Assume* $h \in L^\infty[0, 2\pi_p]$ *is* $2\pi_p$*-periodic and* $g \in C^1(\mathbb{R})$ *is bounded and satisfies the following conditions: the limits* $\lim_{x\to+\infty} g(x) = g(+\infty)$ *and* $\lim_{x\to-\infty} g(x) = g(-\infty)$ *exist and* g *satisfies*

$$g(x) = g(\pm\infty) + c^\pm |x|^{-(p-1)\sigma} sgn x + O(|x|^{-(p-1)\sigma-1})$$

for $|x| \gg 1$*, where* $\sigma \in (0, \frac{1}{p-1})$ *is a constant and* $c^\pm$ *are constants satisfying*

$$D_0 =: \frac{c^+}{\alpha^{\frac{2-(p-1)\sigma}{p}}} + \frac{c^-}{\beta^{\frac{2-(p-1)\sigma}{p}}} \neq 0.$$

Define a $2\pi_p$*-periodic function* λ_1 *as*

$$\lambda_1(t) = \int_0^{2\pi_p} S(\theta)h(t+\theta)d\theta - \frac{2}{p}\left(\frac{g(+\infty)}{\alpha^{\frac{2}{p}}} - \frac{g(-\infty)}{\beta^{2/p}}\right) B\left(\frac{2}{p}, \frac{1}{q}\right),$$

where $q = \frac{p}{p-1}$ *is the conjugate exponent of* p*. Let one of the following conditions be satisfied:*

(I) $\lambda_1(t) \neq 0$ *for all* $t \in \mathbb{R}$*;*

(II) *either* (a) $\lambda_1(t) \geq 0$ *and* $D_0 < 0$ *or* (b) $\lambda_1(t) \leq 0$ *and* $D_0 > 0$.

Then there exists an $\varepsilon_0 > 0$ *such that for any* $\omega \in (n, n+\varepsilon_0)$*, equation* (1) *has a solution* $(x_\omega(t), x'_\omega(t))$ *of Mather type with rotation number* ω.

Theorem 2. *Assume* $g(x) \equiv 0$*,* h *is piece-wise two times differentiable and* $2\pi_p$*-periodic. Assume*

$$\lambda_1(t) = \int_0^{2\pi_p} S(\theta)h(t+\theta)d\theta \equiv 0\,.$$

For $p \neq 2$*, define a* $2\pi_p$*-periodic function* $\lambda_2(t)$ *as*

$$\lambda_2(t) = (p-2)\left[\int_0^{2\pi_p} S(\theta)h(t+\theta)\int_0^{\theta} S(\tau)h'(t+\tau)d\tau d\theta - \int_0^{2\pi_p} S^2(\theta)h^2(t+\theta)d\theta\right].$$

For $p = 2$*, define a* 2π*-periodic function* $\lambda_3(t)$ *as*

$$\lambda_3(t) = -\frac{1}{2}\left[\int_0^{2\pi} S^3(\theta)h^3(t+\theta)d\theta + \int_0^{2\pi} S(\theta)h''(t+\theta)\left(\int_0^{\theta} S(\tau)h(t+\tau)d\tau\right)^2 d\theta\right]$$
$$- \int_0^{2\pi} S^2(\theta)h(t+\theta)h'(t+\theta)\int_0^{\theta} S(\tau)h(t+\tau)d\tau d\theta.$$

Then the conclusions of Theorem 1 *are true, if one of the following conditions holds:*

(I) $p \neq 2$, $\lambda_2(t) \neq 0$ *for all* $t \in \mathbb{R}$*;*

(II) $p = 2$, $\lambda_3(t) \neq 0$ *for all* $t \in \mathbb{R}$.

2. Generalized polar coordinates transformation

If we introduce a new variable $y = \phi_p(x')$, then (1) is equivalent to the planar system

$$x' = \phi_q(y), \quad y' = (p-1)[-\alpha\phi_p(x^+) + \beta\phi_p(x^-) + h(t) - g(x)], \tag{7}$$

where $q = \frac{p}{p-1}$ is the conjugate exponent of p. Let $u = \sin_p t$ be the solution of the initial value problem

$$(\phi_p(u'))' + (p-1)\phi_p(u) = 0, \qquad u(0) = 0,\ u'(0) = 1$$

which for $t \in [0, \frac{\pi_p}{2}]$ can be expressed implicitly by

$$t = \int_0^{\sin_p t} \frac{ds}{(1-s^p)^{\frac{1}{p}}}.$$

Then it follows from [10] that $u = \sin_p t$ can be extended to $\mathbb{R}$ as a $2\pi_p$-periodic odd C^2-function which satisfies $\sin_p t : [0, \frac{\pi_p}{2}] \to [0,1]$ and $\sin_p(\pi_p - t) = \sin_p t$ for $t \in [\frac{\pi_p}{2}, \pi_p]$, $\sin_p(2\pi_p - t) = -\sin_p t$ for $t \in [\pi_p, 2\pi_p]$.

Let the function S be the unique solution of problem (6), then it is not difficult to verify that $S \in C^2(R)$ is $\frac{2\pi_p}{n}$-periodic and can be expressed as

$$S(t) = \begin{cases} \alpha^{-\frac{1}{p}} \sin_p \alpha^{\frac{1}{p}} t, & t \in \big[0,\, \alpha^{-\frac{1}{p}}\pi_p\big) \\ -\beta^{-\frac{1}{p}} \sin_p \beta^{\frac{1}{p}}(t - \alpha^{-\frac{1}{p}}\pi_p), & t \in \big[\alpha^{-\frac{1}{p}}\pi_p,\, \frac{2\pi_p}{n}\big], \end{cases} \tag{8}$$

from which it is easy to verify that the following equality holds:

$$|S'(t)|^p + \alpha(S^+(t))^p + \beta(S^-(t))^p \equiv 1, \quad t \in R. \tag{9}$$

For $\rho > 0$, $\theta(\mathrm{mod}\ 2\pi_p)$, we define the generalized polar coordinates transformation $(\rho, \theta) \to (x, y)$ as

$$x = \rho^{\frac{1}{p}} S\big(\tfrac{\theta}{n}\big), \qquad y = \rho^{\frac{1}{q}} \phi_p\big(S'\big(\tfrac{\theta}{n}\big)\big).$$

Under this transformation and by using (9), (7) is changed into the planar system

$$\rho' = p\rho^{\frac{1}{p}} S'\big(\tfrac{\theta}{n}\big)(h(t) - g(x)), \qquad \theta' = n - n\rho^{-\frac{1}{q}} S\big(\tfrac{\theta}{n}\big)(h(t) - g(x)). \tag{10}$$

If we define $r = \rho^{\frac{1}{q}}$, then (10) can be further simplified as

$$r' = (p-1)S'\big(\tfrac{\theta}{n}\big)(h(t) - g(x)), \quad \theta' = n\left[1 - r^{-1} S\big(\tfrac{\theta}{n}\big)(h(t) - g(x))\right], \tag{11}$$

where $x = r^{\frac{1}{p-1}} S\big(\frac{\theta}{n}\big)$.

Let $(r(t; r_0, \theta_0),\ \theta(t; r_0, \theta_0))$ be the solution of (11) with initial value $(r_0,\ \theta_0)$. Then for large initial value, i.e., $r_0 \gg 1$, by the boundedness of h, g, S, S' and for t in any bounded interval $I \subset [0,\ 2n\pi_p]$, we get $r(t) = r_0 + O(1)$ which yields $r^{-1}(t) = r_0^{-1} + O(r_0^{-2})$. Going back to (11), we get for $t \in I$, $\theta'(t) \geq \frac{1}{2} > 0$. As in [3], we can write (11) in the following equivalent form:

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{1}{n(1 - r^{-1}S\left(\frac{\theta}{n}\right)(h(t) - g(x))} \\ \frac{dr}{d\theta} &= \frac{(p-1)S'\left(\frac{\theta}{n}\right)(h(t) - g(x))}{n(1 - r^{-1}S\left(\frac{\theta}{n}\right)(h(t) - g(x))}\,. \end{aligned} \tag{12}$$

Now let $(r(\theta; r_0, t_0),\ t(\theta; r_0, t_0))$ be the solution of (12) with initial value $(r_0,\ t_0)$ where $t_0 \in I$ and $\theta \in [0,\ 2n\pi_p]$. Then for $r_0 \gg 1$, we obtain $r(\theta) \geq r_0/2 \gg 1$ and (12) can be written as

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{p-1}{n}\Big[S'\left(\tfrac{\theta}{n}\right)(h(t)-g(x)) + r^{-1}(\theta)S'\left(\tfrac{\theta}{n}\right)S\left(\tfrac{\theta}{n}\right)(h(t)-g(x))^2 + \cdots\Big] \\ \frac{dt}{d\theta} &= \frac{1}{n}\Big[1 + r^{-1}(\theta)S\left(\tfrac{\theta}{n}\right)(h(t)-g(x)) + r^{-2}(\theta)S^2\left(\tfrac{\theta}{n}\right)(h(t)-g(x))^2 + \cdots\Big], \end{aligned} \tag{13}$$

where $x = x(\theta) = r_0^{\frac{1}{p-1}} S\left(\frac{\theta}{n}\right) + O(1)$.

3. Lemmas

For the proof of theorems, we need the following lemmas:

Lemma 1. *Assume the conditions of Theorem* 1 *hold, then we have*

$$\begin{aligned} r_1 &= r_0 + \mu_0(t_0) + O(r_0^{-1}) \\ t_1 &= t_0 + 2\pi_p + \lambda_1(t_0)r_0^{-1} + \lambda_{1+\sigma}r_0^{-(1+\sigma)} + O(r_0^{-2}), \end{aligned} \tag{14}$$

where $r_1 = r(2n\pi_p; r_0, t_0)$, $t_1 = t(2n\pi_p; r_0, t_0)$ *and*

$$\begin{aligned} \mu_0(t) &= (p-1)\int_0^{2\pi_p} S'(\theta)f(t+\theta)d\theta \\ \lambda_1(t) &= \int_0^{2\pi_p} S(\theta)f(t+\theta)d\theta - \frac{2}{p}\left(\frac{g(+\infty)}{\alpha^{\frac{2}{p}}} - \frac{g(-\infty)}{\beta^{\frac{2}{p}}}\right)B\left(\frac{2}{p}, \frac{1}{q}\right) \\ \lambda_{1+\sigma} &= -\frac{2}{p}\left(\frac{c^+}{\alpha^{\frac{\tau+1}{p}}} + \frac{c^-}{\beta^{\frac{\tau+1}{p}}}\right)B\left(\frac{\tau+1}{p}, \frac{1}{q}\right) \\ &= -D_0\frac{2}{p}B\left(\frac{\tau+1}{p}, \frac{1}{q}\right), \end{aligned}$$

where $\tau = 1 - (p-1)\sigma \in (0,\ 1)$. *Moreover, we have* $\mu_0(t) = -(p-1)\lambda_1'(t)$.

Proof. It follows from (13) and for $t_0 \in \mathbb{R}$ and $\theta \in [0, 2n\pi_p]$, we have

$$r(\theta) = r_0 + O(1), \qquad t(\theta) = t_0 + \tfrac{\theta}{n} + O(r_0^{-1}). \tag{15}$$

For $r_0 \gg 1$, substituting (15) into (13), then integrating over $[0, 2n\pi_p]$ and letting $r_1 = r(2n\pi_p)$, $t_1 = t(2n\pi_p)$, we obtain

$$\begin{aligned} r_1 &= r_0 + \mu_0(t_0) + O(r_0^{-1}) \\ t_1 &= t_0 + 2\pi_p + \lambda_1(t_0) r_0^{-1} + \lambda_{1+\sigma} r_0^{-(1+\sigma)} + O(r_0^{-2}), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mu_0(t) = \frac{p-1}{n} &\int_0^{2n\pi_p} S'\big(\tfrac{\theta}{n}\big) h\big(t + \tfrac{\theta}{n}\big) d\theta \\ &- \frac{p-1}{n} g(+\infty) \int_I S'\big(\tfrac{\theta}{n}\big) d\theta - \frac{p-1}{n} g(-\infty) \int_J S'\big(\tfrac{\theta}{n}\big) d\theta \\ &= (p-1) \int_0^{2\pi_p} S'(\theta) h(t+\theta) d\theta, \end{aligned}$$

and

$$\begin{aligned} \lambda_1(t) &= \frac{1}{n} \left[\int_0^{2n\pi_p} S\big(\tfrac{\theta}{n}\big) h\big(t + \tfrac{\theta}{n}\big) d\theta - g(+\infty) \int_I S\big(\tfrac{\theta}{n}\big) d\theta \right] - \frac{1}{n} g(-\infty) \int_J S\big(\tfrac{\theta}{n}\big) d\theta \\ &= \int_0^{2\pi_p} S(\theta) h(t+\theta) d\theta - g(+\infty) \int_0^{\frac{\pi_p}{\alpha^{1/p}}} S(\theta) d\theta + g(-\infty) \int_{\frac{\pi_p}{\alpha^{1/p}}}^{\frac{2\pi_p}{n}} |S(\theta)| d\theta \,, \end{aligned}$$

where $I = \{\theta \in [0, 2n\pi_p] : S(\frac{\theta}{n}) > 0\}$ and $J = \{\theta \in [0, 2n\pi_p] : S(\frac{\theta}{n}) < 0\}$.

By using the similar method used in [12], we can show that

$$\begin{aligned} \int_0^{\frac{\pi_p}{\alpha^{1/p}}} S(\theta) d\theta &= \frac{1}{\alpha^{\frac{2}{p}}} \frac{2}{p} B\left(\frac{2}{p}, \frac{1}{q}\right) \\ \int_{\alpha^{1/p}}^{\frac{2\pi_p}{n}} S(\theta) d\theta &= \frac{1}{\beta^{\frac{2}{p}}} \frac{2}{p} B\left(\frac{2}{p}, \frac{1}{q}\right). \end{aligned}$$

From above equalities, we obtain the expressions of $\mu_1(t)$ and $\lambda_1(t)$.

Next, we calculate the value $\lambda_{1+\sigma}$. From (16) and the expression of S in (8), we obtain

$$\begin{aligned} \lambda_{1+\sigma} &= -c^+ \int_0^{\frac{\pi_p}{\alpha^{1/p}}} \big(S(\theta)\big)^\tau d\theta - c^- \int_{\frac{\pi_p}{\alpha^{1/p}}}^{\frac{2\pi_p}{n}} |S(\tau)|^\tau d\theta \\ &= -\left(\frac{c^+}{\alpha^{\frac{\tau+1}{p}}} + \frac{c^-}{\beta^{\frac{\tau+1}{p}}} \right) \int_0^{\pi_p} (\sin_p \theta)^\tau d\theta \end{aligned}$$

and

$$\int_0^{\pi_p} (\sin_p \theta)^\tau d\theta = 2 \int_0^{\frac{\pi_p}{2}} (\sin_p \theta)^\tau d\theta = \frac{2}{p} B\left(\frac{\tau+1}{p}, \frac{1}{q}\right),$$

which yields the expression of $\lambda_{1+\sigma}$. Now the integration by parts yields $\mu_0(t) = -(p-1)\lambda_1(t)$. □

Lemma 2. *Assume the conditions of Theorem 2 hold, then we have*

$$\begin{aligned} r_1 &= r_0 + \mu_1(t_0) r_0^{-1} + O(r_0^{-2}) \\ t_1 &= t_0 + 2\pi_p + \lambda_2(t_0) r_0^{-2} + O(r_0^{-3}), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \mu_1(t) &= -(p-1) \int_0^{2\pi_p} S(\theta) h''(t+\theta) \int_0^\theta S(\tau) h(t+\tau) d\tau d\theta \\ &\quad - 2(p-1) \int_0^{2\pi_p} S^2(\theta) h(t+\theta) h(t+\theta) d\theta \\ \lambda_2(t) &= (p-2) \int_0^{2\pi_p} S(\theta) h'(t+\theta) \int_0^\theta S(\tau) h(t+\tau) d\tau d\theta \\ &\quad - (p-2) \int_0^{2\pi_p} S^2(\theta) h^2(t+\theta) d\theta. \end{aligned}$$

Moreover, we have $(p-2)\mu_1(t) = (p-1)\lambda_2'(t)$.

Proof. Substituting (15) into (13) and integrating over $[0,\, \theta] \subset [0,\, 2\pi_p]$ we, obtain

$$\begin{aligned} r(\theta) &= r_0 + \mu_0(t_0, \theta) + O(r_0^{-1}) \\ t(\theta) &= t_0 + \tfrac{\theta}{n} + \lambda_1(t_0, \theta) r_0^{-1} + O(r_0^{-2}) \\ r^{-1}(\theta) &= r_0^{-1} - \mu_0(t_0, \theta) r_0^{-2} + O(r_0^{-3}), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mu_0(t, \theta) &= \frac{p-1}{n} \int_0^\theta S'\big(\tfrac{\tau}{n}\big) h\big(t + \tfrac{\tau}{n}\big) d\tau \\ \lambda_1(t, \theta) &= \frac{1}{n} \int_0^\theta S\big(\tfrac{\tau}{n}\big) h\big(t + \tfrac{\tau}{n}\big) d\tau. \end{aligned} \tag{19}$$

Substituting (18)–(19) into (13) and integrating over $[0,\, 2n\pi_p]$, we get

$$\begin{aligned} r_1 &= r_0 + \mu_0(t_0) + \mu_1(t_0) r_0^{-1} + O(r_0^{-2}) \\ t_1 &= t_0 + 2\pi_p + \lambda_1(t_0) r_0^{-1} + \lambda_2(t_0) r_0^{-2} + O(r_0^{-3}), \end{aligned}$$

where $\lambda_1(t) = \lambda_1(t, 2n\pi_p)$, $\mu_0(t) = \mu_0(t, 2n\pi_p)$,

$$\begin{aligned}\mu_1(t) = {}& \frac{p-1}{n}\int_0^{2n\pi_p} S'\big(\tfrac{\theta}{n}\big)h'\big(t+\tfrac{\theta}{n}\big)\lambda_1(t,\theta)d\theta \\ &+ \frac{p-1}{n}\int_0^{2n\pi_p} S\big(\tfrac{\theta}{n}\big)S'\big(\tfrac{\theta}{n}\big)h^2\big(t+\tfrac{\theta}{n}\big)d\theta \\ = {}& -(p-1)\int_0^{2\pi_p} S(\theta)h''(t+\theta)\int_0^{\theta} S(\tau)h(t+\tau)d\tau d\theta \\ &- 2(p-1)\int_0^{2\pi_p} S^2(\theta)h(t+\theta)h'(t+\theta)d\theta - (p-1)\lambda_1(t)\lambda_1'(t)\end{aligned}$$

and

$$\begin{aligned}\lambda_2(t) = {}& \frac{1}{n}\int_0^{2n\pi_p} S^2\big(\tfrac{\theta}{n}\big)h^2\big(t+\tfrac{\theta}{n}\big)d\theta - \frac{1}{n}\int_0^{2n\pi_p} S\big(\tfrac{\theta}{n}\big)h\big(t+\tfrac{\theta}{n}\big)\mu_0(t,\theta)d\theta \\ &+ \frac{1}{n}\int_0^{2n\pi_p} S\big(\tfrac{\theta}{n}\big)h'\big(t+\tfrac{\theta}{n}\big)\lambda_1(t,\theta)d\theta \\ = {}& (p-2)\int_0^{2\pi_p} S(\theta)h(t+\theta)\int_0^{\theta} S(\tau)h'(t+\tau)d\tau d\theta \\ &- (p-2)\int_0^{2\pi_p} S^2(\theta)h^2(t+\theta)d\theta + \lambda_1(t)\lambda_1'(t).\end{aligned}$$

From above equalities we obtain after some elementary calculation

$$(p-2)\mu_1(t) = (p-1)\Big[\lambda_2'(t) - \frac{p}{2}\big(\lambda_1'(t)\big)^2 - (p-1)\lambda_1(t)\lambda_1''(t)\Big],$$

which implies that, for $\lambda_1(t) \equiv 0$, we have $(p-2)\mu_1(t) = (p-1)\lambda_2'(t)$. □

Lemma 3. *Assume that the conditions of Theorem 2 hold, and $p = 2$. Then*

$$\begin{aligned} r_1 &= r_0 + \mu_1(t_0)r_0^{-1} + O(r_0^{-2}) \\ t_1 &= t_0 + 2\pi + \lambda_3(t_0)r_0^{-3} + O(r_0^{-4}), \end{aligned} \tag{20}$$

where

$$\mu_1(t) = -\int_0^{2\pi} S(\theta)h''(t+\theta)\int_0^{\theta} S(\tau)h(t+\tau)d\tau d\theta - 2\int_0^{2\pi} S^2(\theta)h(t+\theta)h'(t+\theta)d\theta$$

and $\lambda_3(t)$ is given as in Theorem 2.

Proof. Substituting (15) into (13) and integrating over $[0, \theta] \subset [0, 2n\pi]$, we obtain (18) with μ_0, λ_1 given by (19) with $p = 2$. Substituting (18) into (13) and integrating over $[0,\, \theta] \subset [0,\, 2n\pi]$, we obtain

$$\begin{aligned} r(\theta) &= r_0 + \mu_0(t_0,\theta) + \mu_1(t_0,\theta) r_0^{-1} + O(r_0^{-2}) \\ t(\theta) &= t_0 + \tfrac{\theta}{n} + \lambda_1(t_0,\theta) r_0^{-1} + \lambda_2(t_0,\theta) r_0^{-2} + O(r_0^{-3}) \\ r^{-1}(\theta) &= r_0^{-1} - \mu_0(t_0,\theta) r_0^{-2} + (\mu_0^2(t_0,\theta) - \mu_1(t_0,\theta)) r_0^{-3} + O(r_0^{-4}), \end{aligned} \tag{21}$$

where

$$\begin{aligned} \mu_1(t,\theta) &= \frac{1}{n}\int_0^\theta S'\big(\tfrac{\tau}{n}\big) h'\big(t+\tfrac{\tau}{n}\big)\lambda_1(t,\tau)d\tau \\ &\quad + \frac{1}{n}\int_0^\theta S\big(\tfrac{\tau}{n}\big) S'\big(\tfrac{\tau}{n}\big) h^2\big(t+\tfrac{\tau}{n}\big)d\tau \\ &= \frac{1}{n^2}\int_0^\theta S'(\tfrac{\tau}{n}) h'(t+\tfrac{\tau}{n}) \int_0^\tau S\big(u/n\big) h\big(t+u/n\big) du d\tau \\ &\quad + \frac{1}{n}\int_0^\theta S\big(\tfrac{\tau}{n}\big) S'\big(\tfrac{\tau}{n}\big) h^2\big(t+\tfrac{\tau}{n}\big)d\tau \end{aligned} \tag{22}$$

and

$$\begin{aligned} \lambda_2(t,\theta) &= -\frac{1}{n}\int_0^\theta S\big(\tfrac{\tau}{n}\big) h\big(t+\tfrac{\tau}{n}\big)\mu_0(t,\tau)d\tau + \frac{1}{n}\int_0^\theta S^2\big(\tfrac{\tau}{n}\big) h^2\big(t+\tfrac{\tau}{n}\big)d\tau \\ &\quad + \frac{1}{n}\int_0^\theta S(\tau) h(t+\tau)\lambda_1(t,\tau)d\tau \\ &= \frac{1}{n^2}\int_0^\theta Sh \int_0^\tau Sh'\, du\, d\tau + \frac{1}{n^2}\int_0^\theta Sh' \int_0^\tau Sh\, du\, d\tau \\ &= \frac{1}{n^2}\int_0^\theta S\big(\tfrac{\tau}{n}\big) h\big(t+\tfrac{\tau}{n}\big)d\tau \int_0^\theta S\big(\tfrac{\tau}{n}\big) h'\big(t+\tfrac{\tau}{n}\big)d\tau. \end{aligned} \tag{23}$$

Substituting (21)–(23) into (13) again and integrating over $[0,\, 2n\pi]$, we obtain

$$\begin{aligned} r_1 &= r_0 + \mu_0(t_0) + \mu_1(t_0) r_0^{-1} + O(r_0^{-2}) \\ t_1 &= t_0 + 2\pi + \lambda_1(t_0) r_0^{-1} + \lambda_2(t_0) r_0^{-2} + \lambda_3(t_0) r_0^{-3} + O(r_0^{-4}), \end{aligned}$$

where $\lambda_k(t) = \lambda_k(t,\, 2n\pi)$, $k = 1, 2$, $\mu_i(t) = \mu_i(t,\, 2n\pi)$, $i = 0, 1$, and

$$\begin{aligned} \lambda_3(t) &= \frac{1}{n}\int_0^{2n\pi} S^3 h^3 d\theta - \frac{2}{n}\int_0^{2n\pi} S^2 h^2 \mu_0 d\theta + \frac{2}{n}\int_0^{2n\pi} S^2 h h' \lambda_1 d\theta \\ &\quad + \frac{1}{n}\int_0^{2n\pi} Sh(\mu_0^2 - \mu_1) d\theta - \frac{1}{n}\int_0^{2n\pi} Sh'\lambda_1\mu_0 d\theta \\ &\quad + \frac{1}{n}\int_0^{2n\pi} Sh''\lambda_1^2 d\theta + \frac{1}{n}\int_0^{2n\pi} Sh'\lambda_2 d\theta. \end{aligned} \tag{24}$$

Now, substituting the expressions of μ_0, μ_1, λ_1 and λ_2 into (24) and using $\lambda_1(t) \equiv 0$, we obtain from Lemma 2 that $\lambda_2(t) \equiv 0$ and $\mu_0(t) = -\lambda_1(t) \equiv 0$. After some elementary calculation, we obtain the expression of $\lambda_3(t)$ given in Theorem 2. □

4. Proof of the theorems

Now, we are ready to prove the main results of this paper.

Proof of Theorem 1. Assume the conditions of Theorem 1 hold. If (I) is satisfied, then the Poincaré map $P : (t_0, r_0) \to (t_1, r_1)$ of the solutions of (13) has the following form:

$$\begin{aligned} t_1 &= t_0 + 2\pi_p + \lambda_1(t_0) r_0^{-1} + O(r_0^{-2}) \\ r_1 &= r_0 + \mu_0(t_0) + O(r_0^{-1}), \end{aligned} \tag{25}$$

with $\mu_0(t) = -(p-1)\lambda_1'(t)$.

Now we introduce another action variable u and a positive parameter ε by $r = \frac{1}{u\varepsilon}$ with $u \in [1, 2]$. Then $r \gg 1 \Leftrightarrow \varepsilon \ll 1$. Under this transformation, (25) is changed to the following form:

$$\begin{aligned} t_1 &= t_0 + 2\pi_p + \lambda_1(t_0) u_0 \varepsilon + O(\varepsilon^2) \\ u_1 &= u_0 - \mu_0(t_0) u_0^2 \varepsilon + O(\varepsilon^2). \end{aligned}$$

Let $t_1 = t_0 + \varepsilon R(t_0, u_0, \varepsilon)$, $u_1 = u_0 + \varepsilon W(t_0, u_0, \varepsilon)$, then $R(t, u, \varepsilon) = \lambda_1(t) u + O(\varepsilon)$, $W(t, u, \varepsilon) = -\mu_0(t) u_0^2 + O(\varepsilon)$, and for $t \in [0, 2n\pi_p]$, $u \in [1, 2]$, we have

$$|R(t,u,\varepsilon)| + \left|\frac{\partial R(t,u,\varepsilon)}{\partial t}\right| + \left|\frac{\partial R(t,u,\varepsilon)}{\partial u}\right| \le C_1 \tag{26}$$

and

$$|W(t,u,\varepsilon)| + \left|\frac{\partial W(t,u,\varepsilon)}{\partial t}\right| + \left|\frac{\partial W(t,u,\varepsilon)}{\partial u}\right| \le C_2 \tag{27}$$

for some constants C_1, C_2. Moreover, if $\min_{t\in\mathbb{R}} \lambda_1(t) = d_0 > 0$, we have for $\varepsilon \ll 1, t \in \mathbb{R}$, $u \in [1, 2]$,

$$\frac{\partial R(t,u,\varepsilon)}{\partial u} \ge \frac{d_0}{2} > 0$$

and if $\max_{t\in S^1} \lambda_1(t) = -d_1 < 0$, we have

$$\frac{\partial R(t,u,\varepsilon)}{\partial u} \le -\frac{d_1}{2} < 0.$$

In both cases, the Poincaré map of (25) is a monotone map. Going back to (13), we know that the Poincaré map $Q : (\theta_0, r_0) \to (\theta_1, r_1)$ is also monotone if $r_0 \gg 1$. Using similar arguments as in [11, Section 4], we may construct a map $\bar{Q}$ which is a global monotone twist homeomorphism of the cylinder $S^1 \times \mathbb{R}$ and coincides with Q on $S^1 \times [A_0, +\infty)$ with a fixed constant $A_0 \gg 1$, where $S^1 = \mathbb{R}/2\pi_p\mathbb{Z}$. Therefore, the existence of Mather sets M_ω of $\bar{Q}$ is guaranteed by Aubry–Mather theory (see [11]). Moreover, for some small $\varepsilon_0 > 0$, such invariant sets with rotation $\omega \in (n, n + \varepsilon_0)$ lie in the domain $S^1 \times [A_0, +\infty)$. Hence they are just the Aubry–Mather sets of the Poincaré map of Q. The above discussion shows the existence of Mather sets, this implies that (1) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type. Moreover, if $\omega = \frac{m}{k}$ is a rational, the solutions $(x_\omega(t + 2i\pi_p), x'_\omega(t + 2i\pi_p))$, $1 \le i \le k - 1$, are mutually unlinked periodic solutions of period $2k\pi_p$ and $\lim_{k\to+\infty} \min_{t\in\mathbb{R}} \|(x_\omega(t), x'_\omega(t))\| = +\infty$. If ω is irrational, the solution $(x_\omega(t), x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one.

In case (II), by Lemma 1, the Poincaré map of (13) has the form of (14), under the same transformation $r = \frac{1}{u\varepsilon}$, (14) is of the following form:

$$\begin{aligned} t_1 &= t_0 + 2\pi_p + \varepsilon R_1(t_0, u_0, \varepsilon) \\ u_1 &= u_0 + \varepsilon W_1(t_0, u_0, \varepsilon), \end{aligned}$$

where $R_1(t, u, \varepsilon) = \lambda_1(t)u + \lambda_{1+\sigma}u^{1+\sigma}\varepsilon^\sigma + O(\varepsilon^1)$, and $W_1(t, u, \varepsilon) = -\mu_0(t)u^2 + O(\varepsilon)$. It is easy to see that R_1 and W_1 satisfy the similar inequalities as (26) and (27). Moreover, for $\lambda_1(t) \ge 0$ and $D_0 < 0$, we have for $\varepsilon \ll 1, t \in \mathbb{R}$, $u \in [1, 2]$, $\lambda_{1+\sigma} > 0$ and

$$\frac{\partial R_1(t, u, \varepsilon)}{\partial u} = \lambda_1(t) + (1 + \sigma)\lambda_{1+\sigma}\varepsilon^\sigma + O(\varepsilon^1) \ge \lambda_1(t) + \frac{1}{2}(1 + \sigma)\lambda_{1+\sigma}\varepsilon^\sigma > 0.$$

Similarly, for $\lambda_1(t) \le 0$ and $D_0 > 0$, we have $\lambda_{1+\sigma} < 0$ and

$$\frac{\partial R_1(t, u, \varepsilon)}{\partial u} = \lambda_1(t) + (1 + \sigma)\lambda_{1+\sigma}\varepsilon^\sigma + O(\varepsilon^1) \le \lambda_1(t) + \frac{1}{2}(1 + \sigma)\lambda_{1+\sigma}\varepsilon^\sigma < 0.$$

The rest proof is similar to that of case (I), so we omit it for simplicity. □

Proof of Theorem 2 (a sketch). By Lemma 2 and Lemma 3, the Poincaré map of (13) has the form of (17) or the form of (20). Under the transformation $r = \frac{1}{u\varepsilon}$, (17) and (20) have the forms

$$\begin{aligned} t_1 &= t_0 + 2\pi_p + \lambda_2(t_0)u_0^2\varepsilon^2 + O(\varepsilon^3) \\ u_1 &= u_0 - \mu_1(t_0)u_0^3\varepsilon^2 + O(\varepsilon^3), \end{aligned}$$

and

$$t_1 = t_0 + 2\pi + \lambda_3(t_0)u_0^3\varepsilon^3 + O(\varepsilon^4)$$
$$u_1 = u_0 - \mu_1(t_0)u_0^3\varepsilon^3 + O(\varepsilon^4),$$

respectively. Let

$$t_1 = t_0 + 2\pi_p + \varepsilon^2 R_2(t,u,\varepsilon),\ \ u_1 = u_0 + \varepsilon^2 W_2(t,u,\varepsilon) \quad \text{for } p \neq 2$$
$$t_1 = t_0 + 2\pi + \varepsilon^3 R_3(t,u,\varepsilon),\ \ u_1 = u_0 + \varepsilon^3 W_3(t,u,\varepsilon) \quad \text{for } p = 2,$$

respectively, then it is not difficult to verify that for $0 < \varepsilon \ll 1$, $\frac{\partial R_k(t,u,\varepsilon)}{\partial u} \neq 0$ if $\lambda_k(t) \neq 0$, $t \in \mathbb{R}$ for $k = 2, 3$. The rest proofs are similar to that of Theorem 1, so we omit them for simplicity. □

Example 1. Consider equation (1) with $\alpha = \beta = n = 1$, $h(t) \equiv 1$, $g(x) = \arctan x + |x|^{-\tau}\operatorname{sgn} x$, where $\tau \in (0,1)$. Then Theorem 1 implies that, for all $t \in \mathbb{R}$, $\lambda_1(t) = -\frac{2\pi}{p}B(\frac{1}{p}, \frac{1}{q}) < 0$. Now (I) of Theorem 1 implies that the conclusion of Theorem 1 holds.

Example 2. Consider the following equation

$$(p-1)^{-1}(\phi_p(x'))' + \phi_p(x) + |x|^{-\tau}\operatorname{sgn} x - 2|x|^{-\tau}\operatorname{sgn} x = 1, \tag{28}$$

where $p > 1$, $\tau \in (0,1)$. Then $\alpha = \beta = n = 1$, $c^+ = 1$, $c^- = -2$, $h(t) \equiv 1$, and it is easy to see that $S(t) = \sin_p t$, $\lambda_1(t) \equiv 0$ and $D_0 = c^+ + c^- < 0$. Now (II) of Theorem 1 implies that there exists $\varepsilon_0 > 0$ such that for any $\omega \in (n, n+\varepsilon_0)$, (28) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number ω.

Example 3. Consider a special case of (1):

$$(p-1)^{-1}(\phi_p(x'))' + \phi_p(x) = 1. \tag{29}$$

In this example, $p \neq 2$, $\alpha = \beta = n = 1$, $g(x) \equiv 0$, $h(t) = 1$. Then it can be verified that $\lambda_1(t) \equiv 0$, $\lambda_2(t) = (2-p)\int_0^{2\pi_p} \sin_p^2 \theta d\theta \neq 0$. Now Theorem 2 implies that there exists $\varepsilon_0 > 0$ such that for any $\omega \in (n, n+\varepsilon_0)$, (29) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number ω.

Example 4. Consider the following linear equation

$$x'' + \alpha x^+ - \beta x^- = h(t), \tag{30}$$

where $\alpha \neq \beta$ satisfying (2) with $p = 2$, $n = 1$, and h is piecewise continuous and 2π–periodic such that $h(t) = 1$, $t \in [0, \frac{\pi}{\sqrt{\alpha}}]$; $h(t) = \frac{\beta}{\alpha}$, $t \in (\frac{\pi}{\sqrt{\alpha}}, 2\pi]$. Then it follows from Theorem 2 that $\lambda_1(t) = \lambda_2(t) \equiv 0$ and $\lambda_3(t) \equiv \lambda_3(0) = -\frac{2}{3\alpha^3}(\alpha - \beta) \neq 0$. Hence Theorem 2 implies that the conclusion of Theorem 2 holds.

Remark 1. Let $p = 2$, Theorem 1 reduces to [3, Theorem 1], moreover, our assumption $D_0 \neq 0$ is weaker than the assumption $c^\pm \neq 0$ and $c^+c^- > 0$. In case $g(x) \equiv 0$ and $\lambda_1(t) \equiv 0$, the result of [3] cannot be applied to equation (30), but Theorem 2 gives partial results. Therefore, our results are natural generalization and refinements of the result of [3].

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